L_2 Cohomology of Pseudoconvex Domains with Complete Kähler Metric

HAROLD DONNELLY

1. Introduction

Let Ω be a bounded pseudoconvex domain with smooth boundary in C^n. The Bergman metric of Ω is a complete Kähler metric. Every biholomorphic automorphism of Ω induces an isometry relative to the Bergman metric. We use $\mathcal{H}_2^{p,q}(\Omega)$ to denote the space of square integrable harmonic (p,q) forms, associated to the Bergman metric. The following result was proved several years ago [5].

THEOREM 1.1. *If Ω is strictly pseudoconvex, then*

$$\dim \mathcal{H}_2^{p,q}(\Omega) = \begin{cases} 0 & p+q \neq n, \\ \infty & p+q = n. \end{cases}$$

Ohsawa has developed this work by giving both alternative proofs and applications to extension problems in the analysis of several complex variables [11; 12; 13].

More recently, Gromov [7] studied the L_2 cohomology of complete Kähler manifolds. Suppose that M is a complete, simply connected, Kähler manifold. Assume that the Kähler form $\omega = d\eta$, where η is bounded in L^∞ norm. Under these hypotheses, Gromov proved this next theorem.

THEOREM 1.2. *If M covers a compact manifold, then*

$$\dim \mathcal{H}_2^{p,q}(M) = \begin{cases} 0 & p+q \neq n, \\ \infty & p+q = n. \end{cases}$$

Theorems 1.1 and 1.2 have analogous conclusions. However, of all strictly pseudoconvex domains in C^n endowed with Bergman metrics, only the ball covers a compact manifold. Thus, the hypotheses of both theorems are only satisfied in a single example.

The purpose of the present article is to show that, nevertheless, the techniques of Gromov can be employed to give a more transparent proof of Theorem 1.1. Gromov's ideas lead to a sufficient condition for $\dim \mathcal{H}_2^{p,q}(\Omega) = 0$,

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$p+q \neq n$, in terms of certain extremal problems involving holomorphic functions. This condition is easily verified for strictly pseudoconvex domains, which gives a proof of vanishing L_2 cohomology outside the middle dimension n. The new proof is more elementary than the argument of [5], since it does not use the asymptotic expansion of the Bergman kernel or the difficult analysis of geodesics in the Bergman metric [6]. Our investigations may also represent some progress toward generalizing Theorem 1.1 to the Bergman metric of weakly pseudoconvex domains.

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2. Vanishing Cohomology for Complete Metrics

Let M be a complete Kähler manifold of complex dimension n. Suppose that $\mathcal{H}_2^i(M)$ denotes the space of square integrable harmonic i forms. If one takes the closure of the image of d, then $\mathcal{H}_2^i(M)$ may be identified with the L_2 cohomology of the complex

$$\Lambda_2^0(M) \xrightarrow{d} \Lambda_2^1(M) \xrightarrow{d} \Lambda_2^2(M) \longrightarrow \cdots \xrightarrow{d} \Lambda_2^{2n}(M).$$

That is, $\mathcal{H}_2^i(M) = \ker d/\overline{\operatorname{im} d}$. Gromov [7] made the following brilliant observation.

PROPOSITION 2.1. *Suppose that the Kähler form ω of M can be written as $\omega = d\eta$, where η is bounded in supremum norm. Then one has $\mathcal{H}_2^i(M) = 0$ for $i \neq n$.*

Proof. Consider the operator $L\phi = \omega \wedge \phi$ for $\phi \in \mathcal{H}_2^i M$. If $i < n$, the primitive decomposition theorem [15] implies that L is injective pointwise on each fiber of $\Lambda^i M$. Since Δ commutes with L, $\omega \wedge \phi$ is harmonic. However, $\omega \wedge \phi = d(\eta \wedge \phi)$ and $\eta \wedge \phi$ is square integrable. It follows from the Hodge decomposition [14] for L_2 cohomology of complete manifolds that $\omega \wedge \phi = 0$. This shows that $\mathcal{H}_2^i(M) = 0$ for $i < n$. The case $i > n$ follows by Poincaré duality, since Δ commutes with the Hodge $*$ operator. □

