Bourgain Algebras of Spaces of Harmonic Functions

KEIJI IZUCHI, KAREL STROETHOFF, & KEITH YALE

1. Introduction

Bourgain algebras were introduced by Cima and Timoney [5] in connection with Bourgain's work on the Dunford–Pettis property for certain concrete function algebras [1]. Subsequently, several authors have studied Bourgain algebras [2; 3; 4; 7; 8; 9; 12; 14; 15] with a variety of goals in mind but with attention to the Bourgain algebra determined by an *algebra*. We are concerned with the study of Bourgain algebras of a class, including the space of bounded harmonic functions on the disk, of *linear subspaces*.

Let $\mathbf{D}$ be the open unit disk in the complex plane $\mathbf{C}$ and let $\mathbf{T}=\partial\mathbf{D}$ be the unit circle. The usual spaces of essentially bounded functions with respect to Lebesgue measure are denoted by $L^\infty(\mathbf{T})$ and $L^\infty(\mathbf{D})$. The space of bounded analytic functions on $\mathbf{D}$ is denoted by $H^\infty(\mathbf{D})$, with $H^\infty=H^\infty(\mathbf{T})$ being used to denote the boundary values of $H^\infty(\mathbf{D})$ functions. We will also write $L^\infty=L^\infty(\mathbf{T})$ for brevity. The algebra of bounded continuous functions on $\mathbf{D}$ is denoted by $BC(\mathbf{D})$ and $C=C(\mathbf{T})$ denotes the algebra of continuous functions on $\mathbf{T}$. Each of these algebras is equipped with the (essential) supremum norm $\|\ \|_\infty$.

Let $\mathfrak{X}$ be one of the spaces L^∞, $L^\infty(\mathbf{D})$ or $BC(\mathbf{D})$, and let $\mathcal{Y}\subset\mathfrak{X}$ be a closed linear subspace. We say that $f\in\mathfrak{X}$ belongs to the *Bourgain algebra of* $\mathcal{Y}$ *relative to* $\mathfrak{X}$, and write $f\in\mathcal{Y}_b$, in case for every weakly null sequence $\{f_n\}$ in $\mathcal{Y}$ there exists a sequence $\{g_n\}$ in $\mathcal{Y}$ such that $\|f_nf-g_n\|_\infty\to 0$ as $n\to\infty$. Essentially as shown in [5], $\mathcal{Y}_b$ contains the constants and is a closed subalgebra of $\mathfrak{X}$. Moreover, if $\mathcal{Y}$ is a *subalgebra*, then $\mathcal{Y}\subset\mathcal{Y}_b$. However, there is no known simple relationship between a *subspace* $\mathcal{Y}$ and its Bourgain algebra $\mathcal{Y}_b$. We emphasize that $\mathcal{Y}_b$ is defined relative to a particular overlying space $\mathfrak{X}$ even though this is not reflected in the notation $\mathcal{Y}_b$. Each of the spaces $\mathfrak{X}=L^\infty$, $L^\infty(\mathbf{D})$, $BC(\mathbf{D})$, or $C(\mathfrak{M})$ (when $\mathcal{Y}$ is an algebra with maximal ideal space $\mathfrak{M}$) has a certain claim to naturality; however, the general dependence of $\mathcal{Y}_b$ upon $\mathfrak{X}$ is quite complicated and not fully understood. For $\mathfrak{X}=L^\infty(\mathbf{D})$ and $\mathcal{Y}$ a subalgebra containing the bounded analytic functions,

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the Bourgain algebra $\mathcal{Y}_b$ (relative to $L^\infty(\mathbf{D})$) has been studied; the choice of $L^\infty(\mathbf{D})$ for the ambient space is natural in regard to boundary-value questions and connections with operator theory [3].

For a function $f \in L^\infty$ we write $\hat{f}$ to denote the Poisson extension of f over the unit disk $\mathbf{D}$; that is,

$$\hat{f}(z) = \int_0^{2\pi} f(e^{i\theta}) P_z(\theta) \frac{d\theta}{2\pi}$$

for $z \in \mathbf{D}$, where P_z is the Poisson kernel at z: $P_z(\theta) = (1-|z|^2)/|e^{i\theta}-z|^2$. For any nonempty subset B of L^∞ we write

$$\hat{B} = \{\hat{f} : f \in B\}.$$

If B is an algebra then the set $\hat{B}$ need not be a subalgebra of $L^\infty(\mathbf{D})$. For example, $h^\infty = (L^\infty)\hat{}$, the space of bounded harmonic functions on $\mathbf{D}$, is not a subalgebra of $L^\infty(\mathbf{D})$. However, for any closed linear subspace B of L^∞, the set $\hat{B}$ is a closed linear subspace of $L^\infty(\mathbf{D})$. We will be concerned with finding the Bourgain algebras of the spaces $\hat{B}$ relative to $L^\infty(\mathbf{D})$ and to $BC(\mathbf{D})$.

Our main result, Theorem 1, is discussed in Section 2 and proved in Section 4. The proof requires certain preliminaries which are gathered in Section 3. In Section 5 we determine the second Bourgain algebra in certain settings.

2. Statement of the Main Result

To state our main result we need to review some notation. Let V be the ideal of vanishing functions in $L^\infty(\mathbf{D})$ as defined in [3]; that is,

$$V = \{f \in L^\infty(\mathbf{D}) : \|f\chi_{\mathbf{D}\setminus r\mathbf{D}}\|_\infty \to 0 \text{ as } r \to 1^-\}.$$

The algebra of quasi-continuous functions, QC, is defined by

$$QC = (H^\infty + C) \cap (\overline{H^\infty + C}).$$

The following theorem is the main result of this paper.

