Dual Kadec–Klee Norms and the Relationships between Wijsman, Slice, and Mosco Convergence

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0. Introduction

In this paper we study the relationships between the three most fundamental forms of set convergence. In particular, it is shown that Wijsman and slice convergence coincide precisely when the weak-star and norm topologies agree on the dual sphere. Consequently, a weakly compactly generated Banach space admits a dense set of norms for which Wijsman and slice convergence coincide if and only if it is an Asplund space. We also show that Wijsman convergence implies Mosco convergence precisely when the weak-star and Mackey topologies coincide on the dual sphere. A corollary of these results is that, given a fixed norm on an Asplund space, Wijsman and slice convergence coincide if and only if Wijsman convergence implies Mosco convergence.

All Banach spaces considered here are assumed to be real. Let X be an infinite-dimensional Banach space with a given norm $\|\cdot\|$. When considering a subspace Y of X, we will always assume it is endowed with the relative norm unless stated otherwise. The ball and sphere of X are defined and denoted as follows: $B_X=\{x:\|x\|\le 1\}$ and $S_X=\{x:\|x\|=1\}$. We also use the notation $B_r=\{x:\|x\|\le r\}$. For $x\in X$ and $A,B\subset X$, let $d(x,A)=\inf\{\|x-a\|: a\in A\}$ and let $d(A,B)=\inf\{\|a-b\|: a\in A,\ b\in B\}$. If $A=\emptyset$, the convention is that $d(x,A)=\infty$; similarly, $d(A,B)=\infty$ if A or B is empty. We shall consider the following three notions of set convergence. Let C_α and C be closed convex subsets of X. If $\lim_\alpha d(x,C_\alpha)=d(x,C)$ for all $x\in X$, then C_α is said to *converge Wijsman* to C. More restrictively, C_α is said to *converge slice* to C if $\lim_\alpha d(W,C_\alpha)=d(W,C)$ for all closed bounded convex sets W. We will say C_α *converges Mosco* to C if the following two conditions are satisfied:

M(i) if $x\in C$, then $d(x,C_\alpha)\to 0$;

M(ii) if $x_{\alpha_\beta}\in C_{\alpha_\beta}$ for some subnet is such that $\{x_{\alpha_\beta}\}_\beta$ is relatively weakly compact and $x_{\alpha_\beta}\xrightarrow{w} x$, then $x\in C$.

Received February 8, 1993. Revision received August 10, 1993.

[1] Research supported in part by an NSERC research grant.

[2] NSERC postdoctoral fellow.

Michigan Math. J. 41 (1994).

Notice that M(i) and M(ii) reduce to the usual definition for Mosco convergence in the case of sequences (for M(ii) we use that a weakly convergent sequence is relatively weakly compact). Moreover, this definition is compatible with the Mosco topology as defined in [B2]. As is the usual practice, we only consider these notions for closed convex sets. It is also clear that Wijsman, Mosco, and slice convergence coincide in finite-dimensional spaces, so we will only consider infinite-dimensional spaces. As a matter of terminology, we will say that given a fixed norm on X, *Wijsman convergence implies Mosco (slice) convergence* if C_α converges Mosco (slice) to C whenever C_α converges Wijsman to C with respect to the given norm on X (if C_n converges Mosco (slice) to C whenever C_n converges Wijsman to C, we will say *Wijsman convergence implies Mosco (slice) convergence sequentially*).

It is necessary to stipulate which norm is being used on X when speaking of Wijsman convergence because it depends on the particular norm (see [B4; BF1; BL]). However, it follows from the definitions that Mosco and slice convergence do not depend on the norm being used. One can also easily check, using the definitions, that slice convergence implies Mosco convergence in every space and that they coincide in reflexive spaces. Moreover, if a net of sets converges slice to some set in a Banach space X, then it is not hard to check that the convergence is Wijsman with respect to every equivalent norm; Beer [B4] has recently shown that the converse holds.

The notion of Wijsman convergence was introduced by Wijsman in [Wi], where it is shown to be a useful tool in finite-dimensional spaces. Mosco's fundamental paper [Mo] introduced what is now known as Mosco convergence; see [At] for a plethora of applications of Mosco convergence in reflexive spaces. Unfortunately, it is not well-behaved in nonreflexive spaces; see [BB1]. However, a recent paper of Beer [B3] shows that many of the nice properties of Mosco convergence in reflexive spaces are valid for slice convergence in nonreflexive spaces. Because of this and the fact that Wijsman convergence is simpler to check, it is desirable to know when Wijsman convergence implies slice convergence. It follows from [BF1, Thm. 3.1] that Wijsman and slice convergence coincide in reflexive spaces if and only if the weak and norm topologies coincide on the dual sphere; see also [B2, Thm. 2.5]. Our main goal is to extend this theorem to nonreflexive spaces, thus answering questions posed in [B4] and [BB2].

In the first section we establish some basic technical results. Most notably, it is shown that the relationship between Wijsman and slice convergence is sequentially and separably determined. We also mention a few basic properties of Kadec–Klee norms (which for brevity we call Kadec norms). We will say a norm on X^* is *w^*-Kadec* if $x_\alpha^* \to x^*$ whenever $\|x_\alpha^*\| \to \|x^*\|$ and $x_\alpha^* \xrightarrow{w^*} x^*$; if this holds for sequences we will say the norm is *sequentially w^*-Kadec.* When studying Mosco convergence it will be useful to consider the Mackey topology τ on X^* of uniform convergence on weakly compact sets. A norm will be called *w^*-τ-Kadec* if $x_\alpha^* \xrightarrow{\tau} x^*$ whenever $\|x_\alpha^*\| \to \|x^*\|$ and $x_\alpha^* \xrightarrow{w^*} x^*$; if this holds for sequences then $\|\cdot\|$ is *sequentially w^*-τ-Kadec.* Some differences in these notions will be discussed.

The second section contains the main results. It is shown that Wijsman and slice (Mosco) convergence coincide precisely when the dual norm is w^*-Kadec (w^*-τ-Kadec). Let us mention that wide classes of Banach spaces can be renormed so that the dual norm is w^*-Kadec. A Banach space is said to be *weakly compactly generated* (WCG) if it contains a weakly compact set whose linear span is norm dense. It is clear (from the definition) that WCG spaces include all separable and all reflexive spaces. Every WCG Asplund space, in particular spaces with separable duals, can be renormed so that the dual norm is w^*-Kadec [DGZ, Cor. VII.1.13]. See [DGZ, Chap. VII] for further renorming results along these lines.