Let $\mathfrak{D}$ be a hyperconvex domain in an n-dimensional Stein manifold N. Recall that a hyperconvex domain is defined as a bounded domain which admits a plurisubharmonic exhaustion $\beta_0 \colon \Omega \to [-\infty, 0)$. Given β_0, one may construct an exhaustion $\beta \colon \Omega \to [-1, 0)$ such that β is strictly plurisubharmonic and smooth and $\beta \in C^\infty(\Omega)$ [3]. Every pseudoconvex domain with Lipschitz boundary, in a Stein manifold, is hyperconvex [3]. The next proposition was stated without proof in [7], where it is attributed to J. P. Demailly.

PROPOSITION 2.2. *Every hyperconvex domain in a Stein manifold admits a complete Kähler metric whose Kähler form can be written as $\omega = d\eta$, where η is bounded in supremum norm.*

Proof. Define $\psi=-\log(-\beta)$ and $\omega=\sqrt{-1}\partial\bar\partial\psi$. Evidently $\omega=d\eta$, where $\eta=-\sqrt{-1}\partial\psi$. A calculation in local coordinates on N gives

$$\frac{\partial\psi}{\partial z_j}=-\beta^{-1}\frac{\partial\beta}{\partial z_j},$$

$$\frac{\partial^2\psi}{\partial z_i\partial\bar z_j}=\beta^{-2}\frac{\partial\beta}{\partial z_i}\frac{\partial\beta}{\partial\bar z_j}-\beta^{-1}\frac{\partial^2\beta}{\partial z_i\partial\bar z_j}.$$

Since $\beta:\mathfrak{D}\to[-1,0)$ is a strictly plurisubharmonic exhaustion, ω corresponds to a complete Kähler metric. If X is any complex tangent vector, then

$$\frac{|\eta(X)|^2}{\langle X,X\rangle_\omega}\le\frac{|\beta^{-1}\partial\beta(X)|^2}{|\beta^{-1}\partial\beta(X)|^2}=1$$

by the fact that β is plurisubharmonic. □

3. Bergman Metric

Let Ω be a pseudoconvex domain with smooth boundary in C^n. For general pseudoconvex domains, our plan is to give a sufficient condition, for vanishing of L_2 cohomology outside the middle degree, in terms of certain extremal problems involving holomorphic functions. Consultation with experts suggests that this condition may always be satisfied, but a rigorous proof apparently lies well beyond the scope of available techniques in several complex variables. In the strictly pseudoconvex case, our discussion leads to a more elementary proof of the part of Theorem 1.1 concerning cohomology vanishing. Instead of appealing to the asymptotic expansion of the Bergman kernel, one only needs the weighted L_2 estimates of Hörmander [9].

The Bergman kernel $K(z,w)$ is the kernel of the projection operator from $L_2\Omega$ to the subspace of square integrable holomorphic functions. Here $L_2\Omega$ is the Hilbert space associated to the underlying Euclidean metric. The Kähler form of the associated Bergman metric is $\omega=\sqrt{-1}\partial\bar\partial\log K(z,z)=d\eta$, with $\eta=-\sqrt{-1}\partial\log K=-\sqrt{-1}K^{-1}\partial K$. If g_ω is the Hermitian metric corresponding to the different form ω, then we need to establish the boundedness of $|\eta(X)|^2/g_\omega(X,X)$, for $X\in T\Omega$ a nonzero tangent vector at each basepoint, in order to apply Gromov's criterion for the vanishing of the L_2 cohomology of g_ω.