THEOREM 1. *If B is a closed linear subspace of L^∞ containing C, and B_b is the Bourgain algebra of B relative to L^∞, then the Bourgain algebra of $\hat{B}$ relative to $L^\infty(\mathbf{D})$ is*

$$(\hat{B})_b = (B_b \cap QC)\hat{} + V.$$

We defer the proof of this theorem to Section 4, after we discuss various corollaries in this section and the preliminaries needed for its proof in the next section.

COROLLARY 2. *If B is a closed subalgebra of L^∞ containing QC, then the Bourgain algebra of $\hat{B}$ relative to $L^\infty(\mathbf{D})$ is*

$$(\hat{B})_b = (QC)\hat{} + V.$$

Proof. Clearly $QC \subset B \subset B_b$, so that $B_b \cap QC = QC$. □

In particular, we have the following.

COROLLARY 3. *The Bourgain algebra of h^∞ relative to $L^\infty(\mathbf{D})$ is*

$$(h^\infty)_b = (QC)\hat{} + V.$$

The main result in [3] states that $((H^\infty)\hat{})_b = (H^\infty + C)\hat{} + V$. Note that this algebra is larger than $(\hat{B})_b$ if B is a closed subalgebra of L^∞ containing QC, even though $(H^\infty)\hat{}$ is smaller than $\hat{B}$.

If we write $C_0(\mathbf{D})$ for the space of continuous functions f on $\mathbf{D}$ for which $\lim_{|z|\to 1^-} f(z) = 0$, then $V \cap BC(\mathbf{D}) = C_0(\mathbf{D})$, which can be used to determine the Bourgain algebra relative to $BC(\mathbf{D})$ given the Bourgain algebra relative to $L^\infty(\mathbf{D})$. The next corollary provides an example.

COROLLARY 4. *If B is a closed subalgebra of L^∞ containing QC, then the Bourgain algebra of $\hat{B}$ relative to $BC(\mathbf{D})$ is*

$$(\hat{B})_b = (QC)\hat{} + C_0(\mathbf{D}).$$

Proof. Clearly,

$$(\hat{B})_b = (B_b \cap QC)\hat{} + V \cap BC(\mathbf{D}) = (QC)\hat{} + C_0(\mathbf{D}). \qquad \square$$

In view of the simplicity of this device, we will not continue giving separate statements or proofs for Bourgain algebras relative to $BC(\mathbf{D})$.

COROLLARY 5. *The Bourgain algebra of $(QC)\hat{}$ relative to $L^\infty(\mathbf{D})$ is*

$$((QC)\hat{})_b = (QC)\hat{} + V.$$

Proof. Clearly $QC \subset QC_b$, so that $QC_b \cap QC = QC$. $\square$

To formulate one more corollary, we need to introduce the notion of *essential oscillation* $\omega(f, \zeta)$ of a function $f \in L^\infty(\mathbf{D})$ at the point $\zeta \in \mathbf{T}$. Let $t > 0$ and put

$$\omega(f, \zeta, t) = \operatorname{ess\,sup}\{|f(z) - f(w)| : z, w \in \mathbf{D}, |z - \zeta| < t, \text{ and } |w - \zeta| < t\}.$$

For $f \in L^\infty$, $\zeta \in \mathbf{T}$, and $t > 0$ we put

$$\omega(f, \zeta, t) = \operatorname{ess\,sup}\{|f(\xi) - f(\eta)| : \xi, \eta \in \mathbf{T}, |\xi - \zeta| < t, \text{ and } |\eta - \zeta| < t\}.$$

In either case, we define

$$\omega(f, \zeta) = \lim_{t \to 0^+} \omega(f, \zeta, t).$$

In [12] it is proved that the Bourgain algebra of the disk algebra $A = H^\infty \cap C$ with respect to L^∞ is given by

$$A_b = (H^\infty \cap W) + C,$$

where $W = \{f \in L^\infty : \forall \epsilon > 0 \text{ the set } \{\zeta \in \mathbf{T} : \omega(f, \zeta) \geq \epsilon\} \text{ is finite}\}$. In [4] it is shown that, relative to $L^\infty(\mathbf{D})$, the Bourgain algebra of the disk algebra $A(\mathbf{D}) = H^\infty(\mathbf{D}) \cap C(\bar{\mathbf{D}})$ is given by

$$A(\mathbf{D})_b = (H^\infty(\mathbf{D}) \cap W(\mathbf{D})) + C(\bar{\mathbf{D}}) + V,$$

where $W(\mathbf{D}) = \{f \in L^\infty(\mathbf{D}) : \forall \epsilon > 0$ the set $\{\zeta \in \mathbf{T} : \omega(f, \zeta) \geq \epsilon\}$ is finite$\}$. The following corollary should be compared with these results. Note that $\hat{C}$ is the space of bounded harmonic functions on $\mathbf{D}$ which extend continuously to $\bar{\mathbf{D}}$.

COROLLARY 6. *The Bourgain algebra of $\hat{C}$ relative to $L^\infty(\mathbf{D})$ is*

$$(\hat{C})_b = ((H^\infty(\mathbf{D}) \cap W(\mathbf{D})) + C(\bar{\mathbf{D}})) \cap \overline{((H^\infty(\mathbf{D}) \cap W(\mathbf{D})) + C(\bar{\mathbf{D}}))} + V.$$

Proof. Clearly $C_b = W$ (see [12]), so that

$$C_b \cap QC = ((H^\infty \cap W) + C) \cap \overline{((H^\infty \cap W) + C)}.$$

Hence $(C_b \cap QC)\hat{} = ((H^\infty \cap W) + C)\hat{} \cap \overline{((H^\infty \cap W) + C)}\hat{}$. It follows from the argument in [4] that $((H^\infty \cap W) + C)\hat{} + V = (H^\infty(\mathbf{D}) \cap W(\mathbf{D})) + C(\bar{\mathbf{D}}) + V$, and the result follows. □

3. Preliminaries

The space QC of quasi-continuous functions has an important role in our work. The following characterization of QC will be useful.

