The Isoperimetric Theorem for General Integrands

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1. Introduction

A round soap bubble solves the classical isoperimetric problem; that is, it minimizes surface area for a given volume. From a physical point of view the bubble minimizes total surface energy arising from surface tension in the soap film. On the other hand, the surface energy of a crystal depends on the surface orientation with respect to the underlying crystal lattice and is given by some norm (or more general integrand) Ψ applied to the unit normal $\mathbf{n}$. (The case of area is given by the Euclidean norm $\Psi(x) = |x|$, so that $\Psi(\mathbf{n}) = 1$.) In 1901, Wulff [Wu] gave a construction for the surface-energy-minimizing shape for a given volume of material, now called the *Wulff shape* B_Ψ, most easily defined as the unit ball in the dual norm:

$$B_\Psi = \{x: \Psi^*(x) \le 1\}.$$

That the Wulff shape B_Ψ uniquely minimizes surface energy was finally proved in great generality by Taylor in 1975 [T1; T2], and more recently by Fonseca and Müller [FM]. Both proofs use the Brunn-Minkowski theorem and the existence of the inverse Radon transform. Specifically, they showed the following.

ISOPERIMETRIC THEOREM. *Among all measurable sets $\Omega \subset \mathbb{R}^n$ with the same volume as B_Ψ, the surface energy $\Psi(\partial\Omega)$ is uniquely minimized by $\Omega = B_\Psi$ (up to translation and sets of volume zero).*

More general geometric and physical problems can be reinterpreted as our isoperimetric problem, as noted for crystals on a table by Winterbottom [Wi] and for crystals in corners by Zia, Avron, and Taylor [ZAT] (earlier discussed by Bauer [Ba, p. 418]). For example, the least-area way to enclose given volume inside a convex cone is a spherical cap, because the capped cone is the Wulff shape for the energy which is 0 in the cone directions and 1 in other directions. Since proofs of [ZAT] did not appear, we further remark that to show the summertop construction [ZAT, 2.2(c), 2.4(δ)] uniquely minimizing, note that any better perturbation of the summertop construction would yield a better perturbation of the complementary Winterbottom

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construction (see Fig. 3(c) of [ZAT]). Extending the perturbation beyond the Winterbottom construction cannot pay off unless space can be enclosed for negative cost, which holds only in the complete wetting cases of infinite negative energy.

We further remark that some limiting cases listed under [ZAT, 2.2(a) or 2.4(α, β)] (when some W_i or W_{ij} "has zero volume") actually can have behavior listed under other cases or a preference for wetting 2 or all 3 dihedral corners.

The purpose of this article is to give a new proof of the isoperimetric theorem which we feel is conceptually much simpler than those given by Taylor and Fonseca–Müller. Our proof is new even for the classical isoperimetric theorem for the area integrand. A simple proof for this case using Steiner symmetrization has been given by DeGiorgi [De]; however, his method will not work for general integrands.

We also note that Dacorogna and Pfister [DP] have given a new proof of the isoperimetric theorem in $\mathbb{R}^2$ for Jordan domains with boundaries which can be parametrized by Sobolev functions.

For typical crystals, Ψ is not uniformly convex and the Wulff shape is a polyhedron. Photographs of typical crystals can be found in [HT, p. 181].

On the other hand, suppose Ψ is smooth and uniformly convex. Then the variational problem of minimizing Ψ is elliptic, and the Wulff shape is smooth [M1, Prop. 3.3]. For the related problem of finding a Ψ-minimizing surface of prescribed boundary, the solution is smooth except for a set of codimension 2 in the surface [AScSi, Thm. II.7; M1]. Analogous energy-minimizing interfaces in materials can have larger singular sets (see [M2; LM]).

Method of proof. Our basic method, applying the divergence theorem to a certain volume-preserving map F from a candidate Ω to the Wulff shape B_Ψ, has been used by Gromov to prove the classical theorem for smoothly bounded sets; see [Be, 12.11.4]. The mapping F was defined earlier by Knothe [Kn] for the purpose of deriving a generalized Brunn–Minkowski inequality. The application to general integrands is new.

Here we outline the method for nice planar regions. Consider any planar curve enclosing a region Ω of the same area as B_Ψ. Let F be an area-preserving map from Ω to B_Ψ carrying vertical lines linearly to vertical lines, as shown in Figure 1. Then $\det DF = 1$ and DF is triangular:

$$DF = \begin{bmatrix} a & 0 \\ * & b \end{bmatrix}.$$

Since $\det DF = ab = 1$, $\operatorname{div} F = a + b \geq 2$ by the geometric mean–arithmetic mean inequality. Hence

$$\begin{aligned} \Psi(\partial\Omega) &= \int_{\partial\Omega} \Psi(\mathbf{n}) \geq \int_{\partial\Omega} \Psi(\mathbf{n})\Psi^*(F) \\ &\geq \int_{\partial\Omega} F \cdot \mathbf{n} = \int_{\Omega} \operatorname{div} F \geq 2 \operatorname{area} \Omega = 2 \operatorname{area} B_\Psi, \end{aligned}$$

with equality if $\Omega = B_\Psi$.