If $\phi_i\in L_2\Omega$ is an orthonormal basis for the subspace of square integrable holomorphic functions, then $K(z,w)=\sum_{i=1}^\infty\phi_i(z)\overline{\phi_i(w)}$. Consequently $\partial K(z,z)=\sum_{i=1}^\infty\partial\phi_i\bar\phi_i$ and moreover $\partial\bar\partial K=\sum_{i=1}^\infty\partial\phi_i\bar\partial\bar\phi_i$. The Kähler form of the Bergman metric is given by

$$\begin{aligned}\omega&=\sqrt{-1}\partial K^{-1}\bar\partial K=\sqrt{-1}(K^{-1}\partial\bar\partial K-K^{-2}\partial K\bar\partial K)\\&=\sqrt{-1}K^{-1}\sum_{i=1}^\infty\partial\phi_i\overline{\partial\phi_i}-\sqrt{-1}K^{-2}\Big(\sum_{i=1}^\infty\partial\phi_i\bar\phi_i\Big)\Big(\sum_{i=1}^\infty\overline{\partial\phi_i}\phi_i\Big).\end{aligned}$$

In addition, one also has $\eta=-\sqrt{-1}K^{-1}\partial K=-\sqrt{-1}K^{-1}\sum_{i=1}^\infty\partial\phi_i\bar\phi_i$.

A special choice of orthonormal basis ϕ_i clarifies the meaning of Gromov's criterion. Fix $z \in \Omega$ and $X \in T_z\Omega - 0$. By elliptic regularity, there is a continuous linear functional, on the space of square integrable holomorphic functions, given by $Tf = Xf(z)$. We choose $\phi_2, \phi_3, \ldots$ to span the kernel of T, and ϕ_1 to span the 1-dimensional orthocomplement of the kernel. Evaluating the crucial ratio at X, we get

$$\frac{|\eta(X)|^2}{g_\omega(X,X)} = \frac{K^{-2}|\partial\phi_1(X)|^2|\phi_1|^2}{K^{-2}|\partial\phi_1(X)|^2\sum_{i=2}^\infty|\phi_i|^2} = \frac{|\phi_1|^2}{\sum_{i=2}^\infty|\phi_i|^2}.$$

Clearly, this ratio is bounded above if and only if $\sum_{i=1}^\infty|\phi_i|^2/\sum_{i=2}^\infty|\phi_i|^2$ is bounded above.

The last ratio can be related to certain extremal problems for holomorphic functions. Let $\|f\|_2$ denote the L_2 norm of f, measured with respect to the Lebesgue measure on Ω. If $z \in \Omega$, consider first the supremum over all holomorphic functions f of $|f(z)|^2$ for $\|f\|_2 \le 1$. By elliptic regularity, $\sup\{|f(z)|^2 \mid \|f\|_2 \le 1\}$ is finite. Choose an orthonormal basis so that $\psi_2, \psi_3, \ldots$ lie in the kernel of the evaluation map at z. If $f = \sum a_i\psi_i$, then $\|f\|_2^2 = \sum_{i=1}^\infty|a_i|^2$ and $|f(z)| = a_1|\psi_1(z)|$. So $\sup\{|f(z)|^2 \mid \|f\|_2 \le 1\} = |\psi_1(z)|^2 = K(z,z)$. Consequently, in any orthonormal basis ϕ_i for the subspace of square integrable holomorphic functions, we have $\sup\{|f(z)|^2 \mid \|f\|_2 \le 1\} = K(z,z) = \sum_{i=1}^\infty|\phi_i(z)|^2$.

If $X \in T_z\Omega$ is a fixed tangent vector, we may also consider the extremal problem $\sup\{|f(z)|^2 \mid X_zf = 0 \text{ and } \|f\|_2 \le 1\}$. As above, we select an orthonormal basis $\phi_1, \phi_2, \ldots$ so that $X\phi_i(z) = 0$, $i \ge 2$. Any square integrable holomorphic f satisfying $X_zf = 0$ may be written as $f = \sum_{i=2}^\infty a_i\phi_i$. Moreover,

$$|f(z)|^2 \le \sum_{i=2}^\infty|a_i|^2 \sum_{i=2}^\infty|\phi_i(z)|^2 = \|f\|_2^2 \sum_{i=2}^\infty|\phi_i(z)|^2,$$

with equality if and only if $a_i = c\overline{\phi_i(z)}$ for some constant c. It follows that $\sup\{|f(z)|^2 \mid X_zf = 0 \text{ and } \|f\|_2 \le 1\} = \sum_{i=2}^\infty|\phi_i(z)|^2$.

