A Characterization of Domains Quasiconformally Equivalent to the Unit Ball

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A difficult problem in the theory of quasiconformal mappings is the characterization of domains in Euclidean n-space that can be mapped quasiconformally onto the unit ball $\mathbf{B}^n$. It is a type of n-dimensional Riemann mapping problem. Gehring [Ge] reduced the problem for a domain to a problem for its boundary. He showed that if a quasiconformal mapping exists from a neighborhood of a domain's boundary onto a neighborhood in $\mathbf{B}^n$ of $\mathbf{S}^{n-1}$, then a quasiconformal mapping exists between the domain and $\mathbf{B}^n$. But unlike the Riemann mapping theorem (which solves the problem for $n=2$), no conditions pertaining solely to the boundary have been discovered which guarantee a domain's quasiconformal equivalence to $\mathbf{B}^n$.

Nakai [N1] established an implicit characterization of quasiconformally equivalent domains in terms of function algebras by showing that a quasiconformal mapping exists between two domains in $\mathbf{R}^2$ if and only if their corresponding Royden algebras are isomorphic. Lewis [Le] and Lelong-Ferrand [L-F] extended his proof to higher dimensions and Riemannian manifolds. The work of Nakai and Lewis relies on methods of functional analysis to characterize the maximal ideal space of a domain Ω as a compactification Ω^* of Ω. Lewis showed that a quasiconformal mapping between two domains implies that these so-called *Royden compactifications* are homeomorphic. The converse question—whether a homeomorphism between Royden compactifications implies existence of a quasiconformal mapping between domains—is the subject of this paper.

Nakai [N2] answered the question affirmatively for the 2-dimensional case, and gave a partial answer for higher dimensions by showing [NT] that the restriction of a homeomorphism between Royden compactifications to their underlying domains is quasiconformal in a neighborhood of the boundary. A connection between this result and the theorem of Gehring has not been previously observed in the literature. However, if one of the domains is $\mathbf{B}^n$, then the result of Nakai and Tanaka dovetails nicely with Gehring's theorem to prove that the converse question can be answered affirmatively. Hence, a domain Ω is quasiconformally equivalent to $\mathbf{B}^n$ if and only if Ω^* is

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homeomorphic to $(\mathbf{B}^n)^*$. We prove this theorem and discuss an example of a domain not quasiconformally equivalent to $\mathbf{B}^n$.

1. Preliminaries

Ω denotes a domain in $\mathbf{R}^n$. $L_n^1(\Omega)$ denotes the space of functions whose weak partial derivatives exist and belong to $L^n(\Omega)$; $W_{n,\mathrm{loc}}^1(\Omega)$ denotes the space of functions that belong locally to $L^n(\Omega)$ and $L_n^1(\Omega)$. A *quasiconformal mapping* is a homeomorphism $f: \Omega \to \mathbf{R}^n$ in $W_{n,\mathrm{loc}}^1(\Omega)$ for which there is a constant K such that

$$|Df(x)|^n \leq KJ_f(x)$$

at almost every $x \in \Omega$. Good discussions of quasiconformal mappings are given in [Vä] and [BI].

The *Royden algebra* $A(\Omega)$ consists of all real functions

$$u \in C(\Omega) \cap L^\infty(\Omega) \cap L_n^1(\Omega),$$

with multiplication and addition defined pointwise on Ω. Given the norm

$$\|u\| = \|u\|_\Omega = \|u\|_\infty + \|\nabla u\|_{n,\Omega},$$

$A(\Omega)$ is a commutative Banach algebra; that is, $A(\Omega)$ is a Banach space satisfying $\|uw\| \leq \|u\|\|w\|$ for each $u, w \in A(\Omega)$. Furthermore, if x and y are distinct points in Ω, then there exists $u \in A(\Omega)$ such that $u(x) = 1$ and $u(y) = 0$. Hence, $A(\Omega)$ *separates* points in Ω. For detailed discussions of the Royden algebra, see [Le], [L-F] and [S2]. The relationship between quasiconformal mappings and Royden algebras is summarized in the following theorem.