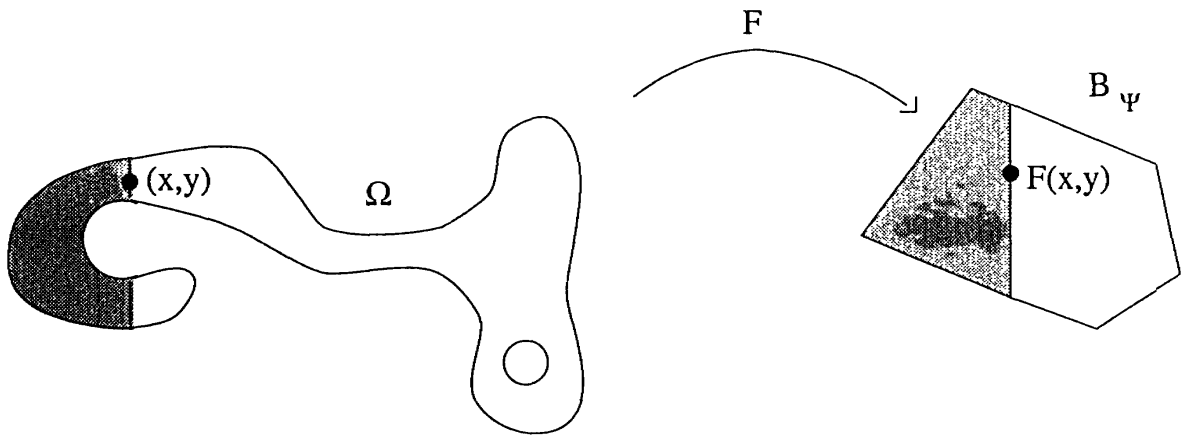

Figure 1 The shaded regions have equal area

The techniques for removing all smoothness hypotheses were developed in [Be]. This work was motivated by the investigation in [BZ], the principal result of which rests upon the classical isoperimetric theorem for sets of finite perimeter.

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2. Sets with Finite Perimeter

This paper usually follows the notation and terminology of [F2]. Ω will denote a measurable subset of $\mathbb{R}^n$ with locally finite perimeter, $n \geq 2$. This means that the gradient in the sense of distribution theory of the characteristic function of Ω is a locally finite Borel measure. In the terminology of currents [F2] this means that $\partial(\mathbb{R}^n \llcorner \Omega)$ is representable by integration, which in turn is equivalent to the current $\mathbb{R}^n \llcorner \Omega$ defined by Ω being an n-dimensional locally integral current [F2, 4.2.16 & 4.5.1].

We will denote by $|A|$ the Lebesgue outer measure of $A \subset \mathbb{R}^n$. The *Lebesgue density of A at $x \in \mathbb{R}^n$* is defined by

$$D(A,x) = \lim_{r \to 0^+} \frac{|A \cap B(x,r)|}{|B(x,r)|},$$

where $B(x,r) = \mathbb{R}^n \cap \{z : |x-z| \leq r\}$.

The *measure-theoretic boundary of A* is defined to be

$$\partial^* A = \mathbb{R}^n \sim (\{x : D(A,x) = 0\} \cup \{x : D(\mathbb{R}^n \sim A, x) = 0\}) \subset \partial A;$$

$\partial^* A$ is Borel. One says that $\nu \in \mathbb{R}^n$ with $|\nu| = 1$ is the *measure-theoretic exterior normal to A at $x_0 \in \mathbb{R}^n$* provided that

$$D(\{x\colon (x-x_0)\cdot\nu > 0\}\cap A, x_0) = 0$$

and

$$D(\{x\colon (x-x_0)\cdot\nu < 0\} \sim A, x_0) = 0.$$

ν is unique if it exists; denote $\nu = \mathbf{n}(A, x_0)$. Clearly, $x_0 \in \partial^* A$. It follows from [F2, 4.5.6 & 4.5.11] that a measurable set Ω has locally finite perimeter if and only if $\mathcal{H}^{n-1}(K\cap\partial^*\Omega) < \infty$ for every compact K, where $\mathcal{H}^{n-1}$ is $(n-1)$-dimensional Hausdorff measure. Furthermore, if the perimeter of Ω is locally finite, then

$$\mathcal{H}^{n-1}(\partial^*\Omega \sim \{x\colon \mathbf{n}(\Omega, x) \text{ exists}\}) = 0. \tag{1}$$

Finally, we remark that if A is not measurable then $\mathcal{H}^{n-1}(K\cap\partial^* A) = \infty$ for some compact K [F2, 4.5.11].

We also note that convex sets have locally finite perimeter.

3. Norms and Integrands

For our purposes it will be convenient to define a *positive, constant coefficient, parametric integrand of degree* $n-1$ *on* $\mathbb{R}^n$ to be a continuous function $\Psi\colon \mathbb{R}^n \to \mathbb{R}$ satisfying the conditions

$$\Psi(v) > 0 \quad \text{for } 0 \neq v \in \mathbb{R}^n;$$

$$\Psi(rv) = r\Psi(v) \quad \text{for } v\in\mathbb{R}^n,\ r \geq 0.$$

Ψ is *convex* if

$$\Psi(v+w) \leq \Psi(v) + \Psi(w) \quad \text{for } v, w \in \mathbb{R}^n.$$

We note that a norm on $\mathbb{R}^n$ is an integrand which is convex and even ($\Psi(-v) = \Psi(v)$).