Combining these observations, we may state the following proposition.

PROPOSITION 3.1. *Let X vary over all nonzero tangent vectors at all points z of Ω. The ratio $|\eta(X)|^2/g_\omega(X,X)$ is uniformly bounded above if and only if the ratio* $\sup\{|f(z)|^2 \mid X_zf = 0$ *and* $\|f\|_2 \le 1\}/\sup\{|f(z)|^2 \mid \|f\|_2 \le 1\}$ *is uniformly bounded below by a positive constant.*

We now assume that Ω is strictly pseudoconvex and establish the validity of the equivalent conditions of Proposition 3.1.

The model for all strictly pseudoconvex domains is the unit ball in C^n. In the case of the unit ball, one has the closed formula

$$K(z,w) = c_n(1-|z|^2)^{-(n+1)}$$

for the Bergman kernel, which is elementary. One calculates

$$\eta = -\sqrt{-1}\,\partial\log K = -\sqrt{-1}(n+1)(1-|z|^2)^{-1}\bar z_j\partial z_j$$

and

$$\omega = \sqrt{-1}\partial\bar{\partial}\log K = \sqrt{-1}(n+1)(1-|z|^2)^{-2} z_j \bar{z}_k \partial z_k \overline{\partial z_j} + \sqrt{-1}(n+1)(1-|z|^2)^{-1}\partial z_j \overline{\partial z_j}.$$

Thus $|\eta(X)|^2/g_\omega(X,X) \le n+1$.

Standard techniques [4] permit one to localize the problem and subsequently to reduce the strictly pseudoconvex case to the case of the unit ball. Before proceeding further, it is convenient to introduce some abbreviations. Set $K(z,\Omega) = \sup\{|f(z)|^2 \mid \|f\|_2 \le 1\}$, where the supremum is over square integrable holomorphic functions defined on Ω. Similarly, for $X \in T_z\Omega$, define $L_X(z,\Omega) = \sup\{|f(z)|^2 \mid X_z f = 0 \text{ and } \|f\|_2 \le 1\}$. The following monotonicity formulas are immediate from the definitions.

LEMMA 3.2. *If $\Omega_1 \subset \Omega_2$ are bounded domains in C^n, then*

(i) $K(z,\Omega_2) \le K(z,\Omega_1)$;
(ii) $L_X(z,\Omega_2) \le L_X(z,\Omega_1)$.

Here $z \in \Omega_1$ and $X \in T_z\Omega_1$.

It is also straightforward [4] to show that the order of these extremal functionals is locally determined.

LEMMA 3.3. *Let Ω be a bounded pseudoconvex domain, and let $z_0 \in \partial\Omega$. Suppose $\bar{U}_1 \subset U_2$, where U_1, U_2 are small open neighborhoods of z_0. Then, for all $z \in U_1 \cap \Omega$,*

(i) $c_1 K(z, U_2\cap\Omega) \le K(z,\Omega)$;
(ii) $c_2 L_X(z, U_2\cap\Omega) \le L_X(z,\Omega)$.

Proof. For either (i) or (ii), the proof is similar, so we do both parts together. In either case, let f be holomorphic on $U_2\cap\Omega$, and assume that f realizes the supremum in the extremal problem for $K(z, U_2\cap\Omega)$ or $L_X(z, U_2\cap\Omega)$, respectively. We showed above that such f exist and that $\int_{U_2\cap\Omega}|f|^2 = 1$.