THEOREM 1.1 [Le, Thms. 3.2, 7.1; L-F, Thm. 11.3]. *Let Ω and Ω' be domains in $\mathbf{R}^n$, $n \geq 2$. Then if $f: \Omega \to \Omega'$ is K-quasiconformal, the transformation $f^*: A(\Omega') \to A(\Omega)$ defined by $f^*v = v \circ f$ for each $v \in A(\Omega')$ is an algebra isomorphism. Conversely, if $T: A(\Omega') \to A(\Omega)$ is an algebra isomorphism, then T induces a $\|T\|^n$-quasiconformal mapping $f: \Omega \to \Omega'$ such that $f^* = T$.*

2. The Royden Compactification

The *Royden compactification* Ω^* *of* Ω is the collection of all nonzero, bounded linear functionals χ on $A(\Omega)$ satisfying $\chi(uw) = \chi(u)\chi(w)$ for each $u, w \in A(\Omega)$. In other words, each $\chi: A(\Omega) \to \mathbf{R}$ must be a continuous homomorphism, so Ω^* lies in the dual space of $A(\Omega)$. The norm of a linear functional on $A(\Omega)$ is given by

$$\|\chi\| = \sup\{|\chi(u)| : u \in A(\Omega),\ \|u\| \leq 1\}.$$

It can be shown that $\|\chi\| = 1$ for each $\chi \in \Omega^*$, and that Ω^* is a compact Hausdorff space in the relative weak* topology generated by $A(\Omega)$ [S2, §4]. Convergence in this topology is characterized by the rule that if $\{\chi_\alpha\}_{\alpha \in \Lambda}$ is a net (generalized sequence) in Ω^*, then $\chi_\alpha \to \chi \in \Omega^*$ if and only if

$$\lim_\alpha |\chi_\alpha(u) - \chi(u)| = 0$$

for each $u \in A(\Omega)$. (For a discussion of nets, see [Ke].) We note that Ω^* is identical to the space of (proper) maximal ideals M of $A(\Omega)$ through the correspondence $M_\chi \leftrightarrow \chi$ defined by

$$M_\chi = \{u \in A : \chi(u) = 0\} = \chi^{-1}(0)$$

[S2, p. 1154].

If $x \in \Omega$, then $\hat{x}$ denotes the *point evaluation* homomorphism defined by $\hat{x}(u) = u(x)$ for each $u \in A(\Omega)$. Clearly, $\hat{x} \in \Omega^*$, and it is not hard to check that the mapping $x \mapsto \hat{x}$ is continuous. In fact, because $A(\Omega)$ separates points in Ω, $x \mapsto \hat{x}$ is a homeomorphism of Ω onto its image $\hat{\Omega} \subset \Omega^*$. The weak* topology on $\hat{\Omega}$ is identical to the Euclidean topology on Ω; therefore the identification of each $x \in \Omega$ with $\hat{x}$ allows Ω to be viewed as a subset of Ω^*. In fact, Ω is dense in Ω^* in the weak* topology [Le, p. 489]. As a result, the set $\Delta = \Delta_\Omega = \Omega^* \backslash \Omega$ constitutes a boundary known as the *Royden ideal boundary of* Ω.

Because Ω is dense in Ω^* in the weak* topology, each $\chi \in \Delta$ is the limit of a net $\{\hat{z}_\alpha\}_{\alpha \in A}$ such that $z_\alpha \in \Omega$ for each α. A net $\{z_\alpha\}$ in Ω for which $\{\hat{z}_\alpha\}$ converges to $\chi \in \Delta$ is called a *Royden net corresponding to* χ. Each $\chi \in \Delta$ may have many corresponding Royden nets, but there is a unique $z \in \partial\Omega$ corresponding to χ such that if $\hat{z}_\alpha \to \chi$ then $z_\alpha \to z$ in the Euclidean topology on $\bar{\Omega}$ [S3, §3]. The set $\Phi_z \subset \Delta$ of all $\chi \in \Delta$ whose Royden nets converge to $z \in \partial\Omega$ is called the *boundary fiber over* z. There is thus a one-to-one correspondence between boundary fibers in Δ and boundary points in $\partial\Omega$.

3. Homeomorphisms of Royden Compactifications

We now present the necessary background for proving our main result. We require the following lemma, which gives a topological characterization of the elements in Ω^* belonging to Δ.

LEMMA 3.1 [Le, Lemma 6.2; S2, Thm. 4.7]. *If $\chi \in \Omega^*$, then $\chi \in \Delta$ if and only if χ has no countable neighborhood basis in Ω^*.*

Lewis showed that a quasiconformal mapping $f : \Omega \to \Omega'$ induces a homeomorphism between Ω^* and Ω'^*. If $T = f^*$ is the induced Royden algebra isomorphism described in Theorem 1.1, he proved that the *adjoint* $T^* : \Omega^* \to \Omega'^*$ of T, defined by $T^*\chi = \chi \circ T$ for each $\chi \in \Omega^*$, is a homeomorphism in the weak* topologies such that $T^*(\Delta) = \Delta'$ and $T^* | \Omega = f$ [Le, p. 490; S2, §5, §6].