The *energy* of a parametric integrand Ψ on the boundary of a measurable set Ω with locally finite perimeter is

$$\Psi(\partial\Omega) = \int_{\partial^*\Omega} \Psi(\mathbf{n}(\Omega, x))\, d\mathcal{H}^{n-1}x.$$

Note that since Ψ is positive, $\Psi(\partial\Omega) < \infty$ if and only if Ω has finite perimeter. Also, it follows from the remark at the end of Section 2 that for A not measurable, any definition of $\Psi(\partial A)$ which involves integration over $\partial^* A$ would give $\Psi(\partial A) = \infty$ because Ψ is positive.

The *parametric area integrand of degree* $n-1$ *on* $\mathbb{R}^n$ is the function $\Psi(v) = |v|$. In this case $\Psi(\partial\Omega) = \mathcal{H}^{n-1}(\partial^*\Omega)$ is the perimeter of Ω.

With each positive parametric integrand Ψ we associate the *dual integrand* Ψ^* defined by

$$\Psi^*(x) = \sup\{x\cdot v\colon \Psi(v) \leq 1\}.$$

Ψ^* is a convex, positive parametric integrand; denote

$$B_\Psi = \{x\colon \Psi^*(x) \leq 1\}.$$

B_Ψ is the *crystal,* or *Wulff shape* of Ψ. We also denote $\Psi_0 = \Psi^{**}$.

Lemma 1. *Let Ψ be a positive integrand.*

(i) $\Psi_0 \leq \Psi$.

(ii) *If* $\mathbf{n}(B_\Psi, x)$ *exists, then*

$$\Psi_0(\mathbf{n}(B_\Psi, x)) = x \cdot \mathbf{n}(B_\Psi, x) = \Psi(\mathbf{n}(B_\Psi, x)). \tag{2}$$

In particular,

$$x \cdot v = 1 \quad \textit{and} \quad \Psi_0(v) \leq 1, \tag{3}$$

where

$$v = \Psi_0(\mathbf{n}(B_\Psi, x))^{-1} \mathbf{n}(B_\Psi, x). \tag{4}$$

Conversely, if $v \in \mathbb{R}^n$ *satisfies* (3) *then* v *is given by* (4).

Proof. (i) is a direct consequence of the definitions of Ψ_0 and Ψ^*.

Assume $\mathbf{n}(B_\Psi, x_0) = n_0$ exists. Then, since $\Psi_0 = \Psi^{**}$,

$$\Psi_0(n_0) = \max\{n_0 \cdot x : x \in B_\Psi\} = n_0 \cdot x_m, \quad x_m \in \partial B_\Psi.$$

It follows that $H = \{y : y \cdot n_0 = \Psi_0(n_0)\}$ is a supporting hyperplane of B_Ψ at x_m. However, H must also be the supporting hyperplane at x_0, whence follows the first equality of (2).

Next observe that $v_0 = \Psi_0(n_0)^{-1} n_0$ satisfies (3) with $x = x_0$. Assume there also exists $v_1 \neq v_0$ satisfying (3). Inasmuch as $v_0^\perp = v_1 - (v_1 \cdot v_0)|v_0|^{-2} v_0$ is tangent to ∂B_Ψ at x_0, for small $t > 0$ there exists $x_t \in \partial B_\Psi$ such that

$$\lim_{t \to 0^+} \frac{(x_0 + t v_0^\perp) - x_t}{t} = 0;$$

set $x_t = x_0 + w_t$. Then

$$\begin{aligned} w_t \cdot v_1 + o(t) &= t v_0^\perp \cdot v_1 \\ &= t[|v_1|^2 - (v_1 \cdot |v_0|^{-1} v_0)^2] \\ &= at, \end{aligned}$$

where $a > 0$ because $|v_1 \cdot |v_0|^{-1} v_0| < |v_1|$ by Schwarz's inequality. We note that v_0 and v_1 are linearly independent since $x_0 \cdot v_0 = 1 = x_0 \cdot v_1$. Thus, by (3),

$$x_t \cdot v_1 = 1 + at - o(t) > 1$$

for t positive and small, which contradicts the fact that $x_t \in B_\Psi$. This completes the proof of uniqueness of v_0.

Finally, $\Psi(v_0) \geq 1$ by (i) and, since $\Psi^*(x_0) = 1$, there exists v_1 such that

$$x_0 \cdot v_1 = 1 = \Psi(v_1). \tag{5}$$

Thus $\Psi_0(v_1) \leq 1$ by (i), hence v_1 satisfies (3) and we conclude $v_1 = v_0$. The second equality of (2) is now equivalent to (5). □

4. The Isoperimetric Theorem for General Norms

Theorem. *Let Ψ be a norm on $\mathbb{R}^n$, or more generally any positive, constant coefficient, parametric integrand of degree $n-1$ (not necessarily even or convex). Let B_Ψ denote the unit ball or Wulff crystal*

$$B_\Psi = \{x: \Psi^*(x) \le 1\}.$$

Let Ω be any measurable subset of $\mathbb{R}^n$ with finite perimeter and of the same volume as B_Ψ. Then

$$\Psi(\partial\Omega) \ge \Psi(\partial B_\Psi), \tag{6}$$

with equality holding if and only if Ω differs from a translate of B_Ψ by a set of volume zero.

We remark that, in view of the discussion in Sections 2 and 3, since Ψ is positive it is reasonable to interpret the left side of (6) as infinity in case Ω either is not measurable or does not have finite perimeter.