Choose an open set U_3 satisfying $\bar{U}_1 \subset U_3$ and $\bar{U}_3 \subset U_2$. Let $\chi \in C_0^\infty(U_2)$, $0 \le \chi \le 1$, be a cut-off function whose restriction to U_3 is identically 1. Define the plurisubharmonic weight functions

$$v_z(\xi) = 2(n+2)\log\|\xi - z\| \quad \text{and} \quad w(\xi) = \log(1+\|\xi\|^2).$$

Set $\beta = \bar{\partial}(\chi f) = f\bar{\partial}\chi$, which extends from $U_2\cap\Omega$ to a C^∞ form on Ω and vanishes identically on $U_3\cap\Omega$. Since $z \in U_1\cap\Omega$, we clearly have

$$\int_\Omega |\beta|^2 \exp(-v_z(\xi))\, d\xi < \infty.$$

Consequently [9, p. 94], there exists $h \in C^\infty\Omega$ satisfying $\bar{\partial}h = \beta$ and

$$\int_\Omega |h|^2 \exp(-v_z - 2w)\, d\xi \le \int_\Omega |\beta|^2 \exp(-v_z)\, d\xi.$$

Since $z \in U_1 \cap \Omega$, the weight $\exp(-v_z)$ is bounded above on the support of β. Moreover, the weight $\exp(-v_z - 2w)$ is bounded below on all of Ω. Thus

$$\int_\Omega |h|^2 \le c_3 \int_\Omega |f|^2 = c_3,$$

where c_3 is independent of z, X, and f.

The function $\hat{f} = \chi f - h$ is holomorphic on all of Ω with

$$\int_\Omega |\hat{f}|^2 \le c_4.$$

Since $\int_\Omega |h|^2 \|\xi - z\|^{-2n-4} d\xi < \infty$, we have $h(z) = 0$ and $X_z h = 0$. The lemma follows since $c_4^{-1/2} \hat{f}$ is an acceptable test function for the extremal problems on Ω. □

Now suppose that Ω is strictly pseudoconvex. If $z_0 \in \partial\Omega$ and U_2 is a sufficiently small neighborhood, then we can map $U_2 \cap \Omega$ biholomorphically to a domain which is strictly convex at the image point of z_0. In the strictly convex case, $U_2 \cap \Omega$ may be squeezed between internally and externally tangent balls. Since the case of the ball was verified by explicit calculations, we have the following result.

PROPOSITION 3.4. *If Ω is any strictly pseudoconvex domain in C^n, then the ratio $|\eta(X)|^2 / g_\omega(X,X)$ is uniformly bounded above.*

Combined with Proposition 2.1, this provides a proof of the following corollary which is independent of any appeal to the asymptotic expansion of the Bergman kernel.

COROLLARY 3.5. *If $\Omega \subset C^n$ is a strictly pseudoconvex domain endowed with its Bergman metric, then $\mathcal{H}_2^i \Omega = 0$ for $i \neq n$.*

Corollary 3.5 was first proved in [5] by a different method. The singular cohomology of a pseudoconvex domain satisfies $H^i(\Omega) = 0$ for $i > n$, but nonzero groups may appear for $i \le n$. In fact, if M is a compact manifold embedded in R^n, then an ϵ neighborhood of M in C^n is strictly pseudoconvex for sufficiently small ϵ. Corollary 3.5 thus stands in stark contrast to the Hodge theorem for compact manifolds. The proof in [5] used the vanishing of singular cohomology for $i > n$, a Mayer–Vietoris argument, and a certain integral formula to control the L_2 cohomology on a neighborhood of $\partial\Omega$ at infinity. Certainly, the criterion of Gromov provides a more transparent explanation of the vanishing theorem for L_2 cohomology.

