Tangents to Curves and a Dyadic Parameterization

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1. Introduction

A rectifiable arc A in the complex plane is usually assumed to be parameterized by arc length along A. The geometric properties of A are then described in terms of the arc-length measure on A. A basic result of the theory is the well-known fact that a rectifiable arc has a tangent almost everywhere with respect to arc-length measure. Our purpose in this paper is to characterize those arcs that have a tangent almost everywhere with respect to Hausdorff linear measure. To this end we introduce a dyadic parameterization of curves, applicable to nonrectifiable Jordan arcs, and our characterization is in terms of this parameterization.

There are many ways of defining a tangent at a point (see Section 3, for example). We shall say that a curve has a tangent at a point when the following definition holds.

DEFINITION 1. Suppose that $A=\{A(x); 0\le x\le L\}$ is a Jordan arc in the plane. For $0<x_0<L$ we say that the real axis is *tangent* to A at $0=A(x_0)$ if for each $\delta>0$ there exists $r>0$ such that for $z=A(x)$ with $|z|<r$, $x<x_0$ implies that $|\arg z-\pi|<\delta$ and $x>x_0$ implies that $|\arg z|<\delta$, for appropriate choice of the argument. The arc A has tangent line T at arbitrary $z_0=A(x_0)$ if, after a translation of z_0 to 0 and rotation about the origin taking T to the real axis, the real axis is tangent to the transformed curve at 0.

The connection between rectifiability of curves and function theory is the theorem of F. and M. Riesz, which states that for rectifiable Jordan curves, harmonic measure and arc length are mutually absolutely continuous: If f is a conformal mapping of the interior of the rectifiable curve C onto the interior of the unit disk, extended continuously to a homeomorphism of the boundaries, then sets of Lebesgue measure zero on the unit circle correspond to sets of arc-length measure zero on C, and conversely.

In the following, the word "curve" shall always denote a closed Jordan curve; otherwise we use the word "arc".

Received October 8, 1992. Revision received September 7, 1993.
Research partially supported by the NATO Collaborative Research Grants Program.
Michigan Math. J. 41 (1994).

If an arc A is not rectifiable, then the geometry of A is often described in terms of Hausdorff measures [F1]. (If A is rectifiable then arc-length measure is the same as the Hausdorff Λ_1 measure, which is also known as linear measure.) However, there is apparently no general "canonical" parameterization of nonrectifiable curves which provides geometric information, although there are parameterizations that are useful for special purposes. For example, quasi-self-similar curves may be parameterized in terms of the Hausdorff measure of their dimension [F2, Thm. 14.17]. Also, Morse [Mo] has developed a general parameterization known as μ-length with applications to the calculus of variations.

In this paper, our approach is in the spirit of the book of Alexandroff and Reshetnyak on irregular curves [AR], in that polygonal approximations to a curve are used. As background we cite the following results of Besicovitch and McMillan, respectively:

(1) If the arc A has a tangent at each point except for a set of linear measure zero (which we write as a.e. (Λ_1)), then A has σ-finite Λ_1 measure, and in particular is of Hausdorff dimension 1.

(2) If T is the set of points on a closed Jordan curve at which there is a tangent, then harmonic measure and linear measure are mutually absolutely continuous on T.

The problem of characterizing the tangent set of an arc is related to the problem of characterizing the set where a function of a real variable is differentiable. The result in this direction which interests us is the following theorem of Stein and Zygmund [SZ], the important point of which is that there is no uniformity in either of conditions (a) or (b).

THEOREM A. *Let $f(x)$ be a real-valued function defined on $\mathbf{R}^1$. Then f has a derivative at almost every point on a set E if and only if both*

(a) $f(x+t)-2f(x)+f(x-t)=O(t)$ *as* $|t|\to 0$, *and*

(b) $\int_{-\delta}^{\delta}|f(x+t)-2f(x)+f(x-t)|^2\,dt/|t^3|<\infty$

for almost every x in E.

Motivated by Theorem A and by work of Beurling and Carleson, Bishop and Jones [BJ] have obtained the following result, of which we will make use. We write $D(z,t)$ to denote the disk of radius t centered at z.

DEFINITION 2. For z on a Jordan arc A, let

$$\beta(z,t)=\inf_L\left\{\sup\frac{\operatorname{dist}(\zeta,L)}{t},\ \zeta\in A\cap D(z,4t)\right\}.$$

where the infimum is taken over all lines L passing through $D(z,t)$.

THEOREM B. *Except for a set of Λ_1 measure 0, $z\in A$ is a tangent point of A if and only if*

$$\int_0^1\beta^2(z,t)\frac{dt}{t}<\infty.$$

In [AP, Thm 3.10], a "dyadic" version of Theorem A was proved, and in [AL] results were established for curves using a geometric version of the second difference which is uniform for the curve. In Section 2 of the present work we shall define the aforementioned dyadic parameterization $z(x)$ for a Jordan arc. In Section 3 we shall define the notion of an approximate tangent and establish a condition that an approximate tangent be a tangent. In Section 4, we show that a geometric "square-function condition" which is expressed in terms of the parameterization implies the existence a.e. (Λ_1) of an approximate tangent. We also show that if A is a quasi-arc then the condition is sufficient for having a tangent a.e. (Λ_1). In this case the condition also implies that the parameterization is absolutely continuous – that is, that sets of measure zero on the parameter domain $[0,1]$ correspond to sets of linear measure zero on the arc, and conversely. (This terminology is not standard; some prefer to say that the parameterization has the Luzin property. See for example [Sa, p. 224].) In Section 5 we relate the square-function condition at a point in the tangent set to Definition 2, and use Theorem B to show that the condition is necessary a.e. (Λ_1) on the tangent set. We also derive absolute continuity properties of $z(x)$ with respect to harmonic measure.