We will present the main construction and central ideas of the proof. These can be mostly given in terms of classical real analysis. Technical geometric measure-theoretic details involving rectifiable currents and their slices are deferred to Section 5.

Proof. Inasmuch as $\Psi_0 \le \Psi$, $\Psi_0^* = \Psi^*$, and $\Psi(\partial B_\Psi) = \Psi_0(\partial B_\Psi)$ by (2), we can clearly assume $\Psi_0 = \Psi$. We also assume that Ω is Borel and that e_n satisfies the conclusion of Lemma 2.

For $t \in \mathbb{R}$ define $a_1(t) = |\Omega \cap \{x: x_1 \le t\}|$ and denote $A_1 = |\Omega|$. For $1 \le k \le n$ define $\pi_k: \mathbb{R}^n \to \mathbb{R}^k$ by $\pi_k(x_1, \ldots, x_n) = (x_1, \ldots, x_k)$, and for $k \ge 2$ define A_k: $\mathbb{R}^{k-1} \to \mathbb{R}$ and $a_k: \mathbb{R}^{k-1} \times \mathbb{R} \to \mathbb{R}$ by

$$A_k(\xi) = \mathcal{H}^{n-k+1}(\Omega \cap \{x: \pi_{k-1}(x) = \xi\}),$$

$$a_k(\xi, t) = \mathcal{H}^{n-k+1}(\Omega \cap \{x: \pi_{k-1}(x) = \xi, x_k \le t\}).$$

(Recall that in $\mathbb{R}^m$, $\mathcal{H}^m$ is equal to Lebesgue measure [F2, 2.10.35].) A_k and a_k are Borel functions with $A_k(\xi) < \infty$ for almost all ξ by Fubini's theorem. Also, a_k is Lipschitzian as a function of x_k. Thus, by Fubini's theorem,

$$\frac{\partial a_k}{\partial x_k} = A_{k+1}, \quad k = 1, \ldots, n-1, \tag{7}$$

almost everywhere in $\mathbb{R}^n$, and

$$\frac{\partial a_n}{\partial x_n} = 1 \tag{8}$$

almost everywhere in Ω because almost every $x \in \Omega$ is a point of linear density 1 in $\pi_{n-1}^{-1}\{\pi_{n-1}(x)\}$ [F2, 3.1.3]. (Where appropriate, we will identify A_k with $A_k \circ \pi_{k-1}$ and a_k with $a_k \circ \pi_k$.) Denote

$$\Omega_{k-1} = \mathbb{R}^{k-1} \cap \{\xi: 0 < A_k(\xi) < \infty\}.$$

Then $t \mapsto a_1(t)/A_1$ is an increasing Lipschitzian map onto $[0,1]$, and for $2 \le k \le n$ and $\xi \in \Omega_{k-1}$, $t \mapsto a_k(\xi, t)/A_k(\xi)$ is an increasing Lipschitzian map onto $[0,1]$. Ω_{k-1} is a Borel set and so we infer, using Fubini's theorem and the assumption $|\Omega| < \infty$, that $|\Omega \sim \pi_{k-1}^{-1}(\Omega_{k-1})| = 0$. We have

$$\bigcap_{i=1}^{k} \pi_i^{-1}(\Omega_i) = D_k \times \mathbb{R}^{n-k}; \tag{9}$$

denote $D_0 = \{0\}$. Denote $\Omega_0 = \Omega \cap (D_{n-1} \times \mathbb{R})$; clearly,

$$|\Omega \sim \Omega_0| = 0. \tag{10}$$

Finally, define $\lambda_1 = \sup\{t: a_1(t) = 0\}$, $\mu_1 = \inf\{t: a_1(t) = A_1\}$, and, for $\xi \in \mathbb{R}^{k-1}$,

$$\lambda_k(\xi) = \sup\{t: a_k(\xi, t) = 0\},$$
$$\mu_k(\xi) = \inf\{t: a_k(\xi, t) = A_k(\xi)\}.$$

We also define the functions $a_k^\Psi, A_k^\Psi, \ldots$, with Ω replaced by B_Ψ. Since B_Ψ is convex, for $\xi \in (B_\Psi)_{k-1}$, $t \mapsto a_k^\Psi(\xi, t)/A_k^\Psi(\xi)$ is a strictly increasing map of $[\lambda_k^\Psi(\xi), \mu_k^\Psi(\xi)]$ onto $[0, 1]$ which has a locally Lipschitzian inverse on $(0, 1)$. It follows that, for each $k = 1, \ldots, n$, there exists $F_k^0: D_{k-1} \times \mathbb{R}^{n-k+1} \to \mathbb{R}$ such that

$$a_1^\Psi(F_1^0(x))/A_1^\Psi = a_1(x_1)/A_1 \tag{11}$$

for $x \in \mathbb{R}^n$ and, for $2 \le k \le n$ and $x \in D_{k-1} \times \mathbb{R}^{n-k+1}$,

$$\frac{a_k^\Psi(F_1^0(x), \ldots, F_k^0(x))}{A_k^\Psi(F_1^0(x), \ldots, F_{k-1}^0(x))} = \frac{a_k(x_1, \ldots, x_k)}{A_k(x_1, \ldots, x_{k-1})}. \tag{12}$$

