An Asymptotic Property of Univalent Mappings

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1. Introduction

Among the many useful consequences of Privalov's theorem on the nontangential asymptotic behavior of holomorphic functions are the criteria that it provides for the vanishing of harmonic functions in the plane. To state a specific result, consider a function h that is harmonic on the upper half plane $\mathbf{U}$, fix $\theta \in (0, \pi/2)$, let E denote a set of real numbers having positive $\mathcal{L}^1$ (Lebesgue) measure, and assume that

$$\lim_{r \downarrow 0} \sup\{|h(u,v)|v^{-1} : (u,v) \in \text{Cone}(x,\theta,r)\} = 0 \qquad (*)$$

whenever $x \in E$. Thus, for each $x \in E$, h tends to zero at the rate $o(y)$ in the vertical cone with vertex $(x,0)$ and aperture 2θ (see Section 2 for all terminology and notation). Then, by using Privalov's theorem [Zy, Thm. 1.9, p. 203], one can prove that h vanishes identically. Theorems of this type extend the basic result that a harmonic function must vanish identically if it tends to zero, along with its normal derivative, upon approach to every point of an analytic arc.

It is interesting to continue this line of investigation and ask whether the criterion (*), which involves control of h in cones, can be lightened so as to only require control of h on vertical segments. A positive result in this direction would have interesting corollaries in several areas: the boundary regularity theory of holomorphic functions; the study of Abel summability of trigonometric series; the study of Cauchy's problem for Laplace's equation (see [La, §2]). The originator of this question, however, is not known to us. We learned of it from Alexandre Weinstein, who indicated that it arose prior to 1930. Also, as far as we know, the only intervening progress has been to establish that h must vanish if it decays at a rate $o(y^{1+\epsilon})$ on vertical segments over a set of positive $\mathcal{L}^1$ measure (see [OS]).

Our present contribution to this problem is directed to the following specific conjecture, which remains open.

Received November 23, 1992. Revision received February 7, 1994.
This research was supported by the Natural Sciences and Engineering Research Council of Canada under grant OGPIN 016.
Michigan Math. J. 41 (1994).

CONJECTURE. Suppose that $h: \mathbf{U} \to \mathbf{R}$, h is a bounded harmonic function, and h is not identically zero. Then

$$\limsup_{y \downarrow 0} |h(x,y)| y^{-1} > 0 \tag{**}$$

for $\mathcal{L}^1$ almost all $x \in \mathbf{R}$.

First we prove that a stronger asymptotic bound than (**) actually holds when the zero level set of h has finite length.

THEOREM 1.1. *Suppose that $h: \mathbf{U} \to \mathbf{R}$, h is harmonic, $\theta \in (0, \pi/2)$, and*

$$\mathcal{H}^1\{z \in \text{Rectangle}(1,1): h(z) = 0\} < \infty,$$

where $\mathcal{H}^1$ denotes linear measure. Then

$$\liminf_{r \downarrow 0} \{|h(u,v)| v^{-1}: (u,v) \in \text{Cone}(x, \theta, r)\} > 0 \tag{1.1}$$

for $\mathcal{L}^1$ almost all $x \in (-1, 1)$.

Subsequently, by using a fundamental theorem of W. K. Hayman and J. M. G. Wu, we show that the imaginary part of a univalent holomorphic function also has this asymptotic property.

THEOREM 1.2. *Suppose that $f: \mathbf{U} \to \mathbf{C}$, f is a univalent holomorphic function, and $\theta \in (0, \pi/2)$. Then*

$$\liminf_{r \downarrow 0} \{|\operatorname{Im} f(u,v)| v^{-1}: (u,v) \in \text{Cone}(x, \theta, r)\} > 0 \tag{1.2}$$

for $\mathcal{L}^1$ almost all $x \in \mathbf{R}$.

These theorems disqualify a variety of constructions that seem to oppose the conjecture. As an example, let S denote the closed segment connecting $(0,0)$ to $(1,0)$, let B be a nowhere dense compact subset of S with linear measure $\mathcal{H}^1(B) > 0$, and let $\mathcal{I}$ be the countable set of connected components of $S \sim B$. Construct a Jordan curve Γ as follows: first connect the endpoints of each $I \in \mathcal{I}$ by a polygonal arc that lies, except for its endpoints, in the lower half plane; then connect $(0,0)$ to $(1,0)$ by a polygonal arc that lies, except for its endpoints, in the upper half plane. Finally, let f map the upper half plane onto Interior(Γ), and set $h \equiv \operatorname{Im} f$. If e is a real number, and $f(e,0)$ is an endpoint of one of the segments in $\mathcal{I}$, and $f(e,0) \notin \{(1,0),(0,0)\}$, then

$$\lim_{y \downarrow 0} h(e,y) y^{-1} = 0. \tag{1}$$

Now the problem is to determine whether h is actually a counterexample to the conjecture, especially in the case that $\mathcal{H}^1(\Gamma) = \infty$. However, by Theorem 1.2, condition (1) must fail at $\mathcal{L}^1$ almost all points $e \in \mathbf{R}$, and h is therefore not a counterexample to the conjecture.