The third section observes some connections between set convergence and differentiability, while the fourth section contains some further results and examples. A slightly expanded version of this paper may be found in the research report [BV].

1. Preliminary Results

We begin by presenting a criterion for checking set convergence which is a variant of [AB, Thm. 3.1]. Let us also introduce the following intermediate notion of set convergence. For closed convex sets C_α and C, we will say C_α *converges weak compact gap* to C if $d(W, C_\alpha) \to d(W, C)$ for all convex weakly compact subsets W of X.

PROPOSITION 1.1. *Let C_α and C be closed convex sets in a Banach space X. Suppose the following two conditions hold.*

(i) *If $x_0 \in C$, then $d(x_0, C_\alpha) \to 0$.*
(ii) *If $x_0^* \in S_{X^*}$ attains its supremum on C, then there exist $x_\alpha^* \in B_{X^*}$ such that x_α^* converges to x_0^* in a topology $\mathfrak{Z}$ on X^* and*

$\limsup_\alpha \{\sup_{C_\alpha} x_\alpha^*\} \leq \sup_C x_0^*$.

If $\mathfrak{Z}$ is the norm (resp. Mackey, w^*) topology, then C_α converges slice (resp. weak compact gap, Wijsman) to C.

Proof. We prove this only for the weak compact gap case, since the other proofs are almost identical. Let W be a weakly compact convex set in X. According to (i),

$$\limsup_\alpha d(W, C_\alpha) \leq d(W, C).$$

So we show that $\liminf_\alpha d(W, C_\alpha) \geq d(W, C)$. If $d(W, C) = 0$ then there is nothing more to do, so suppose $d(W, C) > 0$. Let $\epsilon > 0$ satisfy $2\epsilon < d(W, C)$ and set $r = d(W, C) - 2\epsilon$. By the separation theorem, there exists $\Lambda \in S_{X^*}$ such that

$$\begin{aligned}\sup\{\langle \Lambda, x\rangle : x \in C\} &\leq \inf\{\langle \Lambda, x\rangle : x \in W + B_{r+\epsilon}\}\\ &= \inf\{\langle \Lambda, x\rangle : x \in W + B_r\} - \epsilon.\end{aligned}$$

By a general vision of the Bishop–Phelps theorem [BP, Thm. 2], there is an $x_0^* \in S_{X^*}$ which attains its supremum on C and strictly separates C and $W + B_r$. Thus

$$\inf_W x_0^* - \sup_C x_0^* \geq r.$$

Let x_α^* be given by (ii) with $\mathfrak{J}$ the Mackey topology, and let α_0 be such that

$$\sup_{C_\alpha} x_\alpha^* \leq \sup_C x_0^* + \epsilon \quad \text{and} \quad \inf_W x_\alpha^* \geq \inf_W x_0^* - \epsilon \quad \text{for } \alpha \geq \alpha_0.$$

From this it follows that

$$d(C_\alpha, W) \geq \inf_W x_\alpha^* - \sup_{C_\alpha} x_\alpha^* \geq r - 2\epsilon \geq d(W, C) - 4\epsilon \quad \text{for } \alpha \geq \alpha_0.$$

Since $\epsilon > 0$ was arbitrary, we are done. □

The following theorem will be crucial in our development.

THEOREM 1.2. *Suppose C_α converges Wijsman but not slice (resp. weak compact gap, Mosco) to C in some subspace E of X. Then there is a separable subspace Y of E and a subsequence C_{α_n} such that $C_{\alpha_n} \cap Y$ converges Wijsman but not slice (resp. weak compact gap, Mosco) to $C \cap Y$ as subsets of Y.*

Proof. Since C_α converges Wijsman to C in E, it follows that

$$\limsup_\alpha d(B, C_\alpha) \leq d(B, C) \quad \text{for any } B \subset E.$$

Thus, because C_α does not converge slice to C, by passing to a subnet if necessary, there is a bounded closed convex subset W of E and a $\delta > 0$ such that

$$d(W, C_\alpha) + \delta < d(W, C) \quad \text{for all } \alpha. \tag{1.1}$$

Let Z be an arbitrary separable subspace of E and set $Z_1 = Z$. Fix a dense subset $\{z_{1,i}\}_{i=1}^\infty$ of Z_1 and choose α_1 such that

$$d(z_{1,1}, C) - 1 < d(z_{1,1}, C_{\alpha_1}) < d(z_{1,1}, C) + 1. \tag{1.2}$$

Using (1.1) and (1.2), one can choose $w_1 \in W$, $c_1 \in C_{\alpha_1}$, $x_{1,1}^1 \in C_{\alpha_1}$, and $y_{1,1}^1 \in C$ such that:

$$\|w_1 - c_1\| \leq d(W, C) - \delta;$$

$$\|z_{1,1} - x_{1,1}^1\| \leq d(z_{1,1}, C) + 1;$$

$$\|z_{1,1} - y_{1,1}^1\| \leq d(z_{1,1}, C) + 1.$$

Suppose $\alpha_1 \leq \alpha_2 \leq \cdots \leq \alpha_{n-1}$, $\{z_{i,j}\}_{j=1}^\infty$, c_i, w_i, for $i \leq n-1$, and $\{y_{i,j}^k\}$, $\{x_{i,j}^k\}$ for $i, j \leq k$, $k \leq n-1$, have been chosen. Then set

$$Z_n = \overline{\operatorname{span}}(\{Z_{n-1} \cup \{y_{i,j}^n\} \cup \{x_{i,j}^n\} \cup \{c_n\} \cup \{w_n\} : i \leq n, j \leq n\}). \tag{1.3}$$

Fix a dense collection $\{z_{n,i}\}_{i=1}^\infty \subset Z_n$ and choose $\alpha_n \geq \alpha_{n-1}$ such that

$$d(z_{i,j}, C) - 1/n < d(z_{i,j}, C_{\alpha_n}) < d(z_{i,j}, C) + 1/n \quad \text{whenever } i, j \leq n. \tag{1.4}$$

Using this, for $i, j \leq n$ we choose $x_{i,j}^n \in C_{\alpha_n}$ and $y_{i,j}^n \in C$ such that

$$\|x_{i,j}^n - z_{i,j}\| \leq d(z_{i,j}, C) + 1/n; \tag{1.5}$$

$$\|y_{i,j}^n - z_{i,j}\| \leq d(z_{i,j}, C) + 1/n. \tag{1.6}$$

According to (1.1), let $w_n \in W$ and $c_n \in C_{\alpha_n}$ be chosen so that

$$\|w_n - c_n\| \le d(W, C) - \delta. \tag{1.7}$$

Finally, let Y be the norm closure of $\bigcup_{n=1}^{\infty} Z_n$.