PROPOSITION 7 (Sarason). *Let $f \in L^\infty$. Then*

$$f \in QC \Leftrightarrow \int_0^{2\pi} |f(e^{i\theta}) - \hat{f}(z)| P_z(\theta) \frac{d\theta}{2\pi} \to 0 \text{ as } |z| \to 1^-.$$

Note that VMO is the space of functions $f \in L^1(\mathbf{T})$ satisfying the vanishing condition in the above proposition; a restatement of the above result is thus:

$$QC = VMO \cap L^\infty.$$

As in [3], we use BV to denote the subalgebra of $L^\infty(\mathbf{D})$ consisting of those (equivalence classes) of essentially bounded measurable functions on $\mathbf{D}$ which have nontangential limits at almost all points of the circle $\mathbf{T}$. To be more precise we recall the following notation. For $\zeta \in \mathbf{T}$ and $0 < R < 1$, let $\Gamma_R(\zeta)$ be the interior of the convex hull of z and the set $\{z \in \mathbf{C} : |z| = R\}$. We say that $f \in L^\infty(\mathbf{D})$ has *essentially nontangential limit* L at $\zeta \in \mathbf{T}$ if

$$\operatorname{ess\,sup}\{|f(z) - L| : z \in \Gamma_R(\zeta), |z| > 1 - \delta\} \to 0 \text{ as } \delta \to 0^+$$

for every $0 < R < 1$, in which case we will write $f^*(\zeta)$ for L. We define BV to be the set of $f \in L^\infty(\mathbf{D})$ for which an essential nontangential limit $f^*(\zeta)$ exists for almost every $\zeta \in \mathbf{T}$. If we define the *essential nontangential oscillation* of $f \in L^\infty(\mathbf{D})$ at $\zeta \in \mathbf{T}$ over $\Gamma_R(\zeta)$ to be

$$\omega_R(f, \zeta) = \lim_{\delta \to 0^+} \operatorname{ess\,sup}\{|f(z) - f(w)| : z, w \in \Gamma_R(\zeta), |z - \zeta| < \delta, |w - \zeta| < \delta\},$$

then the space BV is characterized by

$$BV = \{f \in L^\infty(\mathbf{D}) : \omega_R(f, \zeta) = 0 \text{ for all } 0 < R < 1 \text{ for almost every } \zeta \in \mathbf{T}\}.$$

We write $A(\mathbf{D})$ to denote the disk algebra $H^\infty(\mathbf{D}) \cap C(\bar{\mathbf{D}})$. The following result is proved in [3].

THEOREM 8. *Let $\mathcal{Y}$ be a closed linear subspace of $L^\infty(\mathbf{D})$ such that $A(\mathbf{D}) \subset \mathcal{Y} \subset BV$. Then $\mathcal{Y}_b \subset BV$.*

COROLLARY 9. *Let B be a closed linear subspace of L^∞ containing C, and let $(\hat{B})_b$ denote the Bourgain algebra of $\hat{B}$ relative to $L^\infty(\mathbf{D})$. If $f \in (\hat{B})_b$, then f^* exists at almost every point of $\mathbf{T}$ and $f^* \in B_b$, the Bourgain algebra of B relative to L^∞.*

Proof. Clearly $A(\mathbf{D}) = \hat{A} \subset \hat{B}$. Let $f \in (\hat{B})_b$. By Theorem 8, the nontangential limit f^* exists at almost every point of $\mathbf{T}$. If $\{f_n\}$ is a weakly null sequence in B, then $\{\hat{f}_n\}$ is weakly null in $\hat{B}$. Since $f \in (\hat{B})_b$, there exist g_n in B such that $\|f\hat{f}_n - \hat{g}_n\|_\infty \to 0$. Taking nontangential limits, we conclude that $\|f^* f_n - g_n\|_\infty \to 0$; thus $f^* \in B_b$. □

REMARK 10. The above proof shows that we also have: *If $f \in (\hat{B} + V)_b$, then f^* exists at almost every point of $\mathbf{T}$ and $f^* \in B_b$, the Bourgain algebra of B relative to L^∞.*

LEMMA 11. *Let B be a closed linear subspace of L^∞ containing C, and let $(\hat{B})_b$ denote the Bourgain algebra of $\hat{B}$ relative to $L^\infty(\mathbf{D})$. If $f \in (\hat{B})_b$ and $\{f_n\}$ is a weakly null sequence in B, then*

$$\|f\hat{f}_n - (f^* f_n)\hat{}\|_\infty \to 0 \text{ as } n \to \infty.$$

Proof. Since $f \in (\hat{B})_b$, there exist $g_n \in B$ such that $\epsilon_n = \|f\hat{f}_n - \hat{g}_n\|_\infty \to 0$ as $n \to \infty$. By Corollary 9, the nontangential limit f^* exists almost everywhere on the circle $\mathbf{T}$. It is easily seen that $\|f^* f_n - g_n\|_\infty \le \epsilon_n$; thus $\|(f^* f_n)\hat{} - \hat{g}_n\|_\infty \to 0$, and hence $\|f\hat{f}_n - (f^* f_n)\hat{}\|_\infty \to 0$ as $n \to \infty$. □

LEMMA 12. *Let $\{z_n\}$ be a sequence in $\mathbf{D}$ such that $z_n \to \zeta \in \mathbf{T}$. Then there exists a sequence $\{g_n\}$ in C such that:*

(i) *$\|g_n\|_\infty = 1$ for all $n \ge 1$;*

(ii) *$g_n \to 0$ weakly in C; and*

(iii) *$\int_0^{2\pi} |1 - g_n(e^{i\theta})| P_{z_n}(\theta)\, d\theta/2\pi \to 0$ as $n \to \infty$.*