We thank the referee for suggesting that we base the proof of Theorem 1.2 on the Hayman–Wu theorem. Our original proof was more complicated and led to a weaker conclusion. We also wish to acknowledge, with thanks, the many general improvements to our work that derived from the detailed and thoughtful consultation of the editor and the referee.

2. Notation and Terminology

We follow [Fe, esp. pp. 669–671] for all terminology and notation in set theory, measure theory, and general topology. In particular, the *Lebesgue measure* of $E \subset \mathbf{R}$ is denoted by $\mathcal{L}^1(E)$, and the *Hausdorff* 1-*dimensional measure* of $S \subset \mathbf{R}^2$ is denoted by $\mathcal{H}^1(S)$. The usual subsets of the plane are denoted as follows: $\mathbf{U} \equiv \{(x,y) \in \mathbf{R}^2 : y > 0)$ is the *upper half plane;* $\mathbf{D} \equiv \{z \in \mathbf{R}^2 : |z| < 1\}$ is the *unit disk;* $\mathbf{T} \equiv \{z \in \mathbf{R}^2 : |z| = 1\}$ is the *unit circle.*

We say that Γ *is a Jordan curve* if $\Gamma \subset \mathbf{R}^2$ and Γ is homeomorphic to $\mathbf{T}$. If Γ is a Jordan curve, then Interior(Γ) (Exterior(Γ)) denotes the interior region (exterior region) determined by Γ. We say that Γ *has an interior unit normal at b* if the following three conditions hold: (1) Γ is a Jordan curve. (2) $b \in \Gamma$. (3) There exists η having the following three properties: (i) $\eta \in \mathbf{R}^2$ and $|\eta| = 1$; (ii) there exists $\sigma > 0$ such that $b + s\eta \in \text{Interior}(\Gamma)$ and $b - s\eta \in \text{Exterior}(\Gamma)$ whenever $0 < s < \sigma$;

$$\lim_{\substack{z \to b \\ z \in \Gamma}} \operatorname{Re}\left(\eta \cdot \frac{\bar{z} - \bar{b}}{|z - b|}\right) = 0. \tag{iii}$$

When Γ has an interior unit normal at b, there is *exactly one* vector η which meets the three conditions stated in (3); we denote that vector by $n(b, \Gamma)$.

When $x \in \mathbf{R}$, $r \in (0, \infty)$, $s \in (0, \infty)$, and $\theta \in (0, \pi/2)$, we use the following notation to designate certain rectangles and cones in $\mathbf{R}^2$:

$$\text{Rectangle}(r, s) = \{(u, v) \in \mathbf{R}^2 : |u| < r \text{ and } 0 < v < s\},$$

$$\text{Cone}(x, \theta, r) = \{(u, v) \in \mathbf{R}^2 : |u - x| < v \tan\theta \text{ and } 0 < v < r\}.$$

3. Lemmas

Lemma 3.1. *Suppose that* h: Rectangle(1, 1) $\to \mathbf{R}$, *h is harmonic,* $\theta \in (0, \pi/2)$, *and*

$$\mathcal{H}^1\{z \in \text{Rectangle}(1,1) : h(z) = 0\} < \infty.$$

Then S, δ *exist such that the following statements are true:*

(1) $S \subset (-1, 1)$ *and* $\delta : S \to (0, \infty)$.
(2) $\mathcal{L}^1((-1,1) \sim S) = 0$.
(3) $h(z) \neq 0$ *whenever* $z \in \bigcup_{s \in S} \text{Cone}(s, \theta, \delta(s))$.