We now show that $C_{\alpha_n} \cap Y$ converges Wijsman but not slice to $C \cap Y$ as subsets of Y. For any $z_{i,j}$ it follows from (1.6) that $d(z_{i,j}, C) = d(z_{i,j}, C \cap Y)$. Using (1.4) and (1.5), one has $d(z_{i,j}, C \cap Y) = \lim_{n\to\infty} d(z_{i,j}, C_{\alpha_n} \cap Y)$. Because $\{z_{i,j}\}_{i,j}$ is dense in Y and distance functions are 1-Lipschitz, we have $d(y, C_{\alpha_n} \cap Y) \to d(y, C \cap Y)$ for all $y \in Y$. Thus $C_{\alpha_n} \cap Y$ converges Wijsman to $C \cap Y$ as subsets of Y.

However, for $W_1 = W \cap Y$, we have $w_n \in W_1$ and $c_n \in C_{\alpha_n} \cap Y$ for all n (as chosen in (1.7); notice that $w_n, c_n \in Y$ by (1.3)). Hence, using (1.7),

$$d(W_1, C \cap Y) \ge d(W, C) \ge \|w_n - c_n\| + \delta \ge d(W_1, C_{\alpha_n}).$$

So $C_{\alpha_n} \cap Y$ does not converge slice to $C \cap Y$ in Y.

The weak compact gap convergence proof is similar. For the Mosco case, it is clear that M(i) holds because of Wijsman convergence. If M(ii) fails then there is a net $y_\beta \in C_{\alpha_\beta}$ such that $y_\beta \xrightarrow{w} y$, where $y \notin C$ and $\{y_\beta\}_\beta$ is contained in a weakly compact set. Now one can strictly separate y from C; thus, taking a further subnet, we may assume that no subsequence of $\{y_\beta\}_\beta$ has a limit point in C. Hence, using the above construction and weak sequential compactness (Eberlein–Smulian theorem), one can obtain the result for Mosco convergence. □

The following proposition, which is based on ideas from [B1], relates dual Kadec norms to set convergence. If Wijsman convergence of L_α to L implies Mosco convergence of L_α to L for any net (sequence) of sets $L_\alpha = \{x : \langle x_\alpha^*, x\rangle = a\}$ and $L = \{x : \langle x^*, x\rangle = a\}$, where $x_\alpha^*, x^* \in X^*$ and $a \in \mathbf{R}$, then we will say *Wijsman convergence implies Mosco convergence for (sequences of) level sets of functionals.*

Proposition 1.3. (a) *If, in X, Wijsman convergence implies Mosco convergence for (sequences of) level sets of functionals, then the dual norm on X^* is (sequentially) w^*-τ-Kadec.*

(b) *If, in X, Wijsman convergence implies slice convergence for (sequences of) level sets of functionals, then the dual norm on X^* is (sequentially) w^*-Kadec.*

Proof. We only prove (a) for nets, since the other part is similar and is essentially in [BF1, Thm. 3.1]. Moreover, (b) may be essentially found in [B1; B4].

Suppose the dual norm is not w^*-τ-Kadec; then we can find $x_\alpha^*, x^* \in S_{X^*}$ such that $x_\alpha^* \xrightarrow{w^*} x^*$ but $x_\alpha^* \not\xrightarrow{\tau} x^*$. Let $C_\alpha = \{x \in X : \langle x_\alpha^*, x\rangle = 1\}$ and $C = \{x \in X : \langle x^*, x\rangle = 1\}$. Since $d(x, C_\alpha) = |\langle x_\alpha^*, x\rangle - 1|$, it follows that C_α converges Wijsman to C.

We now proceed as in the proof of [BF1, Thm. 3.1]: By passing to a subnet if necessary, there is a weakly compact set $K \subset B_X$ and $\{x_\alpha\}_\alpha \subset K$ such that $|\langle x_\alpha^* - x^*, x_\alpha\rangle| \geq \epsilon$ for some $\epsilon > 0$. Let $x_0 \in X$ be such that $\|x_0\| \leq 3$ and $\langle x^*, x_0\rangle > 2$. Now $\langle x_\alpha^*, x_0\rangle \to \langle x^*, x_0\rangle$ and so, by considering only a tail of the net, we may assume $\langle x_\alpha^*, x_0\rangle \geq 2$ for all α. Consider $v_\alpha = x_0 + x_\alpha$. Then $\langle x_\alpha^*, v_\alpha\rangle \geq 1$. Since $\|v_\alpha\| \leq 4$, we can choose $\frac{1}{4} \leq \lambda_\alpha \leq 1$ such that $\langle x_\alpha^*, \lambda_\alpha v_\alpha\rangle = 1$. By passing to a subnet we have $\lambda_{\alpha_\beta} \to \lambda$ where $\frac{1}{4} \leq \lambda \leq 1$ and $v_{\alpha_\beta} \xrightarrow{w} v$ where $\|v\| \leq 4$. Since $\langle x_\alpha^*, x_0\rangle \to \langle x^*, x_0\rangle$, it follows that

$$\liminf_\beta |1 - \langle x^*, \lambda_{\alpha_\beta} v_{\alpha_\beta}\rangle| = \liminf_\beta |\langle x_{\alpha_\beta}^* - x^*, \lambda_{\alpha_\beta} v_{\alpha_\beta}\rangle| \geq \liminf_\beta \lambda_{\alpha_\beta}\epsilon \geq \epsilon/4.$$

Now, $\langle x^*, \lambda_{\alpha_\beta} v_{\alpha_\beta}\rangle \to \langle x^*, \lambda v\rangle$, and so we have $|1 - \langle x^*, \lambda v\rangle| \geq \epsilon/4$. Consequently, $\lambda v \notin C$; noting that $\{\lambda_\alpha v_\alpha\}_\alpha$ is relatively weakly compact, we see that M(ii) fails. □

We will also need the following known facts (cf. [DS]) about Kadec norms.