Proof. For each open subset J of $\mathbf{T}$ with $\zeta \in J$, we have

$$\int_{\mathbf{T}\setminus J} P_{z_n}(\theta) \frac{d\theta}{2\pi} \to 0 \text{ as } n \to \infty.$$

Indeed, there is a sequence $\{J_n\}$ of open subarcs of $\mathbf{T}$ such that $\zeta \in J_n$, the length of J_n goes to zero, and

$$\int_{J_n} P_{z_n}(\theta) \frac{d\theta}{2\pi} \to 1 \text{ as } n \to \infty,$$

since P_z is an approximate identity. It is not difficult to find a sequence $\{g_n\}$ in C such that

(1) $g_n \geq 0$ and $\|g_n\|_\infty = 1$ for all $n \geq 1$;
(2) $\operatorname{supp}(g_n) \subset J_n \setminus \{\zeta\}$ for all $n \geq 1$; and
(3) $\int_0^{2\pi} g_n(e^{i\theta}) P_{z_n}(\theta)\, d\theta/2\pi \to 1$ as $n \to \infty$.

Since the sequence $\{g_n\}$ is norm bounded and converges pointwise to zero on $\mathbf{T}$, we have property (ii). Property (iii) follows from (1) and (3). □

4. Proof of Theorem 1

The proof will be divided into several steps.

Step 1: We first prove $(B_b \cap QC)\hat{} + V \subset (\hat{B})_b$. The inclusion $V \subset (\hat{B})_b$ is trivial, because if $\hat{f}_n \to 0$ weakly in $\hat{B}$ then $\hat{f}_n \to 0$ uniformly on compact subsets of $\mathbf{D}$.

To prove the inclusion $(B_b \cap QC)\hat{} \subset (\hat{B})_b$, take $f \in B_b \cap QC$ and let $f_n \in B$ be such that $\hat{f}_n \to 0$ weakly in $\hat{B}$, that is, $f_n \to 0$ weakly in B. Then there is a positive constant K such that $\|f_n\|_\infty \leq K$ for all $n \geq 1$. Then for each $z \in \mathbf{D}$ we have

$$\begin{aligned} |\hat{f}(z)\hat{f}_n(z) - (ff_n)\hat{}(z)| &\leq \int_0^{2\pi} |\hat{f}(z) - f(e^{i\theta})||f_n(e^{i\theta})| P_z(\theta)\, \frac{d\theta}{2\pi} \\ &\leq K \int_0^{2\pi} |\hat{f}(z) - f(e^{i\theta})| P_z(\theta)\, \frac{d\theta}{2\pi}. \end{aligned}$$

Since $f \in QC$, we conclude that

$$\sup_{r<|z|<1} \sup_{n \geq 1} |\hat{f}(z)\hat{f}_n(z) - (ff_n)\hat{}(z)| \to 0 \text{ as } r \to 1^-.$$

Since clearly $\hat{f}\hat{f}_n \to 0$ and $(ff_n)\hat{} \to 0$ uniformly on each compact subset of $\mathbf{D}$, we see that $\|\hat{f}\hat{f}_n - (ff_n)\hat{}\|_\infty \to 0$ as $n \to \infty$. Because $f \in B_b$ and $f_n \to 0$ weakly in B, there exist $g_n \in B$ such that $\|ff_n - g_n\|_\infty \to 0$ as $n \to \infty$. Then also $\|(ff_n)\hat{} - \hat{g}_n\|_\infty \to 0$, and therefore $\|\hat{f}\hat{f}_n - \hat{g}_n\|_\infty \to 0$ as $n \to \infty$. This implies that $\hat{f} \in (\hat{B})_b$.

Step 2: We show that if $g \in L^\infty$ and $\hat{g} \in (\hat{B})_b$, then $g \in B_b \cap QC$. Assuming that $g \notin QC$, by Proposition 7 there is a sequence $\{z_n\}$ in $\mathbf{D}$ and $\delta > 0$ such that $|z_n| \to 1$ as $n \to \infty$ and

$$\int_0^{2\pi} |g(e^{i\theta}) - \hat{g}(z_n)| P_{z_n}(\theta)\, \frac{d\theta}{2\pi} > \delta$$

for every $n \geq 1$. By passing to a subsequence of $\{z_n\}$ we may assume that $\hat{g}(z_n) \to c$ for some constant c. Note that also $(g-c)\hat{} = \hat{g} - c \in (\hat{B})_b$, so by replacing g by the function $g - c$ we may assume that $c = 0$. Thus:

(4) $\hat{g}(z_n) \to 0$ as $n \to \infty$; and
(5) $\limsup_{n\to\infty} \int_0^{2\pi} |g(e^{i\theta})| P_{z_n}(\theta)\, d\theta/2\pi \geq \delta$.

By considering a subsequence, we may assume that $z_n \to \zeta$ for some $\zeta \in \mathbf{T}$. By Lemma 12 there is a sequence $\{g_n\}$ in C such that

(6) $\|g_n\|_\infty = 1$ and $g_n \to 0$ weakly in C; and

(7) $\int_0^{2\pi} |1 - g_n(e^{i\theta})| P_{z_n}(\theta)\, d\theta/2\pi \to 0$ as $n \to \infty$.

Let G be a function in L^∞ such that $Gg = |g|$ and $|G| = 1$ almost everywhere on $\mathbf{T}$. For each integer $n \geq 1$ there exists an h_n in C such that $\|h_n\|_\infty \leq 1$ and

(8) $\int_0^{2\pi} |G(e^{i\theta}) - h_n(e^{i\theta})| P_{z_n}(\theta)\, d\theta/2\pi \to 0$ as $n \to \infty$.