A natural converse question is whether a homeomorphism $\phi : \Omega^* \to \Omega'^*$ induces a quasiconformal mapping between Ω and Ω'. However, as observed by the referee, a trivial example shows that the answer is negative. The restriction of any such ϕ to Ω can be deformed on a relatively compact ball $B \subset \Omega$ to a nonquasiconformal homeomorphism of B onto $\phi(B)$ which coincides with ϕ on ∂B. Therefore, allowing for such deformations on compact sets, Nakai and Tanaka proved the "best possible" converse.

THEOREM 3.2 [NT, Thm. 4]. *Let M and N be two Riemannian manifolds of dimension $n \geq 2$. If there exists a homeomorphism ϕ of M^* onto N^*, then there exists a compact subset $F \subset M$ such that the restriction of ϕ to each component of $M \setminus F$ is quasiconformal.*

The more general question of whether or not the existence of a homeomorphism $\phi: \Omega^* \to \Omega'^*$ implies the existence of quasiconformal mapping from Ω to Ω' remains unanswered. However, we can show that if $\Omega' = \mathbf{B}^n$ then there exists a quasiconformal mapping of Ω onto Ω' (not necessarily the restriction of ϕ to Ω). The two main results contributing to the proof are the theorem of Nakai and Tanaka and the following theorem of Gehring.

THEOREM 3.3 [Ge, Thm. 2]. *Suppose that Ω is a domain in $\bar{\mathbf{R}}^n$, that U is a neighborhood of $\partial\Omega$, and that f is a quasiconformal mapping of $\Omega \cap U$ into $\mathbf{B}^n$ such that $|f(x)| \to 1$ as $x \to \partial\Omega$ in $\Omega \cap U$. Then there exists a neighborhood U' of $\partial\Omega$ and a quasiconformal mapping g of Ω onto $\mathbf{B}^n$ such that $g = f$ in $\Omega \cap U'$.*

We combine these theorems to obtain the following.

THEOREM 3.4. *If $\Omega \subset \mathbf{R}^n$, $n \geq 2$, then there exists a quasiconformal mapping $f: \Omega \to \mathbf{B}^n$ if and only if Ω^* is homeomorphic to $(\mathbf{B}^n)^*$.*

Proof. The proof of necessity, as outlined above, is given in [Le, p. 490] and [S2, Thm. 5.1]. For sufficiency, let $\phi: \Omega^* \to (\mathbf{B}^n)^*$ be a homeomorphism. We identify $\hat{\Omega}$ with Ω and $\hat{\mathbf{B}}^n$ with $\mathbf{B}^n$. Then Lemma 3.1 implies $\phi(\Omega) = \mathbf{B}^n$; otherwise, ϕ or ϕ^{-1} maps points in Ω or $\mathbf{B}^n$ that have countable neighborhood bases to points with uncountable neighborhood bases. Furthermore, because the restriction of ϕ to Ω is a homeomorphism, $|\phi(x)| \to 1$ as $x \to \partial\Omega$.

Theorem 3.2 implies that there exists a compact set $F \subset \Omega$ for which the restriction of ϕ to each component of $\Omega \setminus F$ is quasiconformal. Because $\phi(F)$ is compact, there is a number r such that $0 < r < 1$ and $\phi(F) \subset B(0, r)$. Let V be the component of $\Omega \setminus F$ containing the connected set $\phi^{-1}(\mathbf{B}^n \setminus B(0, r))$. We claim $\overline{\Omega \setminus V} \subset \Omega$. If not, there exists

$$z \in \partial(\Omega \setminus V) \cap \partial\Omega.$$

But $z \in \partial(\Omega \setminus V)$ means there is a sequence $z_i \to z$ such that $z_i \in \Omega \setminus V$ for each $i \in \mathbf{N}$; and $z \in \partial\Omega$ means $|\phi(z_i)| \to 1$ as $i \to \infty$. So there exists $m \in \mathbf{N}$ such that

$$\phi(z_i) \in \mathbf{B}^n \setminus B(0, r) \subset \phi(V),$$

and hence $z_i \in V$, for each $i \geq m$, a contradiction.