We thank the referee for his perceptive comments on the first version of this paper.

2. A Dyadic Parameterization

Suppose that C is a closed Jordan curve in the plane. We select two points on C and thus divide C into two subarcs. The following procedure will yield a (dyadic) parameterization of each of the two subarcs, and thus a parameterization of C. Many properties of this parameterization will depend on the choice of the two starting points. However, we shall see that the properties relevant to having a tangent almost everywhere (Λ_1) will not depend on the choice of endpoints. The situation is similar to that of dyadic BMO [GJ], where the advantages greatly outweigh the drawbacks.

Now let $A=\{A(t);\ 0\le t\le 1\}$ be a Jordan arc in the plane given by some parameterization $A(t)$. This parameterization determines a direction along A, and we say that $A(t_1)$ is to the left of $A(t_2)$ if $t_1<t_2$. We use a bisection procedure in order to define a new parameterization $z(x)$ for A. We shall first approximate A by a certain sequence of polygonal paths $\{P_n=\{P_n(x), 0\le x\le 1\},\ n=0,1,2,3,\dots\}$. The points $\{a(n,k),\ 0\le k\le 2^n\}$ will be the corners and endpoints of P_n. Define these points inductively as follows. Let $a(0,0)=A(0)$ and $a(0,1)=A(1)$ (we are assuming that these points are distinct). If $a(n,k)$ and $a(n,k+1)$ are defined, let $a(n+1,2k)=a(n,k)$ and $a(n+1,2k+2)=a(n,k+1)$. Let L be the perpendicular bisector of the chord $C(n,k)$ from $a(n,k)$ to $a(n,k+1)$, and define $a(n+1,2k+1)$ to be the first point of intersection of A with L when moving along A from $a(n,k)$ to $a(n,k+1)$. Now define $P_n=\bigcup_{k=0}^{2^n-1} C(n,k)$ and $P_n(x)$ to be the piecewise linear parameterization of P_n such that $P_n(k/2^n)=a(n,k)$. Next we define $z(x)$ at the dyadic points by $z(k/2^n)=a(n,k)$ for $0\le k\le 2^n$, $n=0,1,2,3,\dots$.

Note that, for dyadic points, $z(x)$ is monotone along A in that $x_1 < x_2$ implies that $z(x_1)$ is to the left of $z(x_2)$.

We need the following simple observations:

$$\begin{aligned}&\text{If } |a(n,k)-a(n,k+1)|>\delta \text{ then}\\ &|a(n+1,2k+1)-a(n+1,2k)|\\ &\qquad =|a(n+1,2k+1)-a(n+1,2k+2)|>\delta/2.\end{aligned} \tag{2.1}$$

Given $\epsilon>0$, there exists $\delta>0$ such that for z, z' on A with $|z-z'|<\delta$, the subarc of A between z and z' has diameter less than ϵ. (2.2)

For $x\in[0,1]$ and for integer $n\geq 1$, define the integer $\kappa_n(x)$ by $x\in[\kappa_n(x)/2^n, (\kappa_n(x)+1)/2^n)$. Similarly, for z on A, define the integer $k_n(z)$ as that value for which z is on the arc of A between $z(k_n(z)/2^n)$ and $z((k_n(z)+1)/2^n)$, including the left but not the right endpoint.

We now extend the function $z(x)$ to all of $[0,1]$. Fix $x\in[0,1]$, x not dyadic. For each integer $n\geq 1$, let t_n be such that $A(t_n)=z(\kappa_n(x)/2^n)$. The sequence $\{t_n\}$ is monotone increasing with limit t, and we define

$$z=A(t)=\lim_{n\to\infty}A(t_n)=\lim_{n\to\infty}z\left(\frac{\kappa_n(x)}{2^n}\right)=z(x), \tag{2.3}$$

the existence of the second limit following from that of the first.

The function z maps $[0,1]$ onto A. To see this, let z on A be given and consider the sequences $\{t_n\}$ and $\{s_n\}$ defined by

$$A(t_n)=z\left(\frac{k_n(z)}{2^n}\right)\quad\text{and}\quad A(s_n)=z\left(\frac{k_n(z)+1}{2^n}\right).$$

These sequences are monotone increasing and decreasing with limits t and s, respectively. Suppose there exists $\epsilon>0$ with $|A(t)-z|>\epsilon$. Then by (2.2) there exists $\delta>0$ with

$$|A(t_n)-A(s_n)|>\delta\quad\text{for all } n. \tag{2.4}$$

Now choose N such that for all $n\geq N$,

$$|A(t_n)-A(t)|<\delta/10\quad\text{and}\quad|A(s_n)-A(s)|<\delta/10. \tag{2.5}$$

By (2.1) and (2.4), either $|A(t_{N+1})-A(t_N)|>\delta/2$ or $|A(s_{N+1})-A(s_N)|>\delta/2$. Assume that the first inequality holds. Then

$$|A(t_{N+1})-A(t)|\geq|A(t_{N+1})-A(t_N)|-|A(t_N)-A(t)|>\delta/2-\delta/10=4\delta/10,$$

which contradicts (2.5). Thus we have

$$z=A(t)=\lim_{n\to\infty}A(t_n)=\lim_{n\to\infty}z\left(\frac{k_n(z)}{2^n}\right)=z(x) \tag{2.6}$$

for $x=\lim_{n\to\infty}(k_n(z)/2^n)$. Equation (2.3) and (2.6) define a strictly monotone function $t=t(x)$ from $[0,1]$ onto $[0,1]$, which must then be a homeomorphism. Then $z(x)=A(t(x))$ is a homeomorphism of $[0,1]$ onto A. Using

(2.2), it may be seen that the functions $P_n(x)$ converge uniformly to $z(x)$ on $[0,1]$.