4. Infinite-Dimensional Spaces of Harmonic Forms

Assume that Ω is a pseudoconvex domain with smooth boundary in C^n. If α is a defining function for Ω (i.e., if $\Omega = \{z \in C^n \mid \alpha(z) > 0\}$), then the Levi form $-\partial^2\alpha/\partial z_i\partial\bar{z}_j$ is positive semidefinite on the complex tangent space of $\partial\Omega$. We require throughout that $\partial\Omega$ be C^2 and $\nabla\alpha \neq 0$ on $\partial\Omega$. Suppose that λ is a convex function of one real variable. One may define $\psi = -\log\alpha + \lambda(\|z\|^2)$. A calculation gives

$$\frac{\partial^2\psi}{\partial z_i\partial\bar{z}_j} = \alpha^{-2}\frac{\partial\alpha}{\partial z_i}\frac{\partial\alpha}{\partial\bar{z}_j} - \alpha^{-1}\frac{\partial^2\alpha}{\partial z_i\partial\bar{z}_j} + \lambda'(\|z\|^2)\delta_{ij} + \lambda''(\|z\|^2)\bar{z}_i z_j.$$

Consequently, if $\lambda'(\|z\|^2)$ is sufficiently large then $\omega = \sqrt{-1}\,\partial\bar{\partial}\psi$ is the Kähler form of a complete metric on M. This leads us to give the following definition.

Definition 4.1. A suitable Kähler metric on a pseudoconvex domain Ω is a metric obtained by the above construction for some choice of α and λ, or a metric quasi-isometric to such a metric.

Applying the criterion of Gromov, it is straightforward to prove the next theorem.

Theorem 4.2. *If $\Omega \subset C^n$ is a pseudoconvex domain with suitable complete Kähler metric, then $\mathcal{H}_2^i\Omega = 0$ for $i \neq n$.*

Proof. Evidently, $\omega = d\eta$ with $\eta = -\sqrt{-1}\,\partial\psi$. It suffices to verify that η is bounded, relative to the complete Kähler metric. If X is any complex tangent vector, then

$$\frac{|\eta(X)|^2}{\langle X, X\rangle_\omega} \le c_1 \frac{|\alpha^{-1}\partial\alpha(X)|^2 + (\lambda'(\|z\|^2)\|z\|\|X\|)^2}{|\alpha^{-1}\partial\alpha(X)|^2 + \lambda'(\|z\|^2)\|X\|^2}$$
$$\le c_1(1+\lambda'(\|z\|^2)\|z\|^2) \le c_2$$

since $\bar{\Omega}$ is compact. □

If Ω is strictly pseudoconvex with C^∞ boundary, then the asymptotic expansion of the Bergman kernel [6] shows that the Bergman metric is quasi-isometric to a suitable Kähler metric. Thus, one has another proof of Corollary 3.5.

Let $\Omega \subset C^n$ be a pseudoconvex domain, with smooth boundary, endowed with a suitable complete Kähler metric g in the sense of Definition 4.1. By $\mathcal{H}_2^{p,q}(\Omega)$ we denote the space of harmonic (p,q) forms that are square integrable with respect to g. If $p+q \neq n$, then Corollary 3.5 implies that $\mathcal{H}_2^{p,q}(\Omega) = 0$. Our aim is to show that $\mathcal{H}_2^{p,q}(\Omega)$ is infinite-dimensional whenever $p+q = n$. If Ω is strictly pseudoconvex and $p+q=n$, then $\dim\mathcal{H}_2^{p,q}(\Omega) = \infty$ was proved in [5; 11]. The alternative demonstration below uses a mixture of ideas from [5] and [7]. Ohsawa's approach [11] is quite different. He relies upon detailed calculations with cleverly chosen test functions.

The preliminary step is to obtain a positive lower bound on the spectrum of $\Delta: L_2\Lambda^i(\Omega) \to L_2\Lambda^i(\Omega)$ for $i \neq n$. This follows from a more general statement concerning complete Kähler manifolds M^n. Suppose that the Kähler form of M^n is ω and consider the operator $L\phi = \omega \wedge \phi$. First we observe the following lemma.

