Duality and Functionals on S

I. M. AL-GROUZ & D. R. WILKEN

Let S be the family of normalized univalent functions, a subset of the space $\mathcal{A}$ of functions analytic on the open unit disk Δ in the complex plane $\mathbf{C}$. That is, $S = \{f \in \mathcal{A} : f \text{ is univalent in } \Delta, f(0)=0, \text{ and } f'(0)=1\}$. Then S is a compact subset of $\mathcal{A}$ in the topology of uniform convergence on compact subsets of Δ.

In successful efforts to construct examples of functions in S which are extreme points of S but not support points of S, Duren and Leung [5] and Hamilton [6] introduced examples of functionals which were linear on a subspace of $\mathcal{A}$ containing S and were continuous on S but were not continuous on the linear span of S. The following definition was given by Duren and Leung [5].

DEFINITION. Let L be a complex-valued linear functional defined on $\operatorname{sp} S$, the linear span of S, such that L is continuous on S. L is called a *continuous linear functional on* S, and we write $L \in S^*$.

Duren and Leung mentioned two ways to exhibit functionals $L \in S^*$:

(i) Let $\{\lambda_n\}$ be a sequence of complex numbers such that $\sum_{n=1}^{\infty} n|\lambda_n| < \infty$. For $f(z) = \sum_{n=1}^{\infty} a_n z^n \in \operatorname{sp} S$, define $L(f) = \sum_{n=1}^{\infty} a_n \lambda_n$. Then $L \in S^*$.

(ii) Let μ be a finite complex regular Borel measure on Δ (not necessarily with compact support) such that

$$\int_{\Delta} \frac{1}{(1-|z|)^2}\, d|\mu|(z) < \infty.$$

For $f \in \operatorname{sp} S$, define $L(f) = \int f\, d\mu$. Then $L \in S^*$.

A functional $L \in S^*$ as defined in (ii) is said to be of *integral type*. In this case let $\lambda_n = \int z^n\, d\mu$. It is easy to see that $\sum_{n=1}^{\infty} n|\lambda_n| < \infty$ and $L(f) = \sum a_n \lambda_n$. Consequently, functionals of integral type also have the form described in (i). Thus far these are the only known examples of continuous linear functionals on S. Duren and Leung raised the question of whether there exist $L \in S^*$ that are not of integral type.

In this paper we construct a new class of continuous linear functionals on S. Moreover, we show that there exist sequences $\{\lambda_n\}$ with $\sum_{n=1}^{\infty} n|\lambda_n| = \infty$

Received October 5, 1992. Revision received June 28, 1993.
Michigan Math. J. 41 (1994).

and such that if $f(z)=\sum_{n=1}^{\infty}a_nz^n\in \operatorname{sp}S$ then $L(f)=\sum_{n=1}^{\infty}a_n\lambda_n$ is defined and continuous on S. In particular, not every functional in S^* is of integral type. We also give a description of all functionals in S^*. Toward this end we consider the recently much-studied integral families $\mathfrak{F}_\alpha$. Let $\mathfrak{M}$ denote the set of finite complex regular Borel measures on the unit circle $\Gamma=\{x:|x|=1\}$.

DEFINITION. Let $\alpha>0$. Then $\mathfrak{F}_\alpha$ is the family of functions

$$f(z)=\int_\Gamma \frac{1}{(1-xz)^\alpha}\,d\mu(x),\quad \mu\in\mathfrak{M}.$$

It can be shown that $\mathfrak{F}_\alpha$ is a Banach space under the norm $\|f\|_{\mathfrak{F}_\alpha}=\inf\|\mu\|$, where the infimum is taken over all measures $\mu\in\mathfrak{M}$ that represent f as above.

MacGregor [9] showed that $S\subset\mathfrak{F}_\alpha$, hence $\operatorname{sp}S\subset\mathfrak{F}_\alpha$, for $\alpha>2$. Our approach is to show that, surprisingly, the Banach space norm that S inherits as a subset of $\mathfrak{F}_\alpha$ gives the same topology on S as the topology of uniform convergence on compact subsets of Δ. It follows that for $\alpha>2$ each continuous linear functional on $\mathfrak{F}_\alpha$ defines an element of S^* when restricted to $\operatorname{sp}S$. By this means one makes available a large pool of functionals in S^* from which we are able to construct our examples.

Our first step is to identify the predual of $\mathfrak{F}_\alpha$ with an appropriate Banach space of analytic functions on Δ. When α is a positive integer, the identification is well-known although not readily accessible in the literature. To simplify matters we utilize the following observation, essentially due to Hibschweiler and MacGregor [7].

Let α be a real number. Let $G_\alpha(z)=\sum_{n=0}^{\infty}(n+1)^{\alpha-1}z^n$, and let $\mathcal{G}_\alpha$ denote the family of functions

$$g(z)=\int_\Gamma G_\alpha(xz)\,d\mu(x),\quad \mu\in\mathfrak{M}.$$

With the norm defined exactly as for $\mathfrak{F}_\alpha$ above, it can be shown that $\mathcal{G}_\alpha$ is also a Banach space. Then $\mathfrak{F}_\alpha=\mathcal{G}_\alpha$ for $\alpha>0$ and the norms are equivalent.

Let A^0 denote the well-known *disk algebra* of functions in $\mathfrak{A}$ which extend continuously to Γ. Let A^0 have the sup-norm $\|f\|_\infty=\sup\{|f(z)|: z\in\Delta\}$.

DEFINITION. Let β be a real number. Then A^β is the Banach space of functions

$$g(z)=\sum_{n=0}^{\infty}\frac{a_n}{(n+1)^\beta}z^n,$$

where $f(z)=\sum_{n=0}^{\infty}a_nz^n\in A^0$ and $\|g\|_{A^\beta}=\|f\|_\infty$.

REMARK. When β is a positive integer, say $\beta=k$, then A^k is the family of functions in $\mathfrak{A}$ whose kth derivatives extend continuously to Γ. The given norm is equivalent to the usual norm on A^k.

The following proposition is proven in detail in [3]. We only sketch the ideas.