Ubiquity of Łojasiewicz's Example of a Nonbasic Semialgebraic Set

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0. Introduction

Thirty years ago, Łojasiewicz gave the first example of a semialgebraic set which is not *basic*, that is, which is not the solution of a simultaneous system of inequalities. The key property of that set was that the Zariski closure of its topological boundary crossed the set. In this short note we show that this is, in fact, the only obstruction.

We fix the following notations. Let $X \subset \mathbf{R}^n$ be a real algebraic set, that is, the zero set of finitely many polynomials. We denote by $\mathcal{R}(X)$ the ring of *regular functions* of X, which are by definition the quotients of polynomials whose denominators have no zeros in X. For the sake of simplicity, we assume that X is *irreducible*, so that $\mathcal{R}(X)$ is a domain and its quotient field $\mathcal{K}(X)$ is the field of *rational functions* of X. We set $d = \dim(X)$. For generalities concerning real algebraic sets, regular and singular points, regular and rational functions and mappings, etc., we refer to [BCR, Chap. 3].

1. Generically Basic Sets

A *semialgebraic set* is a subset $S \subset X$ of the form

$$S = \bigcup_{i=1}^{p} \{x \in X \mid f_{i1}(x) > 0, \dots, f_{is_i}(x) > 0, g_i(x) = 0\}$$

for some regular functions f_{ij}, g_i in $\mathcal{R}(X)$. (For the properties of semialgebraic sets see [BCR, Chaps. 2 & 9].) The set S is said to be *basic* if it has a description of the form

$$S = \{x \in X \mid f_1(x) > 0, \dots, f_s(x) > 0\}$$

with $f_1, \dots, f_s \in \mathcal{R}(X)$. A semialgebraic set S is said to be *generically basic* if there are regular functions $f_1, \dots, f_s$, $h \in \mathcal{R}(X)$, $h \neq 0$, such that

$$S \setminus \{x \in X \mid h(x) = 0\} = \{x \in X \mid f_1(x) > 0, \dots, f_s(x) > 0\} \setminus \{x \in X \mid h(x) = 0\}.$$

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Now we recall that: (a) any proper algebraic subset of X can be defined by the vanishing of a single nonzero polynomial; and (b) any semialgebraic subset of dimension less than d is contained in a proper algebraic subset of X. Hence, "generically basic" means "equal to a basic set in the complement of a subset of lower dimension". On the other hand, a theorem of Bröcker [Br, 6.10] says, roughly, that a semialgebraic set $S \subset X$ is basic if and only if $S \cap Y$ is generically basic for all irreducible algebraic sets $Y \subset X$. Thus, the study of basic sets reduces to that of generically basic sets.

Łojasiewicz's example of a nonbasic semialgebraic set [Ło, no. 15, p. 67] is the set $S = \{x<0\} \cup \{y<0\} \subset X = \mathbf{R}^2$. The special property of S is that *it is crossed by the Zariski closure of its topological boundary.* We will see that, in fact, Łojasiewicz found the only obstruction to basicness. However, this statement requires further clarification.

2. Regularly Open Sets

Given any semialgebraic set $S \subset X$, we consider the set S' consisting of all limits of regular points of dimension d of S. This set S' is closed semialgebraic, and its interior in X will be denoted by S^*. In symbols,

$$S^* = \operatorname{Int}(\overline{\operatorname{Int}(S) \cap \operatorname{Reg}(X)}).$$

It is an easy exercise to check the following:

(1) $S \setminus \{x \in X \mid h(x) = 0\} = T \setminus \{x \in X \mid h(x) = 0\}$ for some regular function h if and only if $S^* = T^*$.
(2) $S^* = S^{**}$.

In particular, if we restrict our attention to semialgebraic sets S such that $S = S^*$, there are no losses concerning generic basicness. We call these special semialgebraic sets *regularly open.* This matches the use of the term in general topology, since a set S which is regularly open in our sense verifies $S = \operatorname{Int}(\bar{S})$. A somewhat dual approach to other generic problems on semialgebraic sets can be found in [Ga, Cap. 2] and [BCR, 7.6].

3. Generic Zariski Boundary

Given a semialgebraic set $S \subset X$, we consider the associated regularly open set S^*. Then the Zariski closure of the regular points of X that are boundary points of S^* is called the *generic Zariski boundary* of S, denoted $\partial_Z S$. In symbols, we have

$$\partial_Z S = \overline{(\overline{S^*} \setminus S^*) \cap \operatorname{Reg}(X)}^{Z}.$$

Clearly, the generic Zariski boundary is an algebraic set of dimension less than d.