LEMMA 4.3. *If $i+k=n$, then $L^k: L_2\Lambda^i(M) \to L_2\Lambda^{i+2k}(M)$ is a bijection and a quasi-isometry.*

Proof. The primitive decomposition theory [15] shows that

$$L^k: \Lambda^i_x M \to \Lambda^{i+2k}_x M$$

is a bijection, pointwise on each fiber. This theory also provides an inverse for L^k given by an explicit formula involving L and its adjoint. Since $\|\omega\|_\infty < \infty$, both L^k and its inverse are bounded on $L_2\Lambda^i(M)$. □

The following proposition is proved in [7], but we reproduce the proof for the reader's convenience.

PROPOSITION 4.4. *Suppose $\omega = d\eta$, with η bounded in supremum norm. Then the Laplacian $\Delta: L_2\Lambda^i(M) \to L_2\Lambda^i(M)$ has a positive lower bound on its spectrum for $i \neq n$.*

Proof. Supose $i < n$ and $i+k=n$. If $\psi \in L_2\Lambda^{i+2k}(M)$ then, by Lemma 4.1, $\psi = L^k\phi$ for the form $\phi \in L_2\Lambda^i(M)$. Moreover, $c_1\|\phi\|_2 \le \|\psi\|_2 \le c_2\|\phi\|_2$.

We write $\psi = d\theta + \psi'$, with $\theta = \eta \wedge \omega^{k-1} \wedge \phi$ and $\psi' = \eta \wedge \omega^{k-1} \wedge d\phi$. Notice that

$$\|\theta\|_2 \le c_3\|\eta\|_\infty \|\phi\|_2 \le c_4\|\eta\|_\infty \|\psi\|_2$$

and, when ψ lies in the domain of Δ,

$$\|\psi'\|_2 \le c_5\|\eta\|_\infty \|d\phi\|_2 \le c_5\|\eta\|_\infty \langle \Delta\phi, \phi\rangle^{1/2} \le c_6\|\eta\|_\infty \langle \Delta\psi, \psi\rangle^{1/2}.$$

(In the last step, we used the spectral theorem and the fact that Δ commutes with L.) Furthermore,

$$\|\psi\|_2^2 = \langle \psi, d\theta + \psi'\rangle = \langle \psi, d\theta\rangle + \langle \psi, \psi'\rangle.$$

For each piece there is an upper bound,

$$\langle \psi, d\theta\rangle = \langle \delta\psi, \theta\rangle \le \|\delta\psi\|_2 \|\theta\|_2 \le \langle \Delta\psi, \psi\rangle^{1/2} \|\theta\|_2 \le c_4\|\eta\|_\infty \|\psi\|_2 \langle \Delta\psi, \psi\rangle^{1/2},$$

and

$$\langle \psi, \psi'\rangle \le \|\psi\|_2 \|\psi'\|_2 \le c_6\|\eta\|_\infty \|\psi\|_2 \langle \Delta\psi, \psi\rangle^{1/2}.$$

Thus $\langle \Delta\psi, \psi\rangle \ge c_7\|\eta\|_\infty^{-2} \|\psi\|_2^2$.

We have now shown that Δ acting on $L_2\Lambda^{i+2k}(M)$ has spectrum bounded below. Since the Hodge $*$ operator commutes with Δ, the spectrum is also bounded below on $L_2\Lambda^i(M)$. Notice that $i+(i+2k) = 2(i+k) = 2n$. This accounts for all degrees except n. □

To continue with the proof of infinite dimensionality in the middle degree, we incorporate ideas from [5]. The simplest example is the ball in C^n.