PROPOSITION 1.4. (a) *If the dual norm on X^* is w^*-τ-Kadec (w^*-Kadec), then for each $Y \subset X$, the dual norm on Y^* is w^*-τ-Kadec (w^*-Kadec).*

(b) *Suppose that B_{X^*} is w^*-sequentially compact. If the dual norm on X^* is sequentially w^*-τ-Kadec (sequentially w^*-Kadec), then it is w^*-τ-Kadec (w^*-Kadec).*

Proof. (a) We prove this for the w^*-τ-Kadec since the other proof is similar. Suppose Y is a subspace of X and its dual norm $\|\cdot\|$ is not w^*-τ-Kadec. Then there is a weakly compact set K, an $\epsilon > 0$, and a net $y_\alpha^* \xrightarrow{w^*} y^*$ such that $\|y_\alpha^*\| = \|y^*\| = 1$ and

$$\sup_K |y_\alpha^* - y^*| > \epsilon \quad \text{for all } \alpha.$$

Let x_α^* denote a norm preserving extension of y_α^*. By Alaoglu's theorem, for some subnet, one has $x_{\alpha_\beta}^* \xrightarrow{w^*} x^*$ for some $x^* \in B_{X^*}$. Observe that $x^*|_Y = y^*$ and so it follows that $\|x^*\| = 1$ and

$$\sup_K |x_{\alpha_\beta}^* - x^*| > \epsilon \quad \text{for all } \beta.$$

Hence the dual norm on X^* is not w^*-τ-Kadec.

For (b), a similar type of argument using w^*-sequential compactness works. □

Recall that a space is said to have the *Schur property* if weakly convergent sequences are norm convergent.

REMARK 1.5. (a) *If the dual norm on X^* is w^*-Kadec then X is Asplund; in particular, B_{X^*} is w^*-sequentially compact.*

(b) *If X has the Schur property then every dual norm on X^* is w^*-τ-Kadec.*

(c) *There are spaces X such that X^* has a dual w^*-τ-Kadec norm, but B_{X^*} is not w^*-sequentially compact.*

(d) *No dual norm on ℓ_1^* is sequentially w^*-Kadec, but every dual norm on ℓ_1^* is w^*-τ-Kadec.*

Proof. For (a), if $Y \subset X$ is separable, then by Proposition 1.4(a) the dual norm on Y^* is w^*-Kadec. Thus it follows easily that Y^* is separable; see

[DS]. Hence X is Asplund and thus has w^*-sequentially compact dual ball; see [Di, Bo]. To see (b), by the Schur property and Eberlein–Smulian theorem weakly compact sets are norm compact; thus it is clear that w^*-convergence is the same as τ-convergence. Easily (c) follows from (b) because $X = \ell_1(\Gamma)$ is Schur but B_{X^*} is not w^*-sequentially compact whenever Γ is uncountable (see [Du, p. 48]). Finally, (d) is a direct consequence of (a) and (b). □

2. Dual Kadec Norms and Set Convergence

We begin this section with our main result. This theorem with [B4, Thm. 3.1] shows that Wijsman convergence with respect to a given norm $\|\cdot\|$ on X implies Wijsman convergence with respect to all other equivalent norms on X if and only if the dual norm of $\|\cdot\|$ is w^*-Kadec; this answers a question from [B4].

THEOREM 2.1. *For a Banach space X, the following are equivalent.*

(a) *The dual norm on X^* is w^*-Kadec.*
(b) *For each separable subspace Y of X, the dual norm on Y^* is sequentially w^*-Kadec.*
(c) *Wijsman convergence implies slice convergence sequentially in every separable subspace of X.*
(d) *Wijsman and slice convergence coincide in every subspace of X.*

Proof. Proposition 1.4(a) shows that (a) implies (b), so we show (b) ⇒ (c). Let Z be any separable subspace of X. Suppose that C_n converges Wijsman to C as subsets of Z. We wish to show that (i) and (ii) with $\mathfrak{I}$, the norm topology in Proposition 1.1, hold. Clearly (i) follows from Wijsman convergence, so we show (ii). Let $z_0^* \in S_{X^*}$ attain its supremum on C, say $\sup_C z_0^* = \langle z_0^*, z_0 \rangle$ where $z_0 \in C$.

Let $\alpha_0 = \langle z_0^*, z_0 \rangle$ and let $L = \{z : \langle z_0^*, z \rangle = \alpha_0 + 1\}$. Since Z and hence L are separable, we can choose a sequence of compact convex sets $\{K_n\}$ such that $K_n \subset L$ for each n, $K_1 \subset K_2 \subset K_3 \subset \cdots$, $d(K_n, z_0) < 1 + 1/n$, and

$$L \text{ is the norm closure of } \bigcup_{n=1}^{\infty} K_n. \tag{2.1}$$

Since compact sets have finite ϵ-nets and since C_n converges Wijsman to C, we deduce that

$$\lim_{j\to\infty} d(K_n, C_j) = d(K_n, C) \quad \text{for each } n$$

(in other words, Wijsman convergence is precisely compact gap convergence). Thus we may choose $j_1 < j_2 < j_3 < \cdots$ such that

$$d(z_0, C_j) < 1/n \quad \text{and} \quad 1 - 1/n < d(K_n, C_j) < 1 + 1/n \quad \text{for } j \ge j_n. \tag{2.2}$$

It follows that $(1 - 1/n)B_Z \cap (K_n - C_j) = \emptyset$ for $j \ge j_n$. Thus, by the separation theorem, there exists $\Lambda_{n,j} \in S_{Z^*}$ such that

$$\sup\{\langle \Lambda_{n,j}, z\rangle : z \in (1-1/n)B_Z\} \le \inf\{\langle \Lambda_{n,j}, z\rangle : z \in K_n - C_j\} \quad \text{for } j \ge j_n.$$

This implies

$$\sup_{C_j} \Lambda_{n,j} + (1-1/n) \le \min_{K_n} \Lambda_{n,j} \quad \text{for } j \ge j_n. \tag{2.3}$$

Now set $z_j^* = 0$ for $j < j_1$ and

$$z_j^* = \Lambda_{n,j} \quad \text{for } j_n \le j < j_{n+1}.$$

Claim: $z_j^* \xrightarrow{w^*} z_0^*$. Assume temporarily that the claim is true. Because

$$d(z_0, K_n) < 1 + 1/n,$$

it follows that

$$\min_{K_n} z_j^* < \langle z_j^*, z_0\rangle + \left(1 + \frac{1}{n}\right). \tag{2.4}$$