Example 1: Suppose that A is the standard von Koch snowflake, which is constructed from the interval $[0,1]$ by replacing the middle third of the interval by a "tent" with sides of length $1/3$ and iterating the procedure [F1, Chap. 9]. Then $z(x)$ is the parameterization of the snowflake in which the digits in the base-4 expansion of x locate the sequence of self-similar pieces of the snowflake which contain the point $z(x)$. This parameterization is Hölder continuous with exponent $(\log 3)/(\log 4)$. It is not absolutely continuous. Note, of course, that the standard snowflake has a tangent at no point.

Example 2: Let A consist of two line segments of length 1 and $\sqrt{3}$, meeting at a right angle. The corner on A will correspond to the point x_0 which has binary expansion 0.101010... . The parameterization $z(x)$ is Lipschitz continuous on each closed interval not containing x_0, and is Hölder continuous at x_0 with exponent $(\log 3)/(\log 4)$. The parameterization is absolutely continuous on $[0,1]$.

These examples show that the absolute continuity of $z(x)$ and of $x(z)$ depend on the geometry of A, as we shall see in Section 3.

The parameterization $z(x)$ is related to the geometry of A as follows. Recall that $C(n,k)$ is the chord from $a(n,k)=z(k/2^n)$ to $a(n,k+1)=z((k+1)/2^n)$. We inductively assign arguments $S(n,k)$ to the segment $C(n,k)$ in such a way that for $n=0,1,2,3,\dots$ and $0\le k\le 2^n-1$,

$$S(n,k)=\arg\left(z\left(\frac{k+1}{2^n}\right)-z\left(\frac{k}{2^n}\right)\right),$$

$$|S(n+1,2k)-S(n,k)|<\pi/2,$$

and

$$\frac{S(n+1,2k)+S(n+1,2k+1)}{2}=S(n,k). \tag{2.7}$$

DEFINITION 3. Given z on A, for $n\ge 1$ define

$$\theta_n(z)=S(n,k_n(z))-S(n-1,k_{n-1}(z)).$$

Then

$$S(n,k_n(z))=\sum_{j=1}^{n}\theta_j(z)+S(0,0).$$

One may consider the values of $S(n,k_n(z))$ as values in a random walk with differences $\theta_n(z)$; the behavior of these differences will reflect the geometry of A.

3. Tangents to Curves

For an arc C and points z_1 and z_2 on C we write $a_C(z_1,z_2)$ to denote the arc of C between z_1 and z_2. We define two weak forms of a tangent to a curve A

at a point z on A. (Compare these to the case when A is the graph of a function considered in [AP].)

DEFINITION 4. A has a dyadic tangent $T_d(z)$ at z on A if

$$\lim_{n\to\infty} S(n, k_n(z)) = T_d(z)$$

exists.

Using the dyadic parameterization $z(x)$, one may (as mentioned in the previous section) consider the values $S(n, k_n(z(x)))$ to be values of a random walk. Then the following proposition, which we state without proof, will follow from a theorem of Gundy [Gu].

PROPOSITION 1. *Suppose that A is a Jordan arc with dyadic parameterization $z(x)$, and that $\{\theta_n(z)\}$ is as in Definition 3. Then, except for a set of Lebesgue measure zero on* $[0,1]$, *A has a dyadic tangent at $z = z(x)$ if and only if*

$$\sum_{n=1}^{\infty} \theta_n^2(z(x)) < \infty.$$

The goal of this paper is to prove a version of this result involving linear measure on A and tangents as defined in Definition 1.

Given two arcs A and Γ, we shall say that Γ *preserves order* on A if for each pair z' and z'' on $\Gamma \cap A$, each point of $\Gamma \cap A$ between z' and z'' on Γ is between z' and z'' on A.

DEFINITION 5. A has an approximate tangent $T_{ap}(z)$ at z on A if there exists a rectifiable curve Γ that preserves order on A, that has a tangent T at z on Γ, and that satisfies

$$\frac{\Lambda_1(\Gamma \cap A \cap D(z,r))}{\Lambda_1(\Gamma \cap D(z,r))} \to 1 \quad \text{as } r \to 0.$$

We note that on Γ, Λ_1 coincides with arc length. If A has a tangent at z then it also has a dyadic tangent at z, although the converse is false (see [AP]). On the other hand, the existence of a tangent does not imply that of an approximate tangent, and the existence of an approximate tangent does not imply that of a tangent. However, if A is a quasi-arc and if Γ satisfies a certain regularity condition at z, then the existence of an approximate tangent does imply that of a tangent. We recall that A is a quasi-arc if there exists a constant $c > 0$ such that, for z separating z' and z'' on A,

$$\frac{|z'-z|+|z-z''|}{|z'-z''|} < c. \tag{3.1}$$

A closed Jordan curve is a quasi-circle if (3.1) holds for z on the arc of smaller diameter between z' and z''. A rectifiable arc Γ is *Ahlfors regular* if there exists a constant $c' > 0$ such that, for each disk $D(z,r)$,

$$\frac{\Lambda_1(\Gamma \cap D(z,r))}{r} < c'.$$

We need a local version of this notion.