LEMMA 4.5. *Consider the ball $B \subset C^n$, where B is endowed with its Bergman metric. Then* $\dim \mathcal{H}_2^{p,q} B = \infty$ *whenever $p+q=n$.*

Proof. The semisimple Lie group $G = SU(n,1)$ acts transitively, holomorphically, and isometrically on B. One has the G-invariant Dolbeault complex:

$$\Lambda_2^{p\,\text{even}}(B) \xrightarrow{\bar\partial+\bar\partial^*} \Lambda_2^{p\,\text{odd}}(B).$$

Let $\mathcal{H}_2^{p,\,\text{even}}$ and $\mathcal{H}_2^{p,\,\text{odd}}$ be the Schwartz kernels of orthogonal projection onto square integrable harmonic forms. The index theorem of [1] or [2] gives

$$\operatorname{Tr} \mathcal{H}_2^{p\,\text{even}}(x,x) - \operatorname{Tr} \mathcal{H}_2^{p\,\text{odd}}(x,x) = T_p(R),$$

where Tr denotes the pointwise trace. Here $T_p(R)$ is a locally computable polynomial in the components of the curvature tensor R. The Hirzebruch proportionality principle [8] guarantees that $T_p(R) \neq 0$ for all $0 \le p \le n$. According to our Theorem 4.2, $\operatorname{Tr} \mathcal{H}_2^{p,q}(x,x) = 0$ whenever $p+q \neq n$. Thus the space $\mathcal{H}_2^{p,q}(B)$ is nonzero when $p+q=n$. Since G acts transitively, $\dim \mathcal{H}_2^{p,q} B = \infty$. □

The main result of this section is given in the following theorem.

THEOREM 4.6. *Let $\Omega \subset C^n$ be a pseudoconvex domain endowed with a suitable Kähler metric, in the sense of Definition 4.1. Then* $\dim \mathcal{H}_2^{p,q}(\Omega) = \infty$ *for $p+q=n$.*

Proof. We first show that zero occurs in the essential spectrum of

$$\Delta: L_2\Lambda^{p,q}(\Omega) \to L_2\Lambda^{p,q}(\Omega) \quad \text{for } p+q=n.$$

Note that Ω has a strictly pseudoconvex point—namely, any point furthest from the origin of coordinates. Near this point [10], the metric and its first derivatives are asymptotic (via normal coordinate comparison) to those of the ball in C^n. For any $\epsilon > 0$, Lemma 4.5 and the minimax principle give a compactly supported $\phi \in \Lambda^{p,q}(B)$ which satisfies $(d\phi, d\phi) + (\delta\phi, \delta\phi) < \epsilon(\phi,\phi)$. Since the suitable metric on Ω is complete, we may transplant infinitely many disjoint copies of ϕ into a small Euclidean neighborhood of the strictly pseudoconvex point of Ω.

Let $\gamma > 0$ be a lower bound for the spectrum of Δ on

$$L_2\Lambda^{p,q-1}(\Omega) \oplus L_2^{p,q+1}(\Omega).$$

If $E_\gamma: L_2\Lambda^{p,q}(\Omega) \to L_2\Lambda^{p,q}(\Omega)$ is the spectral projection, then $E_\gamma\phi \neq 0$ for those transplanted ϕ of the preceding paragraph. However, Proposition 4.4 gives the vanishing of $\bar\partial E_\gamma \phi = E_\gamma \bar\partial \phi = 0$ and $\bar\partial^* E_\gamma \phi = E_\gamma \bar\partial^* \phi = 0$. Consequently, $E_\gamma \phi$ must be harmonic. Since zero occurs in the essential spectrum of Δ on $L_2^{p,q}(\Omega)$ for $p+q=n$, the space of square integrable harmonic forms must be infinite-dimensional. □

Using Theorem 4.6 and the asymptotic expansion of the Bergman kernel [6], we deduce this final result.

COROLLARY 4.7. *Let Ω be a strictly pseudoconvex domain in C^n, endowed with its Bergman metric. Then* $\dim \mathcal{H}_2^{p,q}\Omega = \infty$ *for* $p+q=n$.

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Department of Mathematics
Purdue University
West Lafayette, IN 47907