For $j \in \mathbf{N}$, let n_j denote the number n such that $j_n \le j < j_{n+1}$. Thus, by (2.3) and (2.4), one has

$$\begin{aligned}\sup_{C_j} z_j^* + \left(1 - \frac{1}{n_j}\right) &\le \min_{K_{n_j}} z_j^* \\ &< \langle z_j^*, z_0\rangle + \left(1 + \frac{1}{n_j}\right).\end{aligned}$$

In other words,

$$\sup_{C_j} z_j^* < \langle z_j^*, z_0\rangle + \frac{2}{n_j}.$$

Since $\langle z_j^*, z_0\rangle \to \langle z_0^*, z_0\rangle = \sup_C z_0^*$, this immediately yields

$$\limsup_j \{\sup_{C_j} z_j^*\} \le \sup_C z_0^*.$$

Moreover, the dual norm on Z^* is sequentially w^*-Kadec, so $\|z_j^* - z_0^*\| \to 0$. Thus (ii) with $\mathfrak{J}$, the norm topology in Proposition 1.1, holds—provided our claim is true.

Let us now prove the claim by showing every subsequence of $\{z_j^*\}$ has a subsequence that converges w^* to z_0^*. By abuse of notation, let $\{z_j^*\}$ denote an arbitrary subsequence of $\{z_j^*\}$. From the w^*-sequential compactness of B_{Z^*}, by passing to another subsequence if necessary, we have $z_{j'}^* \xrightarrow{w^*} \Lambda$ for some $\Lambda \in B_{Z^*}$. We now show that $\|\Lambda\| = 1$. Again, we use n_j to denote the n such that $j_n \le j < j_{n+1}$; because $d(z_0, C_j) \le 1/n$ for $j \ge j_n$, it follows that

$$\sup_{C_j} z_j^* \ge \langle z_j^*, z_0\rangle - \frac{1}{n_j}.$$

Let $m \in \mathbf{N}$ and $z \in K_m$ be fixed. Because $z \in K_n$ for $n \ge m$, the above inequality yields

$$\begin{aligned}\langle \Lambda, z - z_0\rangle &= \lim_{j'} \langle z_{j'}^*, z - z_0\rangle \\ &\ge \liminf_j \left(\min_{K_{n_j}} z_j^* - \left(\sup_{C_j} z_j^* + \frac{1}{n_j}\right)\right)\end{aligned}$$

$$\geq \liminf_j \left(1 - \frac{1}{n_j} - \frac{1}{n_j}\right) \quad \text{(by (2.3))}$$
$$= 1.$$

Consequently, we have

$$\min_{K_m} \Lambda \geq \langle \Lambda, z_0 \rangle + 1 \quad \text{for all } m \in \mathbf{N}. \tag{2.5}$$

Since $\lim_n d(K_n, z_0) \to 1$, it also follows that $\|\Lambda\| = 1$.

It now suffices to show that $\Lambda = z_0^*$. Let $H = \{z : \langle z_0^*, z\rangle \geq 0\}$. We claim that $\langle \Lambda, z\rangle \geq 0$ for all $z \in H$. So suppose that $\langle \Lambda, h\rangle \leq -\delta$ for some $\delta > 0$ and some $h \in H$ with $\|h\| \leq 1$. Now consider $z_0 + h$; then $\alpha_0 \leq \langle z_0^*, z_0 + h\rangle \leq 1 + \alpha_0$ and so $d(z_0 + h, L) \leq 1$. Thus, by (2.1), we can find $\tilde{z} \in K_m$ for some m such that $\|\tilde{z} - (z_0 + h)\| \leq 1 + \delta/2$. Hence it follows that

$$\langle \Lambda, \tilde{z}\rangle \leq \|\tilde{z} - (z_0 + h)\| + \langle \Lambda, z_0 + h\rangle \leq (1 + \delta/2) + \langle \Lambda, z_0 + h\rangle \leq \langle \Lambda, z_0\rangle + 1 - \delta/2.$$

This contradicts (2.5). Therefore, $\langle \Lambda, h\rangle \geq 0$ for all $h \in H$. But since $\|\Lambda\| = \|z_0^*\| = 1$, this means $\Lambda = z_0^*$. This shows that the claim holds and thus (b) ⇒ (c).

Now (c) ⇒ (d) follows from Theorem 1.2, and (d) ⇒ (a) is a consequence of Proposition 1.3(b). □

Observe that we've also proved the following variant of [AB, Thm. 3.1].

REMARK 2.2. *Suppose X is a separable Banach space. Then C_n converges Wijsman to C if and only if the following two conditions hold.*

(i) *If $x_0 \in C$, then there exist $x_n \in C_n$ such that $\|x_n - x_0\| \to 0$.*
(ii) *If $x_0^* \in S_{X^*}$ attains its supremum on C, then there exist $x_n^* \in B_{X^*}$ such that $x_n^* \xrightarrow{w^*} x_0^*$ and $\limsup_n \{\sup_{C_n} x_n^*\} \leq \sup_C x_0^*$.*

Proof. If (i) and (ii) hold, then Proposition 1.1 shows that C_n converges Wijsman to C. Conversely, (i) follows directly from Wijsman convergence; moreover, the z_j^*s constructed in the proof of Theorem 2.1 satisfy (ii) with $z_0^* = x_0^*$ (the w^*-Kadec property was used only to go from w^* to norm convergence). □

The next theorem is a Wijsman versus weak-compact-gap variant of Theorem 2.1.