DEFINITION 6. A rectifiable arc Γ is *locally Ahlfors regular* at $z \in \Gamma$ if there exists a constant $M(z) > 0$ such that, for all $r > 0$,

$$\frac{\Lambda_1(\Gamma \cap D(z,r))}{r} < M(z).$$

LEMMA 1. *A rectifiable arc Γ is locally Ahlfors regular at almost every point.*

Proof. Let

$$E = \left\{ z \in \Gamma : \overline{\lim_{r \to 0}} \frac{\Lambda_1(\Gamma \cap D(z,r))}{r} = \infty \right\},$$

and for $C > 0$ let

$$E_C = \left\{ z \in \Gamma : \overline{\lim_{r \to 0}} \frac{\Lambda_1(\Gamma \cap D(z,r))}{r} > C \right\},$$

so that $E \subset E_C$ for all $C > 0$.

Suppose that $z \in E_C$ and that n is a positive integer. Then there exists $r(n,z) < 1/n$ such that

$$\frac{\Lambda_1(\Gamma \cap D(z,r(n,z)))}{r(n,z)} > C.$$

The collection of all of these $D(z, r(n,z))$ is a Vitali class for E_C. Thus, by the Vitali covering theorem [F1, p. 11], for any $\epsilon > 0$ there is a disjoint sequence of these disks $\{D(z_k, r_k)\}$ with

$$\Lambda_1(E_C) < (\Sigma_k 2r_k) + \epsilon.$$

Thus

$$\Lambda_1(E) \le \Lambda_1(E_C) < (\Sigma_k 2r_k) + \epsilon < \frac{2}{C}(\Sigma_k \Lambda_1(\Gamma \cap D(z_k, r_k))) + \epsilon < \frac{2}{C}\Lambda_1(\Gamma) + \epsilon.$$

Since C and ϵ are arbitrarily large and small respectively, the lemma is proved. □

THEOREM 1. *Suppose that A is a quasi-arc with approximate tangent $T_{ap}(z)$ at z, and that the corresponding rectifiable curve Γ is locally Ahlfors regular at z. Then A has a tangent at z.*

Proof. Suppose that $z = 0$, and let $M = M(0)$ be as in Definition 6. Suppose that A does not have a tangent at 0. We may assume that the tangent to the rectifiable Γ at 0 is the real axis. Since the real axis is not tangent to A at 0, there exists $\delta > 0$ such that for any $r > 0$ there exists ζ on A with $|\zeta| \le r$, but for which the requirement in Definition 1 fails. Fix ϵ with $0 < \epsilon < 1/2M$.

Choose $R>0$ to be so small that, for z on Γ with $|z|<R$, $|\arg z|<\delta/2$ or $|\arg z-\pi|<\delta/2$, according as $\operatorname{Re} z$ is positive or negative, and such that for $s<R$

$$\frac{\Lambda_1(\Gamma\cap A\cap D(0,s))}{\Lambda_1(\Gamma\cap D(0,s))}>1-\epsilon. \tag{3.2}$$

Now we may choose ζ on A with $|\zeta|\le r<R/2cM$ and $|\arg\zeta|>\delta$, where c is as in (3.1). Let $R'=cMr$ and let z' and z'' be (respectively) the rightmost and leftmost points of $\Gamma\cap A$ which are separated by ζ. We may assume that $z'\in a_A(0,\zeta)$. Then, by (3.1),

$$|z'|<c|\zeta|=R'/M<R'<R/2.$$

Now Γ preserves order on A, so that $a_\Gamma(z',z'')\subset\Gamma\setminus A$ and we have

$$\Lambda_1((\Gamma\setminus A)\cap D(0,2R'))<\epsilon\Lambda_1(\Gamma\cap D(0,2R'))<2\epsilon MR'<R'. \tag{3.3}$$

If $|z''|\ge 2R'$ then $\Lambda_1((\Gamma\setminus A)\cap D(0,2R'))\ge 2R'-|z'|\ge R'$, which contradicts (3.3). Thus $|z''|<2R'$, and using (3.3) again,

$$|z'-z''|<\Lambda_1(a_\Gamma(z',z''))\le\Lambda_1((\Gamma\setminus A)\cap D(0,2R'))<\epsilon 2MR'=\epsilon 2M^2cr.$$

Since $|z'|<R$ and $|z''|<R$, we also have $|\zeta-z'|>r\sin(\delta/2)$ and $|\zeta-z''|>r\sin(\delta/2)$, so that

$$\frac{|\zeta-z'|+|\zeta-z''|}{|z'-z''|}>\frac{2r\sin(\delta/2)}{\epsilon 2M^2cr}=\frac{\sin(\delta/2)}{\epsilon M^2c}.$$

Since ϵ may be arbitrarily small, this contradicts the fact that A is a quasi-arc, and so the theorem is proved. □

The proof of Theorem 1 requires only that (3.1) holds when z' and z'' are points at which Γ and A diverge, so that the condition that A be a quasi-arc may be weakened.

We now turn to the question of absolute continuity. From the construction above we have

$$\operatorname{diam}\left(a_A\left(z\left(\frac{k}{2^n}\right),z\left(\frac{k+1}{2^n}\right)\right)\right)\ge\frac{1}{2^n}, \tag{3.4}$$

which suggests that $x(z)$ is absolutely continuous on A. However, (3.4) itself does not ensure this. One may construct a homeomorphism $z(x)$ of $[0,1]$ onto an arc A for which a Cantor set of positive measure in $[0,1]$ corresponds to a standard Cantor 1/3-set of measure zero on $A\cap[0,1]$, but for which (3.4) still holds. (We thank D. G. Larman for this example.) However, if A is a quasi-arc we have the following proposition, the proof of which follows a suggestion of C. A. Rogers.