Note that $F_k^0(x)$ depends only on $x_1, \ldots, x_k$. Further, $t \mapsto F_1^0(t, \cdot)$ is continuous and increasing, and is locally Lipschitzian on (λ_1, μ_1), and for $2 \le k \le n$, for $\xi \in D_{k-1}$, $t \mapsto F_k^0(\xi, t, \cdot)$ is continuous and increasing, and is locally Lipschitzian on $(\lambda_k(\xi), \mu_k(\xi))$. Finally, we denote

$$F^0 = (F_1^0, \ldots, F_n^0): D_{n-1} \times \mathbb{R} \to B_\Psi.$$

We also observe that

$$G = (a_1^\Psi/A_1^\Psi, \ldots, a_n^\Psi/A_n^\Psi): B_\Psi \to [0, 1]^n$$

is one-to-one and Lipschitzian, hence a homeomorphism. Thus

$$F^0 = G^{-1} \circ (a_1/A_1, \ldots, a_n/A_n): D_{n-1} \times \mathbb{R} \to B_\Psi$$

is Borel. We next use (7), (8), (11), and (12) to compute, for $k = 1, \ldots, n-1$,

$$\begin{aligned} \frac{\partial F_k^0}{\partial x_k} &= \frac{(A_k^\Psi \circ F^0)_{k+1}}{A_k(A_{k+1}^\Psi \circ F^0)} \quad \text{almost everywhere in } D_{k-1} \times \mathbb{R}^{n-k+1}, \quad \text{and} \\ \frac{\partial F_n^0}{\partial x_n} &= \frac{A_n^\Psi \circ F^0}{A_n} \quad \text{almost everywhere in } D_{n-1} \times \mathbb{R}. \end{aligned} \tag{13}$$

In order to employ the divergence theorem in Section 5 we replace F^0 with a mapping having bounded derivatives. To this end we define $f_k = \min\{n+1, \partial F_k^0/\partial x_k\}$, and for $k = 2, \ldots, n$ and $x \in D_{k-1} \times \mathbb{R}^{n-k+1}$ set

$$F_k(x) = \lambda_k^\Psi(\xi) + \int_{\lambda_k(\xi)}^{x_k} f_k(\xi, t)\, dt, \quad \xi = \pi_{k-1}(x).$$

F_k is clearly Lipschitzian as a function of x_k. For $\epsilon > 0$ and $x_k \le \mu_k(\xi)$ we have

$$\int_{\lambda_k(\xi)+\epsilon}^{x_k} f_k(\xi, t)\, dt \le F_k^0(x) - F_k^0(\xi, \lambda_k(\xi) + \epsilon),$$

hence $F_k(x) \le F_k^0(x)$ because $F_k^0(\xi, t)$ is a right continuous function of t at $t = \lambda_k(\xi)$, a left continuous function of t at $t = \mu_k(\xi)$, and $F_k^0(\xi, \lambda_k(\xi)) = \lambda_k^\Psi(\xi)$. Similarly, $F_1(x) \le F_1^0(x)$ for $x \in \mathbb{R}^n$. We conclude that $F = (F_1, \ldots, F_n)$ is a Borel mapping of $D_{n-1} \times \mathbb{R}$ into B_Ψ.

Now let A be a Borel subset of $\partial^*\Omega$ with $\mathcal{H}^{n-1}(A) > 0$. It is clear from (9) and (10) that $D_{n-1} \subset \Omega_{n-1}$ and $|\Omega_{n-1} \sim D_{n-1}| = 0$. Thus, by Lemma 2,

$$|\pi_{n-1}(A) \sim D_{n-1}| = 0 \quad \text{and} \quad |\pi_{n-1}(A)| > 0. \tag{14}$$

In particular, we see that F is defined $\mathcal{H}^{n-1}$-almost everywhere on $\partial^*\Omega$.

Utilizing the generalization of the relationship between the geometric and arithmetic means we infer, using (13), that almost everywhere on $D_{n-1} \times \mathbb{R}$,

$$\frac{1}{n} \operatorname{div} F^0 \ge \left(\prod_{k=1}^{n} \frac{\partial F_k^0}{\partial x_k} \right)^{1/n} = 1, \tag{15}$$

because $A_1 = A_1^\Psi = |\Omega|$. Using this, we infer from the definition of F that $\operatorname{div} F \ge n$. Thus, by (10), the divergence theorem (21), and the definitions of Ψ^* and B_Ψ,

$$\begin{aligned} n|\Omega| &\le \int_\Omega \operatorname{div} F \\ &= \int_{\partial^*\Omega} F(x) \cdot \mathbf{n}(\Omega, x)\, d\mathcal{H}^{n-1}x \\ &\le \int_{\partial^*\Omega} \Psi^*(F(x))\Psi(\mathbf{n}(\Omega, x))\, d\mathcal{H}^{n-1}x \\ &\le \Psi(\partial\Omega). \end{aligned} \tag{16}$$

(6) is now clear because (2) implies that equality holds in (16) for the case $\Omega = B_\Psi$.