Let $\mathcal{C}(X)$ denote the complement $\bar{\mathbf{R}}^n \setminus X$. If $U = \mathcal{C}(\overline{\Omega \setminus V})$ then the above claim implies $\mathcal{C}(\Omega) \subset U$; since $\partial\Omega \subset \mathcal{C}(\Omega)$, U is a neighborhood of $\partial\Omega$. An application of de Morgan's laws gives

$$U = \mathcal{C}((Q \cap \mathcal{C}(V)) \cup \partial(\Omega \setminus V)) = (\mathcal{C}(\Omega) \cup V) \cap \mathcal{C}(\partial(\Omega \setminus V)).$$

Hence

$$\Omega \cap U = \Omega \cap V \cap \mathcal{C}(\partial(\Omega \backslash V)) = V \backslash \partial(\Omega \backslash V).$$

But because V is open,

$$\partial(\Omega \backslash V) \subset \overline{\Omega \cap \mathcal{C}(V)} \subset \mathcal{C}(V);$$

thus $\Omega \cap U = V$. Because $\phi \mid V$ is quasiconformal, Theorem 3.3 implies there is a neighborhood U' of $\partial\Omega$ and a quasiconformal mapping $f: \Omega \to \mathbf{B}^n$ such that

$$f \mid \Omega \cap U' = \phi \mid \Omega \cap U'. \qquad \square$$

The proof of sufficiency in Theorem 3.4 gives the following, more explicit result.

THEOREM 3.5. *If $\phi: \Omega^* \to (\mathbf{B}^n)^*$ is a homeomorphism, then there is a neighborhood U of $\partial\Omega$ and a quasiconformal mapping f of Ω onto $\mathbf{B}^n$ such that $\phi = f$ in $\Omega \cap U$.*

In a recent conversation, S. Yang reported that he had been able to prove a significant generalization of Gehring's theorem. Let Ω' be a domain *homeomorphic* to the unit ball minus a finite number of interior balls. Yang's generalization replaces $\mathbf{B}^n$ in Theorem 3.3 by Ω': Suppose that Ω is a domain in $\bar{\mathbf{R}}^n$ whose boundary has the same number of components as $\partial\Omega'$, that U is a neighborhood of $\partial\Omega$, and that f is a quasiconformal mapping of $\Omega \cap U$ into Ω' (orientation-preserving in each component of $\Omega \cap U$) such that $f(x) \to \partial\Omega'$ as $x \to \partial\Omega$ in $\Omega \cap U$. Then there is a neighborhood U' of $\partial\Omega$ and a quasiconformal mapping g of Ω onto Ω' such that $g = f$ in $\Omega \cap U'$. This result allows an analogous generalization of Theorem 3.4.

4. An Example of Royden Compactifications with Different Topologies

The theorem of Nakai and Tanaka gives further insight into the topology of Ω^*. For example, it can be used to demonstrate that boundary differences in domains not quasiconformally equivalent are reflected in analogous boundary differences in the Royden compactifications of these domains. For this we require the following theorem, which connects homeomorphic extensions of quasiconformal mappings to the induced homeomorphisms between Royden compactifications.

THEOREM 4.1. *Let $f: \Omega \to \Omega'$ be a quasiconformal mapping and $T = f^*$: $A(\Omega') \to A(\Omega)$ be its corresponding Royden algebra isomorphism. If f has a homeomorphic extension from $\bar{\Omega}$ to $\bar{\Omega}'$, then $T^*(\Phi_z) = \Phi_{f(z)}$.*

Proof. Let $\chi \in \Phi_z$ with corresponding Royden net $\{z_\alpha\}$. Recall that $T^*\hat{x} = \widehat{f(x)}$ for each $x \in \Omega$. Then $T^*(\chi) = \lim_\alpha T^*(\hat{z}_\alpha)$, and so $\{f(z_\alpha)\}$ is a Royden net corresponding to $T^*(\chi)$. Because f has a homeomorphic extension, $f(z_\alpha)$ must converge to $f(z)$ and hence $T^*(\chi)\in\Phi_{f(z)}$. By a similar argument, $T^{*-1}(\chi)\in\Phi_z$ for each $\chi\in\Phi_{f(z)}$. Thus $T^*(\Phi_z)=\Phi_z$. □

It is also necessary to present a definition of transitivity. If Γ is a family of homeomorphisms of a set X onto itself then Γ *acts transitively* on a set $E\subset X$ if for each $a,b\in E$ there exists $g\in\Gamma$ such that $g(a)=b$. For example, the family of 1-quasiconformal self-mappings of $\mathbf{B}^n$ acts transitively on $\mathbf{B}^n$ [GP, Thm. 8.1].

THEOREM 4.2. *The family of self-homeomorphisms of* $(\mathbf{B}^n)^*$ *acts transitively on the set of boundary fibers in* $(\mathbf{B}^n)^*$.