PROPOSITION 2. *Suppose that* $z(x)$ *is a homeomorphism of* $[0,1]$ *onto a quasi-arc* A *for which* (3.4) *holds. Then* $x(z)$ *is absolutely continuous on*

A. That is, sets of linear measure zero on A correspond to sets of Lebesgue measure zero on $[0,1]$.

Proof. Given z and z' on A, let $p=k/2^n$ and $q=(k+1)/2^n$ be such that $[p,q]$ is a maximal dyadic interval contained in $[x(z),x(z')]$. Then

$$|x(z)-x(z')|\leq 3|p-q|\leq 3\,\mathrm{diam}(a_A(z(p),z(q)))\leq 3\,\mathrm{diam}(a_A(z,z')).$$

Suppose now that $E\subset A$ with $\Lambda_1(E)=0$. Let $\{B_i\}_{i=1}^{\infty}$ be a collection of closed sets in the plane which cover E with $\sum_{i=1}^{\infty}\mathrm{diam}(B_i)<\epsilon$. Using the parameterization of A, let z_i and z_{i+1} be the first and last points of A contained in B_i. Since A is a quasi-arc, there is a constant $K\geq 1$ such that for each i

$$\mathrm{diam}(a_A(z_i,z_{i+1}))\leq K|z_i-z_{i+1}|\leq K\,\mathrm{diam}(B_i).$$

Then we have

$$\sum_{i=1}^{\infty}|x(z_i)-x(z_{i+1})|\leq 3K\sum_{i=1}^{\infty}\mathrm{diam}(B_i)<3K\epsilon.$$

Since $x(E)\subset\bigcup_{i=1}^{\infty}[x(z_i),x(z_{i+1})]$, it follows that $x(E)$ has Lebesgue measure zero, as required. □

4. A Sufficient Condition for Tangents a.e.

THEOREM 2. *Suppose that A is a Jordan arc with dyadic parameterization* $z(x)$, *and that for all* $z\in E\subset A$ *and for* $\theta_n(z)$ *as in Definition 3,*

$$\sum_{n=1}^{\infty}\theta_n^2(z)<\infty.$$

Then:

(a) *E has* σ*-finite* Λ_1 *measure;*
(b) *for almost every* $(\Lambda_1)z\in E$, *A has an approximate tangent at* z;
(c) $x(z)$ *is absolutely continuous on* E; *and*
(d) $z(x)$ *is absolutely continuous on* $B=z^{-1}(E)$.

Proof. The proof uses a standard stopping argument similar to that employed in [AP]. We assume that the endpoints of A are 0 and 1. For integers n and k, with $n\geq 0$ and $0\leq k\leq 2^n-1$, we write

$$I(n,k)=\left[\frac{k}{2^n},\frac{k+1}{2^n}\right).$$

Fix $\lambda\geq 1$ and let

$$L_\lambda=\left\{x\in[0,1]\colon \sum_{n=1}^{\infty}\theta_n^2(z(x))<\lambda \text{ and } |\theta_j(x)|<\pi/2-1/\lambda \text{ for all } j\geq 1\right\}.$$

Next define

$$N_\lambda(x)=\min\{n\colon I(n,\kappa_n(x))\cap L_\lambda=\emptyset\},$$

so that $I(N_\lambda(x)-1, \kappa_{N_\lambda(x)-1}(x))$ contains a point x with $\sum_{j=1}^\infty \theta_j^2(z(x))<\lambda$ and $|\theta_j(z(x))|<\pi/2-1/\lambda$ for all j. For each point y on the interval $I(N_\lambda(x), \kappa_{N_\lambda(x)}(x))$, $N_\lambda(y)=N_\lambda(x)$, $\sum_{j=1}^{N_\lambda(x)} \theta_j^2(y)$ is constant, and $|\theta_j(y)|<\pi/2-1/\lambda$ for each $j\le N_\lambda(x)$. Referring to the polygonal paths used in constructing the dyadic parameterization, for each $x\in[0,1]$ let

$$\Gamma_{n,\lambda}(x)=P_{n\wedge N_\lambda(x)}(x)$$

be the parameterization of a polygonal path $\Gamma_{n,\lambda}$. Roughly speaking, the refinement of the sequence of paths P_n stops on intervals where the arc A is rotating too much or where the curve turns too sharply. Note that $\Gamma_{n,\lambda}(x)$ is linear on $I(N_\lambda(x), k_{N_\lambda(x)})$, but not necessarily on $I(N_\lambda(x)-1, k_{N_\lambda(x)-1})$, since $N_\lambda(x)$ is constant on the smaller interval.

LEMMA 2. *For each $\lambda>0$ there exists a constant $c(\lambda)$, independent of A and n, such that $\Lambda_1(\Gamma_{n,\lambda})<c(\lambda)$.*

Proof. The length of any segment $C(m,k)$ is

$$\Lambda_1(C(m,k))=\frac{1}{2^m\prod_{j=1}^m \cos(\theta_j(z(k/2^m)))}.$$

Letting $\beta(n,k)=\min(n, N_\lambda(k/2^n))$, we then have

$$\Lambda_1(\Gamma_{n,\lambda})=\sum_{k=0}^{2^n-1}\frac{1}{2^n\prod_{j=1}^{\beta(n,k)}\cos(\theta_j(z(k/2^n)))}. \tag{4.1}$$