Now assume that equality holds in (6), hence in (16). Then $\operatorname{div} F = n$, hence $F^0 = F$ because each $\partial F_k^0/\partial x_k \ge 0$, and so equality holds in (15) almost everywhere in Ω. Moreover, equality holds for $\mathcal{H}^{n-1}$-almost every $x \in \partial^*\Omega$ in the generalized Schwarz inequality:

$$F(x) \cdot \mathbf{n}(\Omega, x) = \Psi^*(F(x))\Psi(\mathbf{n}(\Omega, x)).$$

Finally, $\Psi^*(F(x)) = 1$ for $\mathcal{H}^{n-1}$-almost every $x \in \partial^*\Omega$. Thus our assumption $\Psi_0 = \Psi$ and the uniqueness in Lemma 1 imply that

$$\mathbf{n}(\Omega, x) = \mathbf{n}(B_\Psi, F(x)) \tag{17}$$

for $\mathcal{H}^{n-1}$-almost every $x \in \partial^*\Omega$ such that $n(B_\Psi, F(x))$ exists.

Equality in (15) implies that

$$\frac{\partial F_1}{\partial x_1} = \cdots = \frac{\partial F_n}{\partial x_n} = 1$$

almost everywhere in Ω. From this we infer the existence of $c_1 \in \mathbb{R}$ and Borel functions $c_k\colon \mathbb{R}^{k-1} \to \mathbb{R}$, $2 \le k \le n$, such that

$$F_1(x) = x_1 - c_1 \quad \text{for } x \in [\lambda_1, \mu_1] \times \mathbb{R}^{n-1} \supset \Omega,$$

and, for almost all $\xi \in D_{k-1}$ and $x_k \in [\lambda_k(\xi), \mu_k(\xi)]$,

$$F_k(\xi, x_k, \cdot) = x_k - c_k(\xi).$$

Using this and repeated application of Fubini's theorem, one verifies that if $B \subset \Omega_0$ and $F(B)$ are measurable then $|F(B)| = |B|$. In particular, if one assumes only that B is measurable and $|B| > 0$, then $|F(B)| > 0$. The structure of F also implies

$$|\Omega \cap I_{\xi_1\xi_2}| = |B_\Psi \cap I_{(\xi_1+c_1)(\xi_2+c_1)}|, \tag{18}$$

where $I_{\xi_1\xi_2} = \pi_1^{-1}[\xi_1, \xi_2]$ for $\xi_1 < \xi_2$. Indeed, application to $F(\Omega_0 \cap I_{\xi_1\xi_2})$ of the iterated integral $\mathcal{I}(F(\Omega_0 \cap I_{\xi_1\xi_2}))$ used above gives $|\Omega_0 \cap I_{\xi_1\xi_2}|$, which is equal to $|\Omega \cap I_{\xi_1\xi_2}|$ by (10). On the other hand, $F(\Omega_0) \subset B_\Psi$, hence

$$\begin{aligned} \mathcal{I}(F(\Omega_0 \cap I_{\xi_1\xi_2})) &\le \mathcal{I}(B_\Psi \cap I_{(\xi_1+c_1)(\xi_2+c_1)}) \\ &= |B_\Psi \cap I_{(\xi_1+c_1)(\xi_2+c_1)}| \end{aligned}$$

by monotonicity. Finally, strict inequality for any choice of $\xi_1 < \xi_2$ would contradict $|\Omega| = |B_\Psi|$.

Again consider a Borel subset A of $\partial^*\Omega$ with $\mathcal{H}^{n-1}(A) > 0$. Since $\pi_{n-1}(A)$ is measurable, so is $B = \Omega_0 \cap (\pi_{n-1}(A) \times \mathbb{R})$. Furthermore, $|B| > 0$ because $|\pi_{n-1}(A) \cap D_{n-1}| > 0$ by (14), hence $|F(B)| > 0$. Denoting

$$F^* = (F_1, \ldots, F_{n-1}) : D_{n-1} \to (B_\Psi)_{n-1},$$

we thus have $\mathcal{H}^{n-1}(F(A)) > 0$ because

$$\pi_{n-1} \circ F = F^* \circ \pi_{n-1} \quad \text{and} \quad |F^* \circ \pi_{n-1}(A)| = |\pi_{n-1} \circ F(B)| > 0.$$

Also, $\mathcal{H}^{n-1}(F(A) \cap \partial^* B_\Psi) > 0$ since $\mathcal{H}^{n-1}(\partial B_\Psi \sim \partial^* B_\Psi) = 0$ because B_Ψ is convex. We thus conclude that (17) holds for $\mathcal{H}^{n-1}$-almost every $x \in \partial^*\Omega$.

Note that we can assume that the centroids (centers of gravity) of Ω and B_Ψ coincide. One uses (18), together with a uniform approximation of the function x_1 by step functions of one variable, to verify that

$$0 = \int_\Omega x_1\,dx = \int_{B_\Psi} y_1\,dy - c_1|B_\Psi| = -c_1|B_\Psi|,$$

hence $c_1 = 0$.

Denote by $\mathcal{B}_n$ the set of ordered orthonormal bases $V = (v_1, \ldots, v_n)$ of $\mathbb{R}^n$ such that v_n satisfies the conclusion of Lemma 2. Note that almost every (in the sense of Haar measure on the orthogonal group) $V \in \mathcal{B}_n$. Denote by F_V the mapping constructed as above with V replacing the standard basis. Choose a countable subset $\mathcal{B}_1 \subset \mathcal{B}_n$ such that the set of $v_1 \in V \in \mathcal{B}_1$ is dense in S^{n-1}, and such that also

$$\mathcal{H}^{n-1}(\partial^*\Omega \sim N) = 0, \tag{19}$$

where N is the set of $x \in \partial^*\Omega$ for which (17) holds for F_V whenever $V \in \mathcal{B}_1$.