Proof. Choose $K > \sec\theta$. Choose ϵ_1, ϵ so that $0 < \epsilon_1 < \epsilon/(8K) < \epsilon < 1$. Choose $\rho \in (0, \epsilon_1)$ so that $\mathcal{H}^1\{z \in \text{Rectangle}(1, \rho) : h(z) = 0\} < \epsilon_1$. Let $\mathcal{G}$ denote the collection of all *path components* of the level set $\{z \in \text{Rectangle}(1, \rho) : h(z) = 0\}$. The collection $\mathcal{G}$ is countable. Moreover, the following three statements are true whenever $G \in \mathcal{G}$:

(4) G is a Borel set, G is $\mathcal{H}^1$ measurable, and $0<\mathcal{H}^1(G)<\epsilon_1$.
(5) If $a, b \in \operatorname{Clos} G$, then $|b-a| \leq \mathcal{H}^1(G)$.
(6) $(\operatorname{Clos} G) \cap (\text{Bdry Rectangle}(1,\rho)) \neq \emptyset$.

We define a partition $(\mathcal{G}_1, \mathcal{G}_2, \mathcal{G}_3)$ of $\mathcal{G}$ as follows:

(7) $\mathcal{G}_1$ is the collection of all $G \in \mathcal{G}$ such that $G \cap \text{Rectangle}(1-\epsilon_1, \rho/2) \neq \emptyset$ and $(\operatorname{Clos} G) \cap ([-1,1] \times \{0\}) = \emptyset$.
(8) $\mathcal{G}_2$ is the collection of all $G \in \mathcal{G}$ such that $G \cap \text{Rectangle}(1-\epsilon_1, \rho/2) \neq \emptyset$ and $(\operatorname{Clos} G) \cap ([-1,1] \times \{0\}) \neq \emptyset$.
(9) $\mathcal{G}_3$ is the collection of all $G \in \mathcal{G}$ such that $G \cap \text{Rectangle}(1-\epsilon_1, \rho/2) = \emptyset$.

If $G \in \mathcal{G}_1$, then $\mathcal{H}^1(G) > \min\{\epsilon_1, \rho/2\} = \rho/2$ (by (5) and (6)). Thus, $\mathcal{G}_1$ is finite. Choose $\rho_1 \in (0, \rho/2]$ such that $G \cap \text{Rectangle}(1-\epsilon_1, \rho_1) = \emptyset$ whenever $G \in \mathcal{G}_1$. For each $G \in \mathcal{G}_2$, choose $\alpha(G) \in (\operatorname{Clos} G) \cap ([-1,1] \times \{0\})$. Each $G \in \mathcal{G}_2$ is contained in a closed disk of center $\alpha(G)$ and radius $\mathcal{H}^1(G)$ (by (5)).

Let S denote the set of all $s \in \mathbf{R}$ such that $|s| < 1 - 2K\epsilon_1$ and $|s-\alpha(G)| > 2K\mathcal{H}^1(G)$ whenever $G \in \mathcal{G}_2$. If $G \in \mathcal{G}_2$ and $s \in S$, then $\text{Cone}(s, \theta, \rho_1)$ does not intersect the closed disk about $\alpha(G)$ of radius $\mathcal{H}^1(G)$ (because $K > \sec\theta$). Thus $G \cap \text{Cone}(s,\theta,\rho_1) = \emptyset$ whenever $s \in S$ and $G \in \mathcal{G}_1 \cup \mathcal{G}_2$. Also, if $G \in \mathcal{G}_3$ and $s \in S$, then $G \cap \text{Cone}(s,\theta,\rho_1) = \emptyset$ (because $\rho_1 < \epsilon_1$ and $\text{Cone}(s,\theta,\rho_1) \subset \text{Rectangle}(1-\epsilon_1, \rho_1)$).

Define $\delta(s) = \rho_1$ for each $s \in S$, and observe that $h(z) \neq 0$ whenever $z \in \bigcup_{s \in S} \text{Cone}(s, \theta, \delta(s))$. Moreover, we have

$$\mathcal{L}^1([-1+2K\epsilon_1, 1-2K\epsilon_1] \sim S) \leq \sum_{G \in \mathcal{G}_2} 4K\mathcal{H}^1(G) \leq 4K\epsilon_1,$$

and consequently,

$$\mathcal{L}^1([-1,1] \sim S) \leq 8K\epsilon_1 < \epsilon.$$

As ϵ is arbitrary, the proof is complete. □

LEMMA 3.2. *Suppose that* Γ *is a Jordan curve,* $h\colon \text{Interior}(\Gamma) \to (0,\infty)$, *$h$ is harmonic, and* $\mathcal{H}^1(\Gamma) < \infty$. *Let* Γ^* *denote the set of all* $b \in \Gamma$ *that meet the following two conditions:* (1) Γ *has an interior unit normal* $n(b,\Gamma)$ *at* b;

$$\liminf_{t \downarrow 0} \frac{h(b + tn(b,\Gamma))}{t} > 0. \tag{2}$$

Then $\mathcal{H}^1(\Gamma \sim \Gamma^*) = 0$.