We put $f_n = g_n h_n$. Then $f_n \in C$, and

(9) $|(|g|g_n)\hat{}\,(z_n) - (gf_n)\hat{}\,(z_n)| \to 0$ as $n \to \infty$

because

$$\begin{aligned} |(|g|g_n)\hat{}\,(z_n) - (gf_n)\hat{}\,(z_n)| &= \left| \int_0^{2\pi} (|g(e^{i\theta})| - g(e^{i\theta}) h_n(e^{i\theta})) g_n(e^{i\theta}) P_{z_n}(\theta) \frac{d\theta}{2\pi} \right| \\ &\leq \|g\|_\infty \int_0^{2\pi} |G(e^{i\theta}) - h_n(e^{i\theta})| P_{z_n}(\theta) \frac{d\theta}{2\pi} \quad \text{(by (6))} \\ &\to 0 \text{ as } n \to \infty \quad \text{(by (8))}. \end{aligned}$$

Since $\|h_n\|_\infty \leq 1$, by (6) we have $f_n \to 0$ weakly in C. Hence $\hat{f}_n \to 0$ weakly in $\hat{B}$. Since $\hat{g} \in (\hat{B})_b$, by Lemma 12 we have

(10) $\|\hat{g}\hat{f}_n - (gf_n)\hat{}\,\|_\infty \to 0$ as $n \to \infty$.

Note that since $\hat{g}\hat{f}_n - (gf_n)\hat{}$ is a continuous function on $\mathbf{D}$, the *essential* supremum norm $\|\hat{g}\hat{f}_n - (gf_n)\hat{}\,\|_\infty$ coincides with the supremum norm of $\hat{g}\hat{f}_n - (gf_n)\hat{}$, so that

$$\begin{aligned} &\limsup_{n\to\infty} \|\hat{g}\hat{f}_n - (gf_n)\hat{}\,\|_\infty \\ &\quad \geq \limsup_{n\to\infty} |(gf_n)\hat{}\,(z_n) - \hat{g}(z_n)\hat{f}_n(z_n)| \\ &\quad = \limsup_{n\to\infty} |(gf_n)\hat{}\,(z_n)| \quad \text{(by (4))} \\ &\quad = \limsup_{n\to\infty} |(|g|g_n)\hat{}\,(z_n)| \quad \text{(by (9))} \\ &\quad = \limsup_{n\to\infty} \int_0^{2\pi} \{|g(e^{i\theta})| - (1 - g_n(e^{i\theta}))|g(e^{i\theta})|\} P_{z_n}(\theta) \frac{d\theta}{2\pi} \\ &\quad \geq \delta \quad \text{(by (5) and (7))}, \end{aligned}$$

contradicting (10).

Step 3: We show that $(\hat{B})_b \subset (B_b \cap QC)\hat{}+ V$, which will complete the proof. Let $f \in (\hat{B})_b$ and put $g = f^*$. If we can show that $f - \hat{g} \in V$, then it will follow that $f - \hat{g} \in (\hat{B})_b$, since $V \subset (\hat{B})_b$. Hence $\hat{g} \in (\hat{B})_b$, so that by Step 2, $g \in B_b \cap QC$, and it follows that $f = \hat{g} + (f - \hat{g}) \in (B_b \cap QC)\hat{}+ V$.

To show that $f - \hat{g} \in V$, assume the contrary. Then there is a sequence $\{r_n\}$ of numbers in $(0,1)$, a sequence $\{A_n\}$ of sets of positive area measure, and a $\delta > 0$ such that:

$r_n \to 1$ as $n \to \infty$;
$A_n \subset \{z \in \mathbf{C}: r_n < |z| < 1\}$;
$|f(z) - \hat{g}(z)| \geq \delta$, for $z \in \bigcup_{n \geq 1} A_n$; and
$|f(z)| \leq \|f\|_\infty$, for $z \in \bigcup_{n \geq 1} A_n$.

Let z_n be a point of density of the set A_n. Then clearly $|z_n| \to 1$ as $n \to \infty$. By passing to a subsequence (if necessary) we may further assume that $z_n \to \zeta \in \mathbf{T}$ as $n \to \infty$. Let $\{g_n\}$ be as in Lemma 12. Note that (iii) of Lemma 12 implies that $\hat{g}_n(z_n) \to 1$ as $n \to \infty$ and $|\hat{g}(z_n)\hat{g}_n(z_n) - (gg_n)\hat{\ }(z_n)| \to 0$ as $n \to \infty$. For each positive integer n, choose $0 < \delta_n < 1 - |z_n|$ such that for $|z - z_n| < \delta_n$:

$|\hat{g}(z_n) - \hat{g}(z)| < \delta/4$;
$|\hat{g}_n(z_n) - \hat{g}_n(z)| < \delta/(4\|f\|_\infty)$; and
$|(gg_n)\hat{\ }(z_n) - (gg_n)\hat{\ }(z)| < \delta/4$.

Because z_n is a point of density of A_n, the set $B_n = A_n \cap \{z \in \mathbf{C}: |z - z_n| < \delta_n\}$ has positive measure. If $z \in B_n$, then

$$\begin{aligned}
&\delta|\hat{g}_n(z_n)| \\
&\quad \leq |f(z) - \hat{g}(z)||\hat{g}_n(z_n)| \\
&\quad \leq |f(z)||\hat{g}_n(z_n) - \hat{g}_n(z)| + |f(z)\hat{g}_n(z) - (gg_n)\hat{\ }(z)| + |(gg_n)\hat{\ }(z) - (gg_n)\hat{\ }(z_n)| \\
&\qquad + |(gg_n)\hat{\ }(z_n) - \hat{g}(z_n)\hat{g}_n(z_n)| + |\hat{g}_n(z_n)||\hat{g}(z_n) - \hat{g}(z)| \\
&\quad \leq \|f\|_\infty \delta/(4\|f\|_\infty) + |f(z)\hat{g}_n(z) - (gg_n)\hat{\ }(z)| + \delta/4 \\
&\qquad + |(gg_n)\hat{\ }(z_n) - \hat{g}(z_n)\hat{g}_n(z_n)| + \delta/4.
\end{aligned}$$