Fix $\nu \in S^{n-1}$ such that $N_\nu = N \cap \{x : \mathbf{n}(\Omega, x) = \nu\} \neq \emptyset$, and denote

$$N_\nu^\Psi = \text{closure}\{y : \mathbf{n}(B_\Psi, y) = \nu\}.$$

By (17) we have $N_\nu^\Psi \neq \emptyset$. Further, N_ν^Ψ is the intersection of ∂B_Ψ with the (unique) support hyperplane with normal ν at any $y \in F_V(N_\nu)$, hence is convex. Since the constant $c_1 = 0$ for each $V \in \mathcal{B}_1$, N_ν lies between the two support hyperplanes $H_\pm = \pi_1^{-1}\{\xi_\pm\}$ of N_ν^Ψ with normal v_1. Inasmuch as the set of such v_1 is dense in S^{n-1}, we conclude that $N_\nu \subset N_\nu^\Psi$. Thus

$$N \subset \text{closure}\, \partial^* B_\Psi = \partial B_\Psi,$$

hence by (19) and $\mathcal{H}^{n-1}(\partial B_\Psi \sim \partial^* B_\Psi) = 0$ we have

$$\mathcal{H}^{n-1}(\partial^*\Omega \sim \partial^* B_\Psi) = 0. \tag{20}$$

On the other hand, it is well known that $\partial(\mathbb{R}^n \llcorner \Omega)$ is a locally rectifiable $(n-1)$-current [F2, 4.5.6], hence by [F2, 4.1.28(4)] the tangent $(n-1)$-vector to $\partial(\mathbb{R}^n \llcorner \Omega)$ at almost all $x \in \partial^*\Omega$ is determined by $\partial^*\Omega$, and similarly for $\partial(\mathbb{R}^n \llcorner B_\Psi)$. Thus, since $\partial(\mathbb{R}^n \llcorner \Omega) = \mathcal{H}^{n-1} \wedge * \mathbf{n}(\Omega, \cdot)$ by [F2, 4.5.6(4)], we infer that $\mathbf{n}(\Omega, x) = \mathbf{n}(B_\Psi, x)$ for $\mathcal{H}^{n-1}$-almost all $x \in \partial^*\Omega \cap \partial^* B_\Psi$. Next we infer, using (20) and $\Psi(\partial\Omega) = \Psi(\partial B_\Psi)$, that $\mathcal{H}^{n-1}(\partial^* B_\Psi \sim \partial^*\Omega) = 0$ because Ψ is positive. Consequently, $\partial(\mathbb{R}^n \llcorner \Omega) = \partial(\mathbb{R}^n \llcorner B_\Psi)$, and it therefore follows by the constancy theorem [F2, 4.1.7] that

$$|\Omega \sim B_\Psi| = 0 = |B_\Psi \sim \Omega|. \qquad \square$$

5. Technical Details

Let Ω be a measurable subset of $\mathbb{R}^n$ with finite perimeter and $|\Omega| < \infty$. We first derive an extension of the Gauss-Green-Federer divergence theorem which is suitable for our purposes. Although this version is essentially known (see [F1] and [F2, 4.1.19]), for completeness we provide a proof using routine application of the slicing theory for currents.

Lemma 2 asserts that $\partial^*\Omega$ is in general position with respect to projection onto almost all hyperplanes.

DIVERGENCE THEOREM. *Let $F = (F_1, \ldots, F_n)$ be a bounded Borel vector field which is defined almost everywhere on Ω, and $\mathcal{H}^{n-1}$-almost everywhere on $\partial^*\Omega$. Assume that, for $k = 1, \ldots, n$ and for almost every line l_k parallel to the x_k axis, $F_k \mid l_k$ is Lipschitzian. Then*

$$\int_\Omega \operatorname{div} F = \int_{\partial^*\Omega} F(x) \cdot \mathbf{n}(\Omega, x)\, d\mathcal{H}^{n-1} x. \tag{21}$$

Proof. As in the proof of the isoperimetric theorem, let $\pi_{n-1} : \mathbb{R}^n \to \mathbb{R}^n$ be the projection, and denote by D_{n-1} the set of $\xi \in \mathbb{R}^{n-1}$ such that

$$\mathcal{H}^{n-k}(\Omega \cap \pi_{n-1}^{-1}\{\xi\}) < \infty$$

and $F_n \mid \pi_{n-1}^{-1}\{\xi\}$ is Lipschitzian. Then $|\Omega \sim (D_{n-1} \times \mathbb{R})| = 0$ by Fubini's theorem.