Proof. Using Carathéodory's theorem [Po, p. 18], we construct a bicontinuous bijection $f\colon \operatorname{Clos} \mathbf{D} \to \Gamma \cup \text{Interior}(\Gamma)$ such that $f \mid \mathbf{D}$ is holomorphic. Let $\mathbf{T}^*$ denote the set of all $w \in \mathbf{T}$ such that (here we use the algebraic notation from $\mathbf{C} \equiv \mathbf{R}^2$)

$$\tau(w) \equiv iw \lim_{\substack{z \to w \\ z \in \mathbf{D}}} \frac{f(z) - f(w)}{z - w} \in \mathbf{C} \sim \{0\}. \tag{3}$$

By the Riesz–Privalov theorem [Po, p. 134], we see that $\mathcal{H}^1(\Gamma \sim f[\mathbf{T}^*]) = 0$. Choose $w \in \mathbf{T}^*$ and set $b = f(w)$. To complete the proof, it suffices to prove statements (1) and (2) for the chosen point b.

Choose $\theta \in \mathbf{R}$ such that $e^{i\theta} = w$. Then $\tau(w) = \frac{d}{dt} f(e^{it})|_{t=\theta} \in \mathbf{C} \sim \{0\}$. It follows that Γ has an interior unit normal at b and $n(b, \Gamma) = i\tau(w)/|\tau(w)|$. Thus, statement (1) is valid for b.

Now choose $\delta \in (0, \infty)$ such that $b + sn(b, \Gamma) \in \text{Interior}(\Gamma)$ whenever $s \in (0, \delta]$. Define $\gamma(s) \equiv f^{-1}(b + sn(b, \Gamma))$ whenever $s \in [0, \delta]$. Substituting $\gamma(s)$ for z and $\gamma(0)$ for w in (3) gives

$$\lim_{s \downarrow 0} \frac{s}{1 - \gamma(s)\bar{w}} = \frac{i\tau(w)}{n(b, \Gamma)} = |\tau(w)|,$$

and it follows that

$$\lim_{s \downarrow 0} \frac{s}{1 - |\gamma(s)|} = |\tau(w)|.$$

Harnack's inequality for the unit disk implies that

$$0 < \frac{1 - |\gamma(s)|}{1 + |\gamma(s)|} h \circ f(0) \leq h \circ f(\gamma(s))$$

whenever $s \in (0, \delta]$, and we conclude

$$0 < \liminf_{s \downarrow 0} \frac{h \circ f(\gamma(s))}{1 - |\gamma(s)|} = \liminf_{s \downarrow 0} \frac{h(b + sn(b, \Gamma))}{1 - |\gamma(s)|}$$

$$= |\tau(w)| \liminf_{s \downarrow 0} \frac{h(b + sn(b, \Gamma))}{s}.$$

Thus, statement (2) is valid for b. □

LEMMA 3.3. *Suppose that* $x \in \mathbf{R}$, $0 < \theta < \phi < \pi/2$, *and* $0 < \delta < \infty$, *and define*

$$K \equiv K(\theta, \phi) \equiv [\cos(\theta - \phi) \sin \phi - \sin \theta][\cos(\theta - \phi) \sin \phi + \sin \theta]^{-1}.$$

Also, suppose that $h \colon \text{Cone}(x, \phi, \delta) \to (0, \infty)$, *h is harmonic, and*

$$\liminf_{y \downarrow 0} h(x, y) y^{-1} \equiv \alpha > 0.$$

Then

$$\liminf_{r \downarrow 0} \{h(u, v) v^{-1} \colon (u, v) \in \text{Cone}(x, \theta, r)\} \geq K(\theta, \phi)\alpha.$$

Proof. Choose $\bar{\alpha} \in (0, \alpha)$ and, as permitted by the hypothesis, choose r so that $0 < 2r \sec^2 \phi < \delta$ and $h(x, s) > \bar{\alpha} s$ whenever $0 < s < r \sec^2 \phi$. Fix $(u, v) \in \text{Cone}(x, \theta, r)$, set $b \equiv |u - x| \tan \phi + v$, $R \equiv b \sin \phi$, $d \equiv \sqrt{(u-x)^2 + (v-b)^2}$, and let Δ denote the open disk centered at (x, b) of radius R. Then $(u, v) \in \Delta \subset \text{Cone}(x, \phi, \delta)$ and $d < b \sin \theta (\cos(\theta - \phi))^{-1} < R$. Also, note that $b \leq r \sec^2 \phi$. Use the inequalities stated above and Harnack's inequality for the restriction of h to the disk Δ to prove that

$$h(u, v) \geq \frac{R - d}{R + d} h(x, b) > K\bar{\alpha} b > K\bar{\alpha} v.$$