We say that *S is crossed by its generic Zariski boundary* if the following equivalent conditions hold:

(1) $\dim(S^* \cap \partial_Z S) = d-1$.
(2) S^* contains some regular point of dimension $d-1$ of $\partial_Z S$.

To justify this terminology, suppose for a moment that X is nonsingular. Then the two conditions above are equivalent to the fact that $\partial_Z S$ disconnects S^*.

The first one to notice the importance of Zariski boundaries in connection with basicness was Bröcker (see [Br]).

4. Birational Models

Now let $Y \subset \mathbf{R}^m$ be another irreducible algebraic set, and let $\varphi: Y \to X$ be a *birational mapping*. In fact, what we have is a biregular diffeomorphism $\varphi: Y \setminus \{g=0\} \to X \setminus \{h=0\}$ for some regular functions g, h. Then, given a semialgebraic set $S \subset X$, its inverse image $T = \varphi^{-1}(S) = \varphi^{-1}(S \setminus \{h=0\}) \subset Y$ is also semialgebraic, and will be called a *birational model* of S. Clearly, S is generically basic if and only if T is generically basic. Hence, to check whether a semialgebraic set is generically basic, it does not matter which birational model we choose; but notice that in changing the birational model we do change the generic Zariski boundary.

5. Łojasiewicz's Obstruction

After this preparation we can state Łojasiewicz's essential obstruction to basicness as follows.

THEOREM. *A semialgebraic set $S \subset X$ is generically basic if and only if no birational model of S is crossed by its generic Zariski boundary.*

The use of a birational model cannot be avoided, as the example of Figure 1 shows.

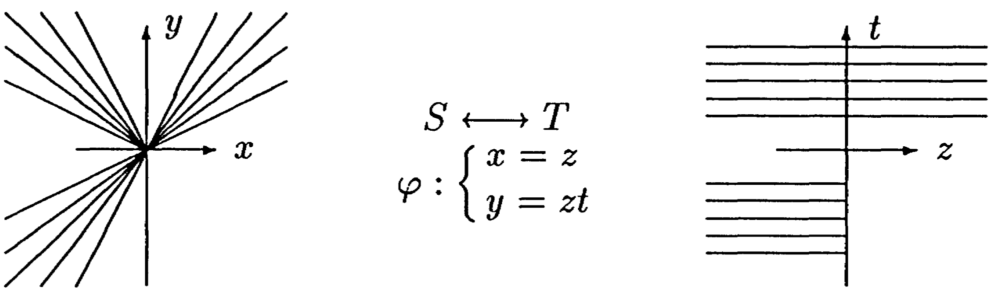

Figure 1

The proof of this theorem is not very difficult, but it requires a special abstract setting that has been available only recently [BCR; Br; ABR]. We describe this setting next.

6. Real Spectra of X [BCR, 4.2, 4.3, & 7.1]

The *real spectrum* of a commutative ring A with 1 is the set of all pairs $\alpha = (\mathfrak{p}_\alpha, \leq_\alpha)$, where $\mathfrak{p}_\alpha$ is a prime ideal of A and $\leq_\alpha$ is an ordering in the residue field $\kappa(\mathfrak{p}_\alpha)$ of $\mathfrak{p}_\alpha$. Thus, given any $f \in A$, we write $f(\alpha) < 0$ (resp. $= 0$, > 0) to mean that $f \bmod \mathfrak{p}_\alpha$ is < 0 (resp. $= 0$, > 0) in the ordered field $(\kappa(\mathfrak{p}_\alpha), \leq_\alpha)$. Now let α and β be prime cones. We say that α *specializes to* β, and write $\alpha \to \beta$, if $f(\beta) > 0$ implies $f(\alpha) > 0$ for $f \in A$; more algebraically, $\alpha \to \beta$ if and only if $\mathfrak{p}_\alpha \subset \mathfrak{p}_\beta$ and the canonical homomorphism $A/\mathfrak{p}_\alpha \to A/\mathfrak{p}_\beta$ sends elements $\geq_\alpha 0$ to elements $\geq_\beta 0$.