$\mathbb{R}^n \llcorner \Omega \in N_n^{\text{loc}}(\mathbb{R}^n)$ and, by [F2, 4.5.6],

$$T = \partial(\mathbb{R}^n \llcorner \Omega) = \mathcal{H}^{n-1} \wedge *\mathbf{n}(\Omega, \cdot). \tag{22}$$

By [F2, 4.3.2],

$$\infty > M(T \llcorner dx_1 \wedge \cdots \wedge dx_{n-1}) = \int M\langle T, \pi_{n-1}, \xi \rangle \, d\xi. \tag{23}$$

Further,

$$\langle T, \pi_{n-1}, \xi \rangle = (-1)^{n-1} \partial \langle \mathbb{R}^n \llcorner \Omega, \pi_{n-1}, \xi \rangle, \tag{24}$$

and $\langle \mathbb{R}^n \llcorner \Omega, \pi_{n-1}, \xi \rangle$ is obtained by integration over $\Omega \cap \pi_{n-1}^{-1}\{\xi\}$ (suitably oriented). Use (23) and (24) to choose $\xi \in D_{n-1}$ such that

$$M(\partial \langle \mathbb{R}^n \llcorner \Omega, \pi_{n-1}, \xi \rangle) < \infty.$$

Inasmuch as $t \mapsto F_n(\xi, t)$ is Lipschitzian, we obtain

$$\begin{aligned} \partial \langle \mathbb{R}^n \llcorner \Omega, \pi_{n-1}, \xi \rangle (F_n) &= \langle \mathbb{R}^n \llcorner \Omega, \pi_{n-1}, \xi \rangle \left(\frac{\partial F_n}{\partial x_n} \, dx_n \right) \\ &= \int_{\Omega \cap \pi_{n-1}^{-1}\{\xi\}} \frac{\partial F_n}{\partial x_n}. \end{aligned}$$

Integrating with respect to ξ and applying (22), (24), [F2, 4.3.2(1)], and Fubini's theorem, we obtain

$$\begin{aligned} \int_{\partial^* \Omega} F_n(x) e_n \cdot \mathbf{n}(\Omega, x) \, d\mathcal{H}^{n-1} x &= (-1)^{n-1} \partial (\mathbb{R}^n \llcorner \Omega)(F_n \, dx_1 \wedge \cdots \wedge dx_{n-1}) \\ &= \int_{D_{n-1}} \partial \langle \mathbb{R}^n \llcorner \Omega, \pi_{n-1}, \xi \rangle (F_n) \, d\xi \\ &= \int_\Omega \frac{\partial F_n}{\partial x_n}. \end{aligned} \tag{25}$$

([F2, 4.3.2(1)] extends to the case where Ψ is bounded Borel because T has finite mass.) Finally, it is clear that (25) similarly holds with n replaced by k, $1 \le k < n$. □

Lemma 2. *For $v \in S^{n-1}$, denote by π_v the projection of $\mathbb{R}^n$ onto the orthogonal complement H_v of $\mathbb{R}v$. Denote*

$$\Omega_v = H_v \cap \{\xi : 0 < \mathcal{H}^1(\Omega \cap \pi_v^{-1}\{\xi\}) < \infty\}.$$

Then, for $\mathcal{H}^{n-1}$-almost all $v \in S^{n-1}$,

$$|\pi_v(\partial^* \Omega) \sim \Omega_v| = 0 = |\Omega_v \sim \pi_v(\partial^* \Omega)|, \tag{26}$$

and for each Borel subset A of $\partial^* \Omega$, $|\pi_v(A)| > 0$ if and only if $\mathcal{H}^{n-1}(A) > 0$.

Proof. Inasmuch as $\mathcal{H}^{n-1}(\partial^* \Omega) < \infty$,

$$\mathcal{H}^{n-1}(\partial^* \Omega \cap \{x : v \cdot \mathbf{n}(\Omega, x) = 0\}) = 0 \tag{27}$$

for $\mathcal{H}^{n-1}$-almost all $v \in S^{n-1}$. Otherwise, by Fubini's theorem,

$$\infty > \mathcal{H}^{n-1} \times \mathcal{H}^{n-1}(S^{n-1} \times \partial^*\Omega \cap \{(v,x)\colon v \cdot \mathbf{n}(\Omega,x) = 0\}) > 0,$$

hence

$$\mathcal{H}^{n-1}(\partial^*\Omega \cap \{x\colon \mathcal{H}^{n-1}\{v\colon v \cdot \mathbf{n}(\Omega,x) = 0\} > 0\}) > 0.$$

But $\mathcal{H}^{n-1}\{v\colon v \cdot \mathbf{n}(\Omega,x) = 0\} > 0$ implies $\mathbf{n}(\Omega,x) = 0$.

Fix v satisfying (27), and let A be a Borel subset of $\mathbb{R}^n$. Applying (22) and (24) with π_v replacing π_{n-1}, we have by [F2, 4.3.2(2)]

$$\begin{aligned} \int_A |v \cdot \mathbf{n}(\Omega,x)|\, d\mathcal{H}^{n-1}x &= \int_A \|\vec{T} \llcorner \pi_v^\# \omega\|\, d\mathcal{H}^{n-1} \\ &= \int_{H_v} \|\langle T, \pi_v, \xi\rangle\|(A)\, d\xi \\ &= \int_{\Omega_v} \|\partial\langle \mathbb{R}^n \llcorner \Omega, \pi_v, \xi\rangle\|(A)\, d\xi. \end{aligned} \tag{28}$$

Here $\vec{T} = *\mathbf{n}(\Omega,\cdot)$ and ω is a volume form of H_v. By (27), the first integral vanishes if and only if $\mathcal{H}^{n-1}(A \cap \partial^*\Omega) = 0$. This, together with (28) for $A = \partial^*\Omega \sim (\Omega_v \times \mathbb{R})$, implies the first equality of (26). Together with (28) it also implies the final statement of the lemma. The second equality of (26) follows from (24). □

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