Thus

$$\inf\{h(u,v)v^{-1}: (u,v)\in \text{Cone}(x,\theta,r)\} > K\bar{\alpha},$$

and the proof is complete. □

4. Proof of Theorem 1.1

Choose $\epsilon > 0$ and $\phi \in (\theta, \pi/2)$. Using Lemma 3.1 and the countable subadditivity of $\mathcal{L}^1$, we construct S^+, S^-, δ so that the following statements are true:

(1) $\delta \in (0,1)$, $S^+ \cup S^- \subset (-1,1)$, and $S^+ \cap S^- = \emptyset$.
(2) $\mathcal{L}^1(S^+ \cup S^-) > 2 - \epsilon$.
(3) $h(z) > 0$ whenever $z \in \bigcup_{s\in S^+} \text{Cone}(s,\phi,\delta)$.
(4) $h(z) < 0$ whenever $z \in \bigcup_{s\in S^-} \text{Cone}(s,\phi,\delta)$.

Choose a closed interval I such that $I \subset (-1,1)$ and $0 < \text{diam}\, I < \delta \tan\phi$; define $E^+ \equiv S^+ \cap I$ and $E^- \equiv S^- \cap I$. To complete the proof of Theorem 1.1, it suffices to prove that inequality (1.1) holds for $\mathcal{L}^1$ almost all $x \in E^+$.

We assume that $\mathcal{L}^1(E^+) > 0$ and define $\Omega \equiv \bigcup_{x\in E^+} \text{Cone}(x,\phi,\delta)$ and $\Gamma \equiv \text{Bdry}\,\Omega$. Because of (1)–(4), the following statements (5)–(9) are also true.

(5) Γ is a Jordan curve (because $\text{diam}\, I < \delta\tan\phi$) and $\Omega = \text{Interior}(\Gamma)$.
(6) $\mathcal{H}^1(\Gamma) < \infty$.
(7) $h(z) > 0$ whenever $z \in \Omega$.
(8) $E^+ \times \{0\} \subset \Gamma \cap (\mathbf{R}\times\{0\})$.

Using Lemma 3.2 and (5)–(8), we deduce that

$$n(b,\Gamma) = i \quad \text{and} \quad \liminf_{t\downarrow 0} h(b+ti)t^{-1} > 0$$

for $\mathcal{H}^1$ almost all $b \in \Gamma \cap (\mathbf{R}\times\{0\})$. Therefore, the following statement is true for $\mathcal{L}^1$ almost all $x \in E^+$:

$$\liminf_{y\downarrow 0} h(x,y)y^{-1} \equiv \alpha(x) > 0. \tag{9}$$

If $x \in E^+$ and (9) is true, then Lemma 3.3 implies that

$$\liminf_{r\downarrow 0}\{h(u,v)v^{-1}: (u,v)\in \text{Cone}(x,\theta,r)\} \ge K(\theta,\phi)\alpha(x) > 0.$$

Thus, inequality (1.1) holds for $\mathcal{L}^1$ almost all $x \in E^+$, and the proof is complete. □

5. Proof of Theorem 1.2

Define $W(z) = -i(z-1)(z+1)^{-1}$ whenever $z \in \mathbf{D}$, and set

$$A \equiv \{z \in \mathbf{D}: \text{Im}\, f(W(z)) = 0,\ \text{Re}\, z > 0\}.$$

As $\{w \in \mathbf{U}: |w| < 1\} = \{W(z): z \in \mathbf{D},\ \text{Re}\, z > 0\}$, it follows that

$$\{w \in \mathbf{U}: |w| < 1,\ \text{Im}\, f(w) = 0\} = W[A].$$

As $|W'(z)| \le 2$ whenever $z \in \mathbf{D}$ and $\text{Re}\, z > 0$, it follows that

$$\mathcal{H}^1(W[A]) \leq 2\mathcal{H}^1(A).$$

The theorem of W. K. Hayman and J. M. G. Wu states that $\mathcal{H}^1(A) < \infty$ ([HW]; also see [Øy] for recent developments). Therefore

$$\mathcal{H}^1(\{w \in \mathbf{U} : |w| < 1, \operatorname{Im} f(w) = 0\}) < \infty.$$

Now it follows from Theorem 1.1 that inequality (1.2) holds for $\mathcal{L}^1$ almost all $x \in (-1/2, 1/2)$, and the proof of Theorem 1.2 is complete. □

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