Thus

$$\delta|\hat{g}_n(z_n)| \leq \tfrac{3}{4}\delta + \|f\hat{g}_n - (gg_n)\hat{\ }\|_\infty + |(gg_n)\hat{\ }(z_n) - \hat{g}(z_n)\hat{g}_n(z_n)|$$

for each positive integer n. Taking the limit inferior, we see that

$$\liminf_{n \to \infty} \|f\hat{g}_n - (gg_n)\hat{\ }\|_\infty \geq \tfrac{1}{4}\delta.$$

But since $f \in (\hat{B})_b$, Lemma 11 tells us that

$$\|f\hat{g}_n - (gg_n)\hat{\ }\|_\infty \to 0 \text{ as } n \to \infty,$$

a contradiction. This completes the proof of Theorem 1. □

5. Second Bourgain Algebras

Given two closed subalgebras $\mathfrak{A}$ and $\mathfrak{B}$ of $\mathfrak{X}$, does $\mathfrak{A} \subset \mathfrak{B}$ imply $\mathfrak{A}_b \subset \mathfrak{B}_b$? The general question of monotonicity appears to be quite sensitive to the ambient space $\mathfrak{X}$ and to properties of the smaller algebra $\mathfrak{A}$. A variety of positive and negative results are known [3; 9; 12], but in our setting the question is open. However, as we will show in our context, $\mathfrak{A}_b = \mathfrak{A}_{bb}$. In particular, even if monotonicity holds, it cannot be strict.

Our determination of second Bourgain algebras depends upon the following result.

THEOREM 13. *The Bourgain algebra of* $(QC)\hat{} + V$ *with respect to* $L^\infty(\mathbf{D})$ *is*

$$((QC)\hat{} + V)_b = (QC)\hat{} + V.$$

COROLLARY 14. *If* B *is a closed subalgebra of* L^∞ *containing* QC, *then for the Bourgain algebras with respect to* $L^\infty(\mathbf{D})$ *we have* $(\hat{B})_{bb} = (\hat{B})_b$.

Proof. This follows immediately from Corollary 4 and Theorem 11. □

For the proof of Theorem 13 we will need to know the Bourgain algebra of QC relative to L^∞.

THEOREM 15. *The Bourgain algebra of* QC *relative to* L^∞ *is* $(QC)_b = QC$.

In the proofs of both Theorem 13 and 15 we will need the following lemma, which produces certain weakly null sequences in QC. Recall that a *thin Blaschke product* is a function of the form

$$\psi(z) = \prod_{n=1}^{\infty} \frac{|z_n|}{z_n} \frac{z_n - z}{1 - \bar{z}_n z}, \quad z \in \mathbf{D},$$

where $|z_n|/z_n$ is defined to equal 1 if $z_n = 0$, and the sequence $\{z_n\}$ in $\mathbf{D}$ satisfies the condition

$$\prod_{\substack{m=1 \\ m \neq n}}^{\infty} \left| \frac{z_m - z_n}{1 - \bar{z}_m z_n} \right| \to 1 \text{ as } n \to \infty.$$

For an algebra $\mathfrak{A}$ we write $\mathfrak{M}(\mathfrak{A})$ to denote the maximal ideal space of $\mathfrak{A}$, and for simplicity we write $\mathfrak{M} = \mathfrak{M}(H^\infty)$. Identifying a thin Blaschke product ψ with its Gelfand transform on the maximal ideal space of $H^\infty + C$, we denote by $Z(\psi)$ the set $\{x \in \mathfrak{M}(H^\infty + C) : \psi(x) = 0\}$. Since ψ is an interpolating Blaschke product we have $Z(\psi) = \mathrm{cl}(\{z_n : n \geq 1\}) \setminus \{z_n : n \geq 1\}$, where $\mathrm{cl}(\{z_n : n \geq 1\})$ denotes the closure of $\{z_n : n \geq 1\}$ in the space $\mathfrak{M}$ [7, p. 379, Lemma 3.3]. We note that $Z(\psi)$ is an infinite set.

If $f \in L^\infty$, then $\hat{f}$ is a bounded harmonic function on $\mathbf{D}$ and thus (by [10, Lemma 4.4]) has a continuous extension to the maximal ideal space $\mathfrak{M}$ of H^∞, which we will again denote by $\hat{f}$. We have the following lemma.

LEMMA 16. *If* ψ *is a thin Blaschke product, then there exists a sequence* $\{f_n\}$ *in* QC *and a sequence* $\{x_n\}$ *in* $Z(\psi)$ *such that:*

(i) $\|f_n\|_\infty = 1$ *for all* $n \geq 1$;
(ii) $f_n \to 0$ *weakly in* QC; *and*
(iii) $\hat{f}_n(x_n) = 1$ *for all* $n \geq 1$.

Proof. Let ψ be a thin Blaschke product. Define $\pi : \mathfrak{M}(H^\infty + C) \to \mathfrak{M}(QC)$ by letting $\pi(\lambda)$ denote the restriction of λ to QC for $\lambda \in \mathfrak{M}(H^\infty + C)$. Then π is a continuous surjective mapping. Moreover, since ψ is thin, from Izuchi [11, Lemma 5] we know that π is injective on $Z(\psi)$. Using that QC is a C*-

algebra, it is not difficult to find sequences $\{x_n\}$ in $Z(\psi)$ and $\{f_n\}$ in QC satisfying (i), (ii), and $\hat{f}_n(x_n)=\hat{f}_n(\pi(x_n))=1$ for every $n\ge 1$. □

LEMMA 17. *Let B be a closed subalgebra of L^∞ such that $QC\subset B\subset H^\infty+C$. Then $B_b\subset H^\infty+C$.*