THEOREM 2.3. *If X is a Banach space then the following are equivalent.*

(a) *The dual norm on X^* is w^*-τ-Kadec.*
(b) *For each separable subspace $Y \subset X$, the dual norm on Y^* is sequentially w^*-τ-Kadec.*
(c) *For each subspace Y of X, Wijsman and weak compact gap convergence coincide.*
(d) *Wijsman convergence implies Mosco convergence in X.*

Proof. Proposition 1.4(a) shows (a) implies (b), so we show (b) ⇒ (c). Let Y be a separable subspace of X. Since the dual norm on Y^* is w^*-τ-Kadec, Remark 2.2 and Proposition 1.1 show that Wijsman and weak compact gap convergence coincide in Y. Combining this with Theorem 1.2 shows that (c) holds. To prove (c) ⇒ (d), observe first that M(i) clearly holds. We now show M(ii): Let $x_\beta \in C_\beta$ for some subnet and suppose $\{x_\beta\}$ is relatively weakly compact. If $x_\beta \xrightarrow{w} x$ and $x \notin C$, then there is an open half-space containing x and a tail of $\{x_\beta\}$ which is strictly separated from C. Let W be the closed convex hull of this tail. Then $d(W, C) > 0$, but $\lim_\beta d(W, C_\beta) = 0$; this contradicts (c). Hence we have (c) ⇒ (d). Finally, Proposition 1.3(a) shows (d) ⇒ (a). ❑

In certain spaces it is enough to know that Wijsman convergence implies Mosco convergence in order to deduce that Wijsman convergence implies slice convergence as the next result shows. By $X \not\supset \ell_1$, we mean that X does not contain an isomorphic copy of ℓ_1.

COROLLARY 2.4. *For a Banach space X, the following are equivalent.*

(a) *The dual norm on X^* is w^*-Kadec.*
(b) *B_{X^*} is w^*-sequentially compact, and Wijsman convergence implies slice convergence sequentially.*
(c) *X is Asplund and Wijsman and slice convergence coincide.*
(d) *$X \not\supset \ell_1$ and Wijsman convergence implies Mosco convergence.*
(e) *$X \not\supset \ell_1$ and its dual norm is w^*-τ-Kadec.*
(f) *$X \not\supset \ell_1$ and, for every separable subspace $Y \subset X$, the dual norm on Y^* is sequentially w^*-τ-Kadec.*

Proof. From Theorem 2.1 and Remark 1.5(a) one immediately obtains that (a) implies (b). If (b) holds, then Proposition 1.3(b) shows the dual norm is sequentially w^*-Kadec. Thus Proposition 1.4(b), Remark 1.5(a), and Theorem 2.1 show that (c) holds. One obtains that (c) implies (d) directly from the definitions. Proposition 1.3(a) shows (d) ⇒ (e), while Proposition 1.4(a) shows (e) ⇒ (f). Finally, suppose (f) holds and let Y be an arbitrary separable subspace of X. Since $Y \not\supset \ell_1$, [Em] shows that every τ-convergent sequence in Y^* is norm convergent (see also [BFa, Thm. 5]). Hence the dual norm on Y^* is sequentially w^*-Kadec. Thus Theorem 2.1 shows that (a) holds. □

We need some more terminology before stating the next corollary. For a Banach space X one can define a metric ρ on the space P of all equivalent norms on X as follows. Fix a norm on X with unit ball B_1. For $\mu, \nu \in P$, define $\rho(\mu, \nu) = \sup\{|\nu(x) - \mu(x)| : x \in B_1\}$. It is shown in [DGZ, p. 52] that (P, ρ) is a Baire space. We will say a set is *residual* if it contains a dense G_δ set.

COROLLARY 2.5. *If X is a WCG Banach space, then the following are equivalent.*

(a) *X is Asplund.*
(b) *There is a residual collection of norms in (P, ρ) for which Wijsman convergence implies slice convergence (in any subspace of X).*

(c) *$X \not\supset \ell_1$, and there is an equivalent norm on X for which Wijsman convergence implies Mosco convergence sequentially.*

Proof. It follows from [DGZ, Cor. VII.1.13, Thm. II.4.1] that the collection of norms with dual w^*-Kadec norms is residual in (P, ρ). Invoking Theorem 2.1 shows (a) $\Rightarrow$ (b). Corollary 2.4 shows (b) $\Rightarrow$ (c). If (c) holds then B_{X^*} is w^*-sequentially compact, because X is WCG (see [Di, p. 228]). Now Proposition 1.3(a) and Proposition 1.4(b) show that the dual norm on X^* is w^*-τ-Kadec, thus (a) follows from Corollary 2.4. □

REMARK 2.6. (a) The proof of [DGZ, Thm. II.4.1] shows for each Banach space that the set of norms in (P, ρ) whose duals are w^*-Kadec (w^*-τ-Kadec) is either empty or residual.

(b) One cannot drop the assumption "X is WCG" in Corollary 2.5, since $C[0, \omega_1]$ is an Asplund space that even admits a Fréchet differentiable norm [DGZ, Thm. VII.5.4]; the proof of [DGZ, Thm. VII.5.2] shows that it cannot admit a dual sequentially w^*-Kadec norm.

(c) Corollary 2.4 does not imply that a sequence of sets converges slice if and only if it converges Mosco and Wijsman (even in Asplund spaces). Indeed, [BL, Thm. 6] shows that each nonreflexive separable Banach space can be renormed so that a decreasing sequence of subspaces converges Wijsman and Mosco but not slice.

(d) There are spaces that are neither Asplund nor Schur which can be renormed so that the dual norm is w^*-τ-Kadec. Indeed, for Ω a σ-finite measure space, [BF2, Thm. 2.4] shows there is a norm on $L_1(\Omega)$ whose dual norm is w^*-τ-Kadec.

3. Set Convergence and Differentiability

Recall that a function is said to be *weak Hadamard differentiable* at a point if its Gateaux derivative exists at the point and is uniform on weakly compact sets. The following proposition shows that this notion is related to Wijsman and Mosco convergence in non-Asplund spaces. Indeed, notice that property (*) ensures that X contains an isomorphic copy of ℓ_1; see [BFa].

PROPOSITION 3.1. *Let X be a Banach space. Then the following are equivalent.*

(a) *For every equivalent norm on X, Wijsman convergence implies Mosco convergence for sequences of level sets of functionals.*
(b) *The following property is satisfied:*

$$\langle x_n^*, x_n\rangle \to \langle x^*, x\rangle \quad \text{whenever } x_n^* \xrightarrow{w^*} x^* \text{ and } x_n \xrightarrow{w} x. \qquad (*)$$

(That is, w^*-convergent sequences in X^* are τ-convergent.)
(c) *Weak Hadamard and Gateaux differentiability coincide for continuous convex functions on X.*

Proof. The equivalence of (b) and (c) follows from the results of [BFa]; see also [BFV].