Proof. Let Φ_z and Φ_y be boundary fibers in $\Delta_{\mathbf{B}^n}$. Then there is a rotation $f:\mathbf{R}^n\to\mathbf{R}^n$ for which $f(z)=y$. The mapping $f|\mathbf{B}^n$ is certainly quasiconformal, and so $T=f^*: A(\mathbf{B}^n)\to A(\mathbf{B}^n)$ is a Royden algebra isomorphism. It follows that $T^*:(\mathbf{B}^n)^*\to(\mathbf{B}^n)^*$ is a homeomorphism, so Theorem 4.1 implies $T^*(\Phi_z)=\Phi_y$. □

We now given an example of a domain for which this type of transitivity does not exist because certain properties of the Royden boundary directly reflect properties of the Euclidean boundary. This domain is the *inward-directed spire domain* Σ described by Gehring and Väisälä [GV]. It is a classic example of a domain not quasiconformally equivalent to $\mathbf{B}^n$. Let $h:[0,\infty)\to\mathbf{R}$ be a function satisfying the following conditions:

(1) h is continuous, $h(0)=0$, $h(x)>0$ for $x>0$, and $h(x)=h(1)$ for $x\ge 1$;
(2) h' is continuous and increasing in $(0,1)$; and
(3) $\lim_{x\to 0}h'(x)=0$.

Then

$$\Sigma=\mathbf{R}^3_+\setminus\{x=(r,\theta,x_3): 0\le r\le h(1-x_3),\ 0\le x_3\le 1\},$$

where (r,θ,x_3) are cylindrical coordinates in $\mathbf{R}^3$ and $\mathbf{R}^3_+=\{x=(x_1,x_2,x_3): x_3>0\}$, is an inward-directed spire domain with vertex $z=e_3$.

Näkki [Nä, Cor. 10.6] showed that each quasiconformal self-mapping f of Σ extends homeomorphically to $\partial\Sigma$. In fact, his proof of the existence of an extension to $\partial\Sigma$ requires only that there be a neighborhood U of $\partial\Sigma$ for which f is quasiconformal on $U\cap\Sigma$ and $f(x)\to\partial\Sigma$ as $x\to\partial\Sigma$. He observed that each such extension fixes the vertex z of Σ, and so the family of (extended) quasiconformal self-mappings of Σ does not act transitively on $\partial\Sigma$. Considering the correspondence between points in the Euclidean boundary and fibers in the Royden boundary, it is reasonable to expect that each self-homeomorphism of Σ^* will fiix Φ_z. Combining the theorem of Nakai and Tanaka with Näkki's extension result shows this to be true.

THEOREM 4.3. *If z is the vertex of Σ, then each self-homeomorphism of Σ^* fixes Φ_z.*

Proof. Let $\phi: \Sigma^* \to \Sigma^*$ be a homeomorphism. By Theorem 3.2, there is a compact set $F \subset \Sigma$ such that $\phi: \Sigma \setminus F \to \Sigma$ is quasiconformal. Let V be the component of $\Sigma \setminus F$ bounded by $\partial\Sigma$. Then $\phi(\partial V \setminus \partial\Sigma) \subset \Sigma$, because $\partial V \setminus \partial\Sigma \subset \Sigma$ and $f = \phi \mid V$ is quasiconformal mapping of V into Σ for which $f(x) \to \partial\Sigma$ as $x \to \partial\Sigma$. By Näkki's result, f can be extended homeomorphically to $\partial\Sigma$, and $f(z) = z$. Let $T = f^*: A(f(V)) \to A(V)$ be the induced Royden algebra isomorphism. Then $T^* \mid \hat{V} = \phi \mid \hat{V}$ and for each $\chi \in \Delta_\Sigma$ there is a corresponding Royden net $\{z_\alpha\}$ in V, which allows us to write

$$\phi(\chi) = \phi(\lim_\alpha \hat{z}_\alpha) = \lim_\alpha T^*(\hat{z}_\alpha) = T^*(\lim_\alpha \hat{z}_\alpha) = T^*(\chi).$$

Thus $\phi \mid \Delta_\Sigma = T^* \mid \Delta_\Sigma$, and Theorem 4.1 implies $\phi(\Phi_z) = \Phi_z$. □

That Σ and $\mathbf{B}^n$ are not quasiconformally equivalent means Σ^* and $(\mathbf{B}^n)^*$ are not homeomorphic. This is reflected in the fact that the family of self-homeomorphisms of Σ^* fixes the boundary fiber Φ_z, while the family of self-homeomorphisms of $(\mathbf{B}^n)^*$ is transitive on its set of boundary fibers.

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