Proof. Let $f\in B_b$ and assume that $f\notin H^\infty+C$. By the Chang–Marshall theorem there is a thin Blaschke product ψ such that $\bar{\psi}\in H^\infty[f]$, the closed subalgebra of L^∞ generated by H^∞ and the function f. Let $\{f_n\}$ and $\{x_n\}$ be as in Lemma 16. Since $\bar{\psi}\in H^\infty[f]$, for each integer $n\ge 1$ there is an integer $m_n\ge 1$ and there are h_{nj} in H^∞ $(0\le j\le m_n)$ such that

$$\left\|\bar{\psi}-\sum_{j=0}^{m_n} h_{nj}f^j\right\|_\infty \to 0 \text{ as } n\to\infty.$$

Because B_b is an algebra, for each j we have $f^j\in B_b$, and thus there is a sequence $\{g_{jn}\}$ in B such that $\|f_nf^j-g_{jn}\|_\infty\to 0$ as $n\to\infty$. Then $\|\bar{\psi}f_n-G_n\|_\infty\to 0$ as $n\to\infty$, where $G_n=\sum_{j=0}^{m_n}h_{nj}g_{jn}\in H^\infty+C$. Then also $\epsilon_n=\|f_n-\psi G_n\|_\infty=\|\psi(\bar{\psi}f_n-G_n)\|_\infty\to 0$ as $n\to\infty$. Since $f_n,\psi G_n\in H^\infty+C$ and $x_n\in\mathfrak{M}(H^\infty+C)$, we have

$$1=|\hat{f}_n(x_n)-\psi(x_n)\hat{G}_n(x_n)|=|(f_n-\psi G_n)\hat{\ }(x_n)|\le\epsilon_n,$$

contradicting that $\epsilon_n\to 0$ as $n\to\infty$. □

Proof of Theorem 15. By Lemma 17, $QC_b\subset H^\infty+C$. It is easily seen that QC_b is a C*-algebra, thus also $QC_b\subset\overline{H^\infty+C}$. Hence

$$QC_b\subset(H^\infty+C)\cap(\overline{H^\infty+C})=QC. \qquad\square$$

Proof of Theorem 13. If $f\in L^\infty(\mathbf{D})$ belongs to $((QC)\hat{\ }+V)_b$, then by Remark 10 we see that $f^*\in(QC)_b$. By Theorem 15 we have $f^*\in QC$, so that $\widehat{f^*}\in(QC)\hat{\ }$. The proof is completed if we show that $g=f-\widehat{f^*}\in V$, because then we will have $f=\widehat{f^*}+g\in(QC)\hat{\ }+V$.

Assuming that $g\notin V$, there is a sequence $\{r_n\}$ in $(0,1)$, a sequence $\{A_n\}$ of sets of positive area measure in $\mathbf{D}$, and a positive number δ such that:

$r_n\to 1$ as $n\to\infty$;
$A_n\subset\{z\in\mathbf{C}: r_n<|z|<1\}$; and
$|g(z)|\ge\delta$ for $z\in\bigcup_{n\ge 1}A_n$.

Let z_n be a point of density of the set A_n. Then clearly $|z_n|\to 1$ as $n\to\infty$. By passing to a subsequence (if necessary) we may further assume that the sequence $\{z_n\}$ is a thin interpolating sequence. Write ψ for the Blaschke product with zeros $\{z_n\}$. Let $\{x_n\}$ and $\{f_n\}$ be as in Lemma 16. Since $g\in((QC)\hat{\ }+V)_b$, there are sequences $\{g_n\}$ in QC and $\{v_n\}$ in V such that $\|g\hat{f}_n-(\hat{g}_n+v_n)\|_\infty\to 0$ as $n\to\infty$. Taking nontangential limits, we conclude that $\|g_n\|_\infty\to 0$; thus $\|\hat{g}_n\|_\infty\to 0$ as $n\to\infty$. Hence $\epsilon_n=\|g\hat{f}_n-v_n\|_\infty\to 0$ as $n\to\infty$. Fix an integer n and let $0<r<1$. Because $x_n\in Z(\psi)$, there is a net $\{w_\alpha\}$ in $\{z_m: m\ge 1\}$ such that $w_\alpha\to x_n$ in $\mathfrak{M}$. By the continuity of $\hat{f}_n$ on $\mathfrak{M}$ we have $\hat{f}_n(w_\alpha)\to\hat{f}_n(x_n)=1$.

Note that also $|w_\alpha|\to 1$. Choose an index α for which $|\hat{f}_n(w_\alpha)|>\frac{1}{2}$ and $|w_\alpha|>r$. Pick $0<\eta<|w_\alpha|-r$ so that $|\hat{f}_n(w)|>\frac{1}{2}$ for all w in $\mathbf{D}$ with $|w-w_\alpha|<\eta$. Writing $w_\alpha=z_m$, w_α is a point of density of the set A_m, and thus the set $B=A_m\cap\{w\in\mathbf{C}: |w-w_\alpha|<\eta\}$ has positive measure. Noting that also $B\subset \mathbf{D}\backslash r\mathbf{D}$, we can choose a $w\in B$ for which $|g(w)\hat{f}_n(w)|\le\|g\hat{f}_n\chi_{\mathbf{D}\backslash r\mathbf{D}}\|_\infty$. We then have

$$\delta/2\le|g(w)||\hat{f}_n(w)|\le\|g\hat{f}_n\chi_{\mathbf{D}\backslash r\mathbf{D}}\|_\infty\le\epsilon_n+\|\nu_n\chi_{\mathbf{D}\backslash r\mathbf{D}}\|_\infty.$$

Letting $r\to 1^-$, we obtain $\delta/2\le\epsilon_n$, contradicting that $\epsilon_n\to 0$ as $n\to\infty$. □

Ghatage, Sun, and Zheng showed in [7] that for Bourgain algebras relative to $C(\mathfrak{M})$, $H^\infty(\mathbf{D})_{bb}=H^\infty(\mathbf{D})_b$. In fact, we can prove the same result for Bourgain algebras relative to the larger space $L^\infty(\mathbf{D})$.