We must now estimate the products in the denominators. It is not difficult to see that the product is minimized by taking as many θ_j as possible equal to $\pi/2-1/\lambda=(\pi\lambda-2)/2\lambda$, subject to the condition $\sum_{j=1}^{\beta(n,k)}\theta_j^2(z(k/2^n))<\lambda$. Thus, taking $q=[\lambda(2\lambda/(\pi\lambda-2))^2]+1$, we have

$$\prod_{j=1}^{\beta(n,k)}\cos\left(\theta_j\left(\frac{k}{2^n}\right)\right)\ge\prod_{j=1}^{q}\cos\left(\frac{\pi\lambda-2}{2\lambda}\right)=\left(\sin\left(\frac{1}{\lambda}\right)\right)^q, \tag{4.2}$$

which, when applied to (4.1), proves the lemma. □

Now define

$$\Gamma_\lambda(x)=\lim_{n\to\infty}\Gamma_{n,\lambda}(x)=\lim_{n\to\infty}P_{n\wedge N_\lambda(x)}(x).$$

If $N_\lambda(x)=\infty$ then $\Gamma_\lambda(x)=z(x)$. If $N_\lambda(x_0)=n_0$ then $\Gamma_\lambda(x)=P_{n_0}(x)$ for all $x\in I(n_0+1, \kappa_{n_0+1}(x_0))$. By the lower semicontinuity of arclength, we then have $\Lambda_1(\Gamma_\lambda)\le c(\lambda)$, where Γ_λ is the curve parameterized by $\Gamma_\lambda(x)$ for $0\le x\le 1$. It is also worth noting that the proof shows that if $\sum_{j=1}^\infty\theta_j^2(z)\le\lambda$ for some $\lambda>0$ and for all z on E, then E is a subset of a rectifiable curve with length not exceeding $c(\lambda)$ (cf. [BJ, Thm. 1]).

An important property of Γ_λ is given by Lemma 3.

LEMMA 3. *For each $\lambda\ge1$, the correspondence of arc lengths under $\Gamma_\lambda(x)$ is a bi-Lipschitz mapping.*

Proof. Let $z'_m = z(k/2^m)$ and $z''_m = z((k+1)/2^m)$. From the above argument we see that the chord length between these points is at most

$$\frac{1}{2^m \prod_{i=1}^{\beta(m,k)} \cos(\theta_i(z(k/2^m)))},$$

where $\beta(m,k) = \min(m, N_\lambda(k/2^m))$. Note that if $N_\lambda(k/2^m) = n_0 \le m$, then $\Gamma_{n,\lambda}$ is not refined along this arc as n increases.

Now consider the refinement of $C(m,k)$. There will possibly be corners on $\Gamma_{n,\lambda}$ at $z(k/2^m + j/2^n)$, $0 \le j \le 2^{n-m}-1$. For each j, the connecting segments will have length

$$\frac{1}{2^m \prod_{i=1}^{\beta(m,k)} \cos(\theta_i(z(k/2^m)))} \frac{1}{2^{n-m} \prod_{i=m+1}^{\beta(n,k2^{n-m}+j)} \cos(\theta_i(z(k/2^m + j/2^n)))}$$
$$= \frac{1}{2^n \prod_{i=1}^{\beta(n,k2^{n-m}+j)} \cos(\theta_i(z(k/2^m + j/2^n)))}.$$

The length of the part of $\Gamma_{n,\lambda}$ from z'_m to z''_m is

$$\sum_{j=1}^{2^{n-m}} \frac{1}{2^n \prod_{i=1}^{\beta(n,k2^{n-m}+j)} \cos(\theta_i(z(k/2^m + j/2^n)))},$$

and each product, as in (4.2), is at least $(\sin(1/\lambda))^q$; the length of the arc is then at most $c(\lambda)/2^m$. Thus the arc of $\Gamma_{n,\lambda}$ from $\Gamma_{n,\lambda}(k/2^m)$ to $\Gamma_{n,\lambda}(j/2^m)$, $j > k$, has length at most

$$\frac{j-k}{2^m} c(\lambda) = |x''_m - x'_m| c(\lambda),$$

where $x''_m = j/2^m$ and $x'_m = k/2^m$.

Now let $n \to \infty$, so that

$$\Lambda_1(a_{\Gamma_\lambda}(\Gamma_\lambda(x'_m), \Gamma_\lambda(x''_m))) \le |x''_m - x'_m| c(\lambda).$$

Next, for any pair of points $x' < x''$ on $[0,1]$, let $x'_m = k_m(x')/2^m$ and $x''_m = (k_m(x'')+1)/2^m$. Letting $m \to \infty$, we have

$$\Lambda_1(a_{\Gamma_\lambda}(\Gamma_\lambda(x'), \Gamma_\lambda(x''))) \le |x'' - x'| c(\lambda).$$

Thus arc length on Γ_λ is a Lipschitz function of length on $[0,1]$. It is clear that length on $[0,1]$ is a contraction of arc length on Γ_λ, and the lemma is proved. □

To finish the proof of Theorem 2, recall again that $x \in L_\lambda$ implies that $\Gamma_\lambda(x) = z(x)$. For $\lambda = 1, 2, 3, \ldots$ let $E_\lambda = z(L_\lambda) \cap E$, so that $E_\lambda \subset \Gamma_\lambda$ and

$$E = \bigcup_{\lambda=1}^{\infty} E_\lambda, \tag{4.3}$$

where $\Lambda_1(E_\lambda) \le c(\lambda)$. We see that E has σ-finite linear measure, so that the Hausdorff dimension of E is at most 1. If $E = A$ then A has dimension 1, and part (a) of Theorem 2 is established.