In the real spectrum we can impose sign conditions on the elements of A to define the *constructible sets* to be the sets

$$C = \bigcup_{i=1}^{p} \{\alpha \mid f_{i1}(\alpha) > 0, \ldots, f_{is_i}(\alpha) > 0, g_i(\alpha) = 0\}.$$

The constructible sets of the form

$$C = \{\alpha \mid f_1(\alpha) > 0, \ldots, f_s(\alpha) > 0\}$$

generate the so-called *Harrison topology* of the real spectrum, in terms of which the specialization relation introduced above behaves as a limit. For instance, if C is an open constructible set, $\beta \in C$, and $\alpha \to \beta$, then $\alpha \in C$. We will also use that a prime cone α is adherent to a constructible set C if and only if there is some $\beta \in C$ with $\beta \to \alpha$.

In our case we consider the ring $\mathcal{R}(X)$ and denote its real spectrum by Ω. Also, since X is irreducible, we have the real spectrum of the field $\mathcal{K}(X)$, denoted by Σ. Note that the prime cones of Σ are exactly the prime cones α of Ω whose support is $\mathfrak{p}_\alpha = (0)$; these prime cones are in fact the orderings of the field $\mathcal{K}(X)$. Thus, Σ is a subspace of Ω.

7. Tilde Operators [BCR, 7.1, 7.5, & 7.6]

The *tilde operator* $S \mapsto \tilde{S}$ sends a semialgebraic set $S \subset X$ to the constructible set $\tilde{S} \subset \Omega$ defined by any formula that also defines S. By Tarski's principle and some refinements of it, this definition is consistent and we obtain a bijection that preserves inclusions and topological operations. We will use the following fact concerning dimension. Let $\alpha \in \Omega$ have *support* $\mathfrak{p}_\alpha$, and denote by $Z \subset X$ its zero set. Then $\dim(Z) = \dim(Z \cap S)$ for every semialgebraic set $S \subset X$ with $\alpha \in \tilde{S}$.

The tilde operator is used to reformulate many problems concerning X in terms of Ω. However, we need a further modification, since we are to discuss generic basicness.

Indeed, the tilde operator induces a mapping $S \mapsto \tilde{S} \cap \Sigma$, which is *generically* injective: If $S, T \subset X$ are semialgebraic sets, $\tilde{S} \cap \Sigma = \tilde{T} \cap \Sigma$ if and only if $S \setminus \{x \in X \mid h(x) = 0\} = T \setminus \{x \in X \mid h(x) = 0\}$ for some regular function $h \in \mathcal{R}(X)$. In particular, $\tilde{S} \cap \Sigma \neq \emptyset$ if and only if S is Zariski dense in X.

We conclude that, according to our definition of generic basicness, what is relevant for us is this *generic tilde operator.* Furthermore, note that this map is in fact injective if we restrict ourselves to regularly open sets. As remarked above, this is enough for our purposes.

8. Fans and Basicness

We will not define arbitrary fans, but only a special type of fan which is associated with hypersurfaces. First we start with a *real prime divisor* V of the field $\mathcal{K}(X)$; that is, V is a discrete valuation ring of $\mathcal{K}(X)$ whose residue field k is a formally real finitely generated extension of $\mathbf{R}$ of transcendence degree $d-1$. (This residue field is thus the field of rational functions of some real algebraic set of dimension $d-1$, which is a hypersurface of some birational model of X.) Let t be a uniformizer of V. Then every element $f \in V$ can be written in the form $f = ut^n$, where n is the *value* of f and u is a unit of V, so that its residue class $\bar{u}$ in k is not zero. Now we pick two distinct orderings τ_1 and τ_2 in k. It is easy to lift τ_1 and τ_2 to get four orderings σ_1, σ_3 and σ_2, σ_4 of V, and so of its quotient field $\mathcal{K}(X)$. Namely, for an $f = ut^n$ as above, let

$$\sigma_1(f) = \tau_1(\bar{u}), \quad \sigma_3(f) = (-1)^n \tau_1(\bar{u}),$$

$$\sigma_2(f) = \tau_2(\bar{u}), \quad \sigma_4(f) = (-1)^n \tau_2(\bar{u}),$$

where for an ordering σ and an element a we let $\sigma(a)$ denote the sign of a in σ. In other words, the units of V have the sign of their residue classes, and the uniformizer is declared either positive or negative. It follows that $\sigma_4 = \sigma_1 \cdot \sigma_2 \cdot \sigma_3$ (orderings are multiplied as signatures), and the four-element set $F = \{\sigma_1, \sigma_2, \sigma_3, \sigma_4\} \subset \Sigma$ is said to be a *fan of X associated with V.*

Note that $\sigma_1, \sigma_2, \sigma_3, \sigma_4, \tau_1, \tau_2$ belong all to the real spectrum of the ring V, and in that real spectrum the construction gives the specialization relations $\sigma_1, \sigma_3 \to \tau_1$ and $\sigma_2, \sigma_4 \to \tau_2$. We also deduce from the construction that the σ_is are the unique prime cones that specialize to the τ_js.