THEOREM 18. *Taking Bourgain algebras relative to $L^\infty(\mathbf{D})$, we have*

$$H^\infty(\mathbf{D})_{bb}=H^\infty(\mathbf{D})_b.$$

Proof. Suppose $f\in H^\infty(\mathbf{D})_{bb}=(H^\infty(\mathbf{D})+C(\bar{\mathbf{D}})+V)_b$. Write $g=f^*$. Then it is easily seen that $g\in(H^\infty+C)_b=H^\infty+C$, so that $\hat{g}\in H^\infty(\mathbf{D})+C(\bar{\mathbf{D}})$. We claim that $f-\hat{g}\in V$. It then follows that

$$f=\hat{g}+(f-\hat{g})\in H^\infty(\mathbf{D})+C(\bar{\mathbf{D}})+V=H^\infty(\mathbf{D})_b.$$

Assuming $f-\hat{g}\notin V$, there is a sequence $\{r_n\}$ in $(0,1)$, a sequence $\{A_n\}$ of sets of positive area measure in $\mathbf{D}$, and a positive number δ such that:

$r_n\to 1$ as $n\to\infty$;
$A_n\subset\{z\in\mathbf{C}: r_n<|z|<1\}$;
$|f(z)-\hat{g}(z)|\ge\delta$ for $z\in\bigcup_{n\ge1}A_n$; and
$|f(z)|\le\|f\|_\infty$ for $z\in\bigcup_{n\ge1}A_n$.

Let z_n be a point of density of the set A_n. Then clearly $|z_n|\to 1$ as $n\to\infty$. By passing to a subsequence (if necessary) we may further assume that the sequence $\{z_n\}$ is interpolating. Let $\{f_n\}$ be a Beurling sequence in $H^\infty(\mathbf{D})$; that is,

$f_n(z_k)=\delta_{nk}$ for all positive integers n and k, and
$\sum_{n=1}^\infty|f_n(z)|<M$ for all $z\in\mathbf{D}$,

where M is a finite constant. Let $\{N_k: k\ge1\}$ be a partition of the positive integers such that each set N_k is infinite. For each positive integer k, put

$$F_k=\sum_{n\in N_k} f_n.$$

Then each $F_k\in H^\infty(\mathbf{D})$, and because $\sum_{k=1}^\infty|F_k(z)|<M$, for all $z\in\mathbf{D}$, $F_k\to0$ weakly in $H^\infty(\mathbf{D})$ (here we use [3, Lemma 1]), thus in $H^\infty(\mathbf{D})_b$. It follows that there exist $h_k\in H^\infty(\mathbf{D})$, $\varphi_k\in C(\bar{\mathbf{D}})$ and $\nu_k\in V$ such that

$$\|(f-\hat{g})F_k-h_k-\varphi_k-\nu_k\|_\infty\to 0 \text{ as } k\to\infty.$$

Taking nontangential limits we conclude that $\|h_k^*+\varphi_k^*\|_{L^\infty}\to 0$, which implies that $\|h_k+(\varphi_k^*)\hat{}\,\|_\infty\to 0$, so that

$$\|(f-\hat{g})F_k+(\varphi_k^*)\hat{} - \varphi_k - \nu_k\|_\infty \to 0 \text{ as } k\to\infty,$$

Fix an integer k such that $\|(f-\hat{g})F_k+(\varphi_k^*)\hat{} - \varphi_k - \nu_k\|_\infty \le \delta/4$. Note that for $n\in N_k$ we have $F_k(z_n)=1$, so we can choose $0<\delta_n<1-|z_n|$ such that $|F_k(z)|>\frac{1}{2}$ for $|z-z_n|<\delta_n$. Because z_n is a point of density of A_n, the set $B_n=A_n\cap\{z\in\mathbf{C}: |z-z_n|<\delta_n\}$ has positive measure, and we may assume that

$|(f(z)-\hat{g}(z))F_k(z)+(\varphi_k^*)\hat{}(z)-\varphi_k(z)-\nu_k(z)|\le\delta/4$ for all $z\in B_n$, and
$|\nu_k(z)|\le\|\nu_k\chi_{\mathbf{D}\setminus r_n\mathbf{D}}\|_\infty$ for all $z\in B_n$.

Then, for all $z\in B_n$:

$$\begin{aligned}\delta/2 &\le |f(z)-\hat{g}(z)||F_k(z)|\\ &\le |(f(z)-\hat{g}(z))F_k(z)+(\varphi_k^*)\hat{}(z)-\varphi_k(z)-\nu_k(z)|\\ &\quad +|(\varphi_k^*)\hat{}(z)-\varphi_k(z)|+|\nu_k(z)|\\ &\le \delta/4+|(\varphi_k^*)\hat{}(z)-\varphi_k(z)|+\|\nu_k\chi_{\mathbf{D}\setminus r_n\mathbf{D}}\|_\infty.\end{aligned}$$

Hence

$$|(\varphi_k^*)\hat{}(z_n)-\varphi_k(z_n)|\ge\delta/4-\|\nu_k\chi_{\mathbf{D}\setminus r_n\mathbf{D}}\|_\infty.$$

Taking the limit as $n\in N_k$ and $n\to\infty$, we obtain $0\ge\delta/4-0$, a contradiction. □

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K. Izuchi
Department of Mathematics
Faculty of Science
Niigata University
Niigata 950-21
Japan

K. Stroethoff
Department of Mathematical Sciences
University of Montana
Missoula, MT 59812-1032

K. Yale
Department of Mathematical Sciences
University of Montana
Missoula, MT 59812-1032