Since $E_\lambda \subset \Gamma_\lambda$, Γ_λ has a tangent at almost every point z of E_λ. At each point of density z_0 of E_λ, this is the approximate tangent to A, since such z_0 is a point of density of $\Gamma_\lambda \cap A$ on Γ_λ, so that

$$\frac{\Lambda_1(\Gamma_\lambda \cap A \cap D(z_0, r))}{\Lambda_1(\Gamma_\lambda \cap D(z_0, r))} \to 1 \quad \text{as } r \to 0$$

[F1, p. 20]. Part (b) of the theorem then follows from (4.3).

Finally, part (c) follows from Lemma 3 and (4.3), while part (d) follows from Lemma 3 and the fact that $B = \bigcup_{\lambda=1}^{\infty} L_\lambda$. □

THEOREM 3. *If A is a quasi-arc such that $\sum_{n=1}^{\infty} \theta_n^2(z) < \infty$ for all $z \in E \subset A$, then A has a tangent a.e. (Λ_1) on E.*

Proof. By Theorem 2, for each $\lambda = 1, 2, 3, \ldots$, A has an approximate tangent almost everywhere on $A \cap E_\lambda$. By Lemma 1 each Γ_λ is almost everywhere locally Ahlfors regular, so that A has a tangent a.e. on $A \cap E_\lambda$, by Theorem 1. The theorem follows from (4.3). □

In the spirit of Theorem A, it seems that the uniform condition in Theorem 1 and Theorem 3 that A be a quasi-arc might be replaced by a corresponding local condition. Such conditions have been studied (see e.g. [RW]), but these are inappropriate to our problem. However, these results are valid if A is assumed to be a "dyadic quasi-arc". This means that (3.1) holds, not for all choices of z' and z'', but merely when z' and z'' are dyadic points of the same generation. The reason is that the curves Γ_λ and A used in the application of Theorem 1 to the proof of Theorem 3 diverge only at such points. It can also be shown that each Γ_λ is a dyadic quasi-arc, with the constant c in (3.1) dependent on λ.

5. A Necessary Condition for Tangents a.e.

Suppose that x is a real number contained in the dyadic interval $[k/2^n, (k+1)/2^n] = [x_n, x_n']$. Then for $t_n = \max\{|x - x_n|, |x - x_n'|\}$ we have $\frac{1}{7} t_n \le t_{n+1} \le \frac{1}{2} t_n$. If the arc A has a tangent at z then the next lemma shows that dyadic intervals on A have a similar property.

LEMMA 4. *Suppose that the Jordan arc A has a tangent at $z = 0$, and that $z_n = z(k_n(z)/2^n)$ and $z_n' = z((k_n(z)+1)/2^n)$ are the endpoints of a dyadic interval on A which contains $z = 0$. Let $t_n = \max\{|z_n|, |z_n'|\}$. Then for any $\epsilon > 0$ there exists N such that $n > N$ implies that*

$$t_n/7 \le t_{n+2} \le (\tfrac{1}{2} + \epsilon) t_n.$$

Proof. We may assume that A is tangent to the real axis at $z = 0$, and we take δ as in Definition 1 to be small. Suppose that z_n is in the left cone and z_n' is in the right cone with vertex at 0. Let z' be the dyadic point on A obtained by "bisecting" $a_A(z_n, z_n')$. We may assume, without loss of generality, that 0 is on $a_A(z', z_n')$, so that $t_n = |z_n|$, and we let z'' be the next dyadic point, obtained by bisecting $a_A(z', z_n')$. The triangle with vertices z', z'' and z_n' has $|z'-z''| = |z''-z_n'|$, and for $\tau =$ the angle with vertex at z' we have $\tau < 2\delta$. By the law of cosines,

$$|z''-z_n'|^2 = |z'-z''|^2 + |z'-z_n'|^2 - 2|z'-z''||z'-z_n'|\cos\tau,$$

so that

$$|z'-z''| = |z''-z_n'| = \frac{|z'-z_n'|}{2\cos\tau} < (\tfrac{1}{2}+\epsilon)|z'-z_n'| \tag{5.1}$$

for n sufficiently large.

The proof of the lemma consists of two cases, which depend on the position of z'' relative to 0. We let $K = (\frac{1}{2}+\epsilon)$, as in (5.1).

Case 1: Suppose that $z \in a_A(z', z'')$. Then $t_{n+2} = |z'|$ or $|z''|$. In either case, by (5.1)

$$t_{n+2} \le |z'-z''| \le K|z'-z_n'| = K|z'-z_n| \le Kt_n.$$

Furthermore,

$$t_{n+2} \ge \frac{|z'-z''|}{2} \ge \frac{|z'-z_n'|}{4} = \frac{|z'-z_n|}{4},$$

so that

$$t_n \le |z_n - z'| + |z'| \le t_{n+2}(4+1).$$

Case 2: Suppose that $z \in a_A(z'', z_n')$. Then $t_{n+2} = |z''|$ or $|z_n'|$. In either case, by (5.1)

$$t_{n+2} \le |z''-z_n'| \le K|z'-z_n'| = K|z_n - z'| \le Kt_n.$$

We also have

$$t_{n+2} \ge \frac{|z_n'-z''|}{2} = \frac{|z'-z''|}{2} \ge \frac{|z_n'-z'|}{4} = \frac{|z_n-z'|}{4}$$

and

$$t_n \le |z_n - z'| + |z'-z''| + |z''| \le t_{n+2}(4+2+1),$$

and the lemma is proved. □

Now recall from Definition 2 that

$$t\beta(z,t) = \inf_L \sup\{\text{dist}(\zeta, L), \zeta \in A \cap D(z, 4t)\},$$

where the infimum is taken over all lines passing through $D(z,t)$. Thus $A \cap D(z, 4t)$ lies inside a strip of width $2t\beta(z,t)$. If A has a tangent at z then $\beta(z,t) \to 0$ as $t \to 0$. Furthermore, it follows from this that $t\beta(z,t) \to 0$ monotonically as $t \to 0$, for t sufficiently small, for the following reason.