After this preparation we can state the following theorem.

THEOREM [AR, Thm. 2]. *A semialgebraic set $S \subset X$ is generically basic if and only if $\#(\tilde{S} \cap F) \neq 3$ for every fan F of X associated with a real prime divisor.*

Now we are ready to prove the theorem of Section 5.

9. Proof of the "Only If" Part

Clearly, it is enough to show that if a semialgebraic set S is crossed by its generic Zariski boundary, then S is not generically basic. Replacing S by S^*, we can assume that S is regularly open. Then the hypothesis means that

$\partial_Z S$ has an irreducible component H of dimension $d-1$ such that $D = S \cap H$ is Zariski dense in H. Note that also $E = (\bar{S} \backslash S) \cap \operatorname{Reg}(X) \cap H$ is Zariski dense in H, and that $D \cap E = \emptyset$. Let $\mathfrak{p} \subset \mathcal{R}(X)$ be the ideal of H. By the very definition of $\partial_Z S$, H is not contained in the singular locus of X, and consequently the localization $\mathcal{R}(X)_\mathfrak{p}$ is a local regular ring. Furthermore, H is a hypersurface of X, so that $\mathfrak{p}$ is a height-1 ideal. Hence $V = \mathcal{R}(X)_\mathfrak{p}$ is a discrete valuation ring whose residue field k is the field of rational functions of H. Thus we are ready to construct a fan associated to V, as explained in Section 8. To that end we pick two orderings of the field k, $\tau_1 \in \tilde{D}$ and $\tau_2 \in \tilde{E}$. They exist because D and E are Zariski dense in H, and they are different because $D \cap E = \emptyset$ and the tilde operator is bijective. Finally, let $F = \{\sigma_1, \sigma_2, \sigma_3, \sigma_4\} \subset \Sigma$ be the fan associated with V obtained by lifting τ_1, τ_2, so that $\sigma_1, \sigma_3 \to \tau_1$ and $\sigma_2, \sigma_4 \to \tau_2$ in the real spectrum of V. Now, the ring V is the localization $\mathcal{R}(X)_\mathfrak{p}$, so that its real spectrum is in fact the subspace of Ω consisting of the prime cones whose supports are contained in $\mathfrak{p}$. Hence we can see the above specializations in Ω.

Thus we have $\sigma_1, \sigma_3 \to \tau_1 \in \tilde{D} \subset \tilde{S}$. Then, since specialization behaves as a limit and $\tilde{S}$ is open, we conclude that $\sigma_1, \sigma_3 \in \tilde{S}$.

On the other hand, $\tau_2 \in \tilde{E} \subset \bar{\tilde{S}}$, so that there is $\alpha \in \tilde{S}$ with $\alpha \to \tau_2$. But we also have

$$\tau_2 \in \tilde{E} \subset \Omega \backslash \tilde{S} = \Omega \backslash \operatorname{Int}(\bar{\tilde{S}}) = \overline{\Omega \backslash \bar{\tilde{S}}},$$

so there exists $\beta \in \Omega \backslash \bar{\tilde{S}}$ with $\beta \to \tau_2$. Clearly, α, β, and τ_2 are all distinct. But there are at most two prime cones that specialize to τ_2, and we conclude (up to permutation)

$$\sigma_2 = \alpha \in \tilde{S} \quad \text{and} \quad \sigma_4 = \beta \in \Omega \backslash \tilde{S}.$$

Therefore, $\#(\tilde{S} \cap F) = 3$ and S is not generically basic, by the theorem in Section 8. □

10. Proof of the "If" Part

Let S be a semialgebraic set which is not generically basic. We have to find a birational model T crossed by its generic Zariski boundary, and to do that we may replace X by a compact model.