(a) ⇒ (b): Suppose (∗) fails. Then we can find $x_n^* \xrightarrow{w^*} x^*$ and $x_n \xrightarrow{w} x$, but $|\langle x_n^*, x_n\rangle - \langle x^*, x\rangle| \geq \epsilon$ for all n and some $\epsilon > 0$. We now show that X admits an equivalent norm whose dual is not sequentially w^*-τ-Kadec. Notice that we may assume $\|x_n^*\| \leq 1$ for all n. If $\|x^*\| = 1$ then $\|\cdot\|$ is not sequentially w^*-τ-Kadec. So suppose $\|x^*\| < 1$. We may assume $x^* = 0$ and that $\|x_n^*\| \leq 1$ for all n. Now let $y \in X$ satisfy $\|y\| = 1$. By replacing y with $-y$ if necessary, we have $\langle x_j^*, y\rangle \leq 0$ for all $j \in J$, where J is an infinite subset of $\mathbf{N}$. Now choose $y^* \in X^*$ satisfying $\langle y^*, y\rangle = \|y^*\| = 1$. Define a convex w^*-compact subset of X^* by

$$B = \{\Lambda \in X^*: |\langle \Lambda, y\rangle| \leq 1\} \cap \{\Lambda \in X^*: \|\Lambda\| \leq 2\}.$$

Let $|||\cdot|||$ denote the dual norm on X^* whose unit ball is B. Observe that $|||y^* + x^*||| = 1$ and $|||y^* + x_j^*||| \leq 1$ for all $j \in J$. Hence $|||\cdot|||$ is not sequentially w^*-τ-Kadec, since $y^* + x_j^* \xrightarrow{w^*} y^* + x^*$ but $\langle y^* + x_j^*, x_j\rangle \not\to \langle y^* + x^*, x\rangle$. Thus, by Proposition 1.3(a), (a) does not hold.

(b) ⇒ (a): Let $\|\cdot\|$ be any equivalent norm on X. If C_n converges Wijsman to C, where $C_n = \{x: \langle x_n^*, x\rangle = \alpha\}$ and $C = \{x: \langle x^*, x\rangle = \alpha\}$, then (by [Bel, Thm. 4.3]) $x_n^* \xrightarrow{w^*} x^*$ and $\|x_n^*\| \to \|x^*\|$. Now suppose $x_j \in C_j$ for $j \in J \subset \mathbf{N}$ and $x_j \xrightarrow{w} x$. By property (∗) we have $\langle x_j^*, x_j\rangle \to \langle x^*, x\rangle$, which means $\langle x^*, x\rangle = \alpha$ and $x \in C$. Thus M(ii) holds. Since M(i) always holds in the presence of Wijsman convergence, we are done. □

COROLLARY 3.2. *Suppose that every separable subspace of X is contained in a complemented subspace whose dual ball is w^*-sequentially compact. Then the following are equivalent.*

(a) *For every equivalent norm on X, Wijsman convergence implies Mosco convergence for sequences of level sets of functionals.*
(b) *X has the Schur property.*

Proof. (a) ⇒ (b): This follows from Proposition 3.2 and [BFV, Cor. 3.5].

(b) ⇒ (a): This is always true. □

The condition in the preceding corollary is, of course, satisfied in all spaces whose dual balls are w^*-sequentially compact (in particular, WCG spaces) as well as in much more general cases; see [BFV]. In addition, there are many Grothendieck $C(K)$ spaces that satisfy property (∗) but are not Schur; see [BFV] and the references therein.

4. Further Examples and Results

Let us first observe that Mosco and slice convergence are preserved in superspaces.

PROPOSITION 4.1. *Let Y be a Banach space, and suppose C_α converges slice (Mosco) to C in Y. If X is a superspace of Y, then C_α converges slice (Mosco) to C in X.*

Proof. It is clear from the definition that this holds for Mosco convergence. We will prove that slice convergence is preserved in superspaces. Suppose that $C_\alpha, C \subset Y$, where Y is a subspace of X, and that C_α does not converge slice to C in X. We will show that C_α does not converge slice to C in Y. We may suppose C_α converges Wijsman to C in Y, since otherwise we are done. Wijsman convergence implies directly that for any $B \subset X$, $\limsup d(B, C_\alpha) \leq d(B, C)$. Thus, because C_α does not converge slice to C, by passing to a subnet if necessary we find a bounded closed convex $W \subset X$ such that

$$d(W, C_\alpha) + 3\delta \leq d(W, C) \quad \text{for all } \alpha \text{ and some } \delta > 0.$$

Let

$$r = d(W, C) - 2\delta.$$

Then $(W + B_r) \cap C_\alpha \neq \emptyset$ for all α, while $(W + B_{r+\delta}) \cap C = \emptyset$. Using the separation theorem, we find a $\Lambda \in S_{X^*}$ such that

$$\sup\{\langle \Lambda, x\rangle : x \in W + B_{r+\delta}\} \leq \inf\{\langle \Lambda, x\rangle : x \in C\}.$$

Let $a = \sup\{\langle \Lambda, x\rangle : x \in W + B_r\}$; then

$$a + \delta = \sup\{\langle \Lambda, x\rangle : x \in W + B_{r+\delta}\} \leq \inf\{\langle \Lambda, x\rangle : x \in C\} \tag{4.1}$$

and

$$W + B_r \subset \{x : \langle \Lambda, x\rangle \leq a\}.$$

Now let $m > 0$ be chosen such that $W + B_r \subset B_m$. Because $(W + B_r) \cap C_\alpha \neq \emptyset$, there exists $y_\alpha \in C_\alpha \subset Y$ such that

$$y_\alpha \in \{y \in Y : \langle \Lambda, y\rangle \leq a\} \cap B_m.$$

We set $W_1 = \{y \in Y : \langle \Lambda, y\rangle \leq a\} \cap B_m$. Hence, $W_1 \cap C_\alpha \neq \emptyset$ for all α. However, according to (4.1), $d(W_1, C) \geq \delta$ and so C_α does not converge slice to C in Y. □

Proposition 4.1 with Theorem 2.1 shows that for nice norms, Wijsman convergence is preserved in superspaces. In general, however, this can fail dramatically.