Fix t_1 and suppose that L is a line minimizing $t_1\beta(z, t_1)$. Then L is the centerline of a strip of width $2t_1\beta(z, t_1)$ which also contains z. Then for $t_2 < t_1$, with $t_2 > t_1\beta(z, t_1)$, we see that $L \cap D(z, t_2)$ is not empty, and that for $\zeta \in A \cap D(z, 4t_2)$ we have $\text{dist}(\zeta, L) \le t_1\beta(z, t_1)$. Thus

$$\sup\{\mathrm{dist}(\zeta, L), \zeta \in A \cap D(z, 4t_2)\} \le t_1 \beta(z, t_1).$$

From this we conclude that $t_2\beta(z,t_2) \le t_1\beta(z,t_1)$. The reasoning is valid for t so small that $\beta(t_1) < 1$.

We next compare the quantity $\beta(z,t)$ to $\theta_n(z)$ at a point z at which A has a tangent. Assume that $z=0$ and let z_n, z_n', and t_n be as in Lemma 4. Again let z' be the dyadic point obtained by "bisecting" $a_A(z_n, z_n')$. Clearly z_n and z_n' are in $D(0,t_n)$, and since A has a tangent at 0 it follows that, for t_n sufficiently small, z' is also in $D(z,t_n)$. Thus, each of z_n, z_n', and z' is in a strip of width $2(t_n/4)\beta(t_n/4)$. Consequently, again for $\beta(t_n)$ sufficiently small, we have

$$\theta_n \le \tan\theta_n \le \frac{(t_n/2)\beta(t_n/4)}{|z_n - z_n'|} \le \frac{(t_n/2)\beta(t_n/4)}{t_n} = \tfrac{1}{2}\beta(t_n/4),$$

that is,

$$\theta_n \le \tfrac{1}{2}\beta(t_n/4). \tag{5.2}$$

We can now prove the following theorem.

THEOREM 4. *Suppose that A is a Jordan arc with dyadic parameterization $z(x)$ and that A has a tangent at each $z \in T \subset A$. Then a.e. $(\Lambda_1) z \in T$ belongs to $E = \{z \in A : \sum_{n=1}^{\infty} \theta_n^2(z) < \infty\}$. Furthermore, $z(x)$ is absolutely continuous on $B = z^{-1}(E)$ and $x(z)$ is absolutely continuous on E.*

Proof. The second assertion will follow from the first and from Theorem 2(c). By Theorem B, we have

$$\int_0^1 \beta^2(z,t)\,\frac{dt}{t} < \infty$$

for a.e. $(\Lambda_1)z$ in E. By (5.2) it suffices to show that this implies that

$$\sum_{n=1}^{\infty} \beta^2\left(z, \frac{t_n}{4}\right) < \infty.$$

We break this sum into two sums, with odd and even indices respectively. Using Lemma 4, it suffices to show that for a function f defined on $[0,1]$ with $xf(x) \to 0$ monotonically as $x \to 0$, and for a sequence $\{x_n\}$ in $[0,1]$ with

$$\frac{x_n}{7} \le x_{n+1} \le \frac{3x_n}{4},$$

we have

$$\int_0^1 f^2(x)\,\frac{dx}{x} \ge K \sum_{n=1}^{\infty} f^2(x_n)$$

for some $K > 0$.

Let $x = e^{-u}$. Then, since $e^{-u}f(e^{-u})$ is monotone decreasing as $u \to \infty$, we have

$$\int_0^1 f^2(x)\frac{dx}{x} = \int_0^\infty f^2(e^{-u})\,du = \int_0^\infty e^{2u}(e^{-u}f(e^{-u}))^2\,du$$

$$\geq \sum_{n=1}^{\infty} e^{2u_n}(e^{-u_{n+1}}f(e^{-u_{n+1}}))^2(u_{n+1}-u_n)$$

$$= \sum_{n=1}^{\infty} e^{2(u_n-u_{n+1})}(u_{n+1}-u_n)f^2(e^{-u_{n+1}})$$

$$\geq \frac{\log(4/3)}{49}\sum_{n=1}^{\infty} f^2(x_n),$$

and the proof of Theorem 4 is complete. □

The following Corollary summarizes some of the above results.

COROLLARY. *Suppose that C is a quasi-circle and that T is the set of points on C at which there is a tangent. Then:*

(a) *except for a set of* Λ_1 *measure zero,* $z \in T$ *if and only if* $z \in E = \{z \in A: \sum_{n=1}^{\infty}\theta_n^2(z) < \infty\}$;
(b) *the set E is almost everywhere* (Λ_1) *independent of the choice of end-points at which the parameterization is begun;*
(c) *if* $S \subset T$ *with harmonic measure* $\omega(S) = 0$ *then Lebesgue measure* $m(x(S)) = 0$; *and*
(d) *if* $S \subset z^{-1}(E) = B$ *with* $m(S) = 0$ *then* $\omega(z(S)) = 0$.

Proof. Part (a) follows from Theorems 3 and 4. For part (b), given one dyadic parameterization, the convergence of the sum corresponding to that parameterization implies a.e. existence of the tangent which implies a.e. convergence of the sum for the second parameterization. Parts (c) and (d) follow from McMillan's theorem, Proposition 2 and Theorem 2(d).

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