First of all, by the theorem in Section 8, there is a real prime divisor V of $\mathcal{K}(X)$ and a fan F associated to V such that $\#(\tilde{S} \cap F) = 3$; we also know that the residue field k of V is a finitely generated extension of $\mathbf{R}$ of transcendence degree $d-1$. Then there are elements $h_1, \ldots, h_r \in V \subset \mathcal{K}(X)$ whose residue classes in k generate k over $\mathbf{R}$, and we consider the domain $A = \mathcal{R}(X)[h_1, \ldots, h_r] \subset \mathcal{K}(X)$. We denote by B the integral closure of A in $\mathcal{K}(X)$. This domain B is a finitely generated $\mathbf{R}$-algebra and consequently corresponds to an irreducible algebraic set $Y \subset \mathbf{R}^m$, whose field of rational functions is again $\mathcal{K}(X)$ and whose ring of regular functions $\mathcal{R}(Y)$ is a ring of fractions of B, and consequently a normal domain. Thus we get a birational mapping $f: Y \to X$ with Y normal. Furthermore, $V \supset \mathcal{R}(Y)$ and the maximal ideal $\mathfrak{m}$ of V lies over a prime ideal $\mathfrak{p}$ of $\mathcal{R}(Y)$; since $h_1, \dots, h_r \in \mathcal{R}(Y)$, the residue field of $\mathfrak{p}$ is k. This implies that $\mathfrak{p}$ is a height-1 prime ideal, and since $\mathcal{R}(Y)$ is normal, the localization $\mathcal{R}(Y)_\mathfrak{p}$ is a discrete valuation ring; as V dominates $\mathcal{R}(Y)_\mathfrak{p}$, both rings coincide. Finally, let $H \subset Y$ be the zero set of $\mathfrak{p}$, which is a real hypersurface of Y. Summing up, we have four orderings σ_i of the field $\mathcal{K}(Y)$ that specialize to two orderings τ_j of $\mathcal{K}(H)$ in the form $\sigma_1, \sigma_3 \to \tau_1$, $\sigma_2, \sigma_4 \to \tau_2$, and (say) $\sigma_1, \sigma_2, \sigma_3 \in \tilde{T}$, $\sigma_4 \notin \tilde{T}$, where $T = f^{-1}(S) \subset Y$. We are to show that the birational model T of S is crossed by its generic Zariski boundary. Again, we can assume that T is regularly open.

First we claim that $H \subset \partial_Z T$. Indeed, since $\sigma_2 \in \tilde{T}$ and $\sigma_2 \to \tau_2$, we get $\tau_2 \in \bar{\tilde{T}}$. Were it the case that $\tau_2 \in \tilde{T}$, we would get $\sigma_4 \in \tilde{T}$, since $\tilde{T}$ is open and $\sigma_4 \to \tau_2$. Hence $\tau_2 \in \bar{\tilde{T}} \setminus \tilde{T}$. But τ_2 is an ordering of the field $\mathcal{K}(H)$, so that $H \cap \bar{T} \setminus T$ is Zariski dense in H. By the definition of generic Zariski boundary we deduce that $H \subset \partial_Z T \cup \operatorname{Sing}(Y)$. Finally, since Y is normal, its singular locus has dimension $\leq d-2 < \dim(H)$, so that $H \subset \partial_Z T$ as claimed.

We now turn our attention to the specializations $\sigma_1, \sigma_3 \to \tau_1$; we will deduce from them that $\dim(H \cap T) = d-1$. To that end we consider a semialgebraic triangulation of Y by open simplices compatible with T and H. Since σ_1 and σ_3 are in $\tilde{T}$, and T is a union of simplices, using the tilde operator we find two open simplices $\Delta_1, \Delta_3 \subset T$ of dimension d such that

$$\sigma_1 \in \tilde{\Delta}_1 \quad \text{and} \quad \sigma_3 \in \tilde{\Delta}_3.$$

Similarly, since $\tau_1 \in \tilde{H}$, there is a third simplex $\Gamma \subset H$ of dimension $d-1$ with $\tau_1 \in \tilde{\Gamma}$. Now $\sigma_i \to \tau_1$ implies

$$\tau_1 \in \tilde{\Gamma} \cap \bar{\tilde{\Delta}}_i,$$

which implies that the dimension of $\Gamma \cap \bar{\Delta}_i$ is at least $d-1$. It follows that Γ is a face of dimension $d-1$ of Δ_i, $i = 1, 3$, or (in other words) that the two simplices Δ_1 and Δ_3 of dimension d share the face Γ of dimension $d-1$. Hence the set $W = \Delta_1 \cup \Gamma \cup \Delta_3$ is open and $W \cap H = \Gamma$. Clearly $W \subset \bar{T}$, and since T is regularly open, it follows that $W \subset T$. In conclusion, $\Gamma \subset H \cap T$ and $\dim(H \cap T) = d-1$, as desired.

Also, since $\partial_Z T \supset H$, we have $T \cap \partial_Z T \supset T \cap H$; the set $T \cap \partial_Z T$ has dimension $d-1$, which means that T is crossed by its generic Zariski boundary (Section 3). This completes the proof. □

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