EXAMPLE 4.2. *Wijsman convergence in X is not necessarily preserved in $X \times \mathbf{R}$.*

Proof. Let X be c_0 endowed with the norm $|||\cdot|||$ which is defined for $x = (x_n)_{n=0}^{\infty}$ as follows:

$$|||x||| = |x_0| \vee |x_1| \vee (\sup_{n \geq 2} |x_n + x_1|).$$

Let $Y=\{x\in X: x_0=x_1\}$ and define $\hat{f}_n\in X^*$ by

$$\hat{f}_n(x)=x_1+x_n \quad \text{and} \quad \hat{f}_\infty(x)=x_1.$$

Then

$$|||\hat{f}_n|||=\sup\{x_1+x_n: |x_1+x_n|\le 1, |x_1|\le 1\}=1$$

and

$$|||\hat{f}_\infty|||=\sup\{x_1: |x_1+x_n|\le 1, |x_1|\le 1\}=1.$$

Now $\hat{f}_n(x)\to\hat{f}_\infty(x)$ for all x, since $x_n\to 0$. It follows directly that $\hat{f}_n^{-1}(1)$ converges Wijsman to $\hat{f}_\infty^{-1}(1)$; see [B1, Thm. 4.3].

Let $f_n=\hat{f}_n|_Y$. Then $|||f_n|||=|||f_\infty|||=1$, and so similarly it follows that

$$f_n^{-1}(1) \text{ converges Wijsman to } f_\infty^{-1}(1) \text{ in } Y.$$

However, $f_n^{-1}(1)$ does not converge Wijsman to $f_\infty^{-1}(1)$ in X. Indeed, consider $z^0=(0,\frac{1}{2},0,0,\ldots)$ and $z^n=\frac{1}{2}e_0+\frac{1}{2}e_1+\frac{1}{2}e_n$. Then $z^n\in f_n^{-1}(1)$ and

$$\tfrac{1}{2}=|||z^0-z^n|||\ge d(z^0,f_n^{-1}(1)).$$

On the other hand, if $x=(x_i)_{i=0}^\infty\in f_\infty^{-1}(1)$, then $x_0=x_1=1$ and consequently one has $|||x-z^0|||\ge|x_0|=1$ which means $d(z^0,f_\infty^{-1}(1))\ge 1$. □

It appears to be unknown if there is a norm whose dual is sequentially w^*-Kadec norm but not w^*-Kadec. Moreover, if such a norm exists, we do not know if Wijsman convergence with respect to it implies slice convergence sequentially. However, for w^*-τ-Kadec norms we have the following example (see also [Tr] for connections between w-Kadec and sequentially w-Kadec norms).

EXAMPLE 4.3. *There is a norm on ℓ_∞ whose dual norm is sequentially w^*-τ-Kadec but not w^*-τ-Kadec.*

Proof. (a) Since w^*-convergent sequences in ℓ_∞^* are w-convergent [Di, p. 103] and since ℓ_∞ has the Dunford–Pettis property [Di, p. 113], it follows that w^*-convergent sequences in ℓ_∞^* are τ-convergent (see Proposition 3.1). Thus the dual of every norm on ℓ_∞ is sequentially w^*-τ-Kadec. Let $|||\cdot|||$ be an equivalent norm on c_0 whose dual is not sequentially w^*-Kadec (see [BFa]). It follows as in Corollary 2.4 that the dual norm of $|||\cdot|||$ is not sequentially w^*-τ-Kadec. Now let $|||\cdot|||$ denote the second dual of this norm on ℓ_∞. Then the dual of $|||\cdot|||$ is not w^*-τ-Kadec on ℓ_∞^*, by the equivalence of (b) and (a) in Theorem 2.3 (a simple proof of this equivalence can be found in [BV, Prop. 1.1]). □

We now show that, with some restrictions on the limit set, one can deduce slice convergence from Wijsman convergence in spaces whose duals need not admit sequentially w^*-Kadec norms (by Remark 2.6(a)); see [BV, Thm. 3.12] for a more general result. Recall that a space is said to have the *Radon–*

Nikodym property (RNP) if every closed bounded convex set has slices of arbitrarily small diameter; see [Bo].

PROPOSITION 4.4. (a) *Suppose X has the RNP and its norm is Fréchet differentiable. If C_α converges Wijsman to a closed bounded convex set C, then C_α converges slice to C.*

(b) *Suppose the norm on X is Fréchet differentiable and C is weakly compact. If C_α converges Wijsman to C, then C_α converges slice to C.*

Proof. (a) Since Fréchet differentiable norms are inherited by subspaces, Theorem 1.2 implies that it suffices to show this result for sequences of sets. We show that Proposition 1.1(ii) with $\mathfrak{I}$ the norm topology holds since Wijsman convergence implies Proposition 1.1(i).

Let x_0^* attain its supremum on C. Since X has the RNP, the functionals which simultaneously attain their suprema on B_X and C are dense in S_{X^*}; see [Bo, Cor. 3.7.9]. Thus we find functionals $\phi_n \in S_{X^*}$ such that ϕ_n attains its supremum on C at (say) x_n, on B_X at (say) u_n, and

$$\|\phi_n - x_0^*\| \le 1/n \quad \text{and} \quad \sup_C \phi_n \le \sup_C x_0^* + 1/n. \tag{4.2}$$

It follows that $1 = \phi_n(u_n) \le d(x_n + u_n, C) \le 1$. Let $v_n = x_n + u_n$ and choose $k_{n,m}$ such that $k_{n,m+1} > k_{n,m}$ and

$$d(v_n, C_k) > 1 - 1/m \quad \text{for } k \ge k_{n,m}.$$

By the separation theorem, for $k_{n,m} \le k < k_{n,m+1}$, choose $\phi_{n,k} \in S_{X^*}$ such that

$$\sup_{C_n} \phi_{n,k} + \left(1 - \frac{1}{m}\right) \le \phi_{n,k}(v_n). \tag{4.3}$$

From here we conclude that $\lim_k \phi_{n,k}(u_n) = 1$ and since the norm is Fréchet differentiable, $\lim_k \|\phi_{n,k} - \phi_n\| = 0$ (see [DGZ, Thm. I.1.4]). Thus (4.2) and (4.3) show

$$\limsup_k \{\sup_{C_k} \phi_{n,k}\} \le \sup_C x_0^* + \frac{1}{n}.$$

A standard diagonalization argument now produces the desired sequence.

(b) For C weakly compact, the same proof works because the Bishop-Phelps theorem ensures there is a dense set in S_{X^*} which simultaneously supports C and B_{X^*}. □

There are also simple proofs, somewhat like the above argument, showing that if the dual norm is locally uniformly rotund (which is a much stronger condition than having a dual w^*-Kadec norm), then Wijsman and slice convergence coincide. The definition and properties of locally uniformly rotund norms can be found in [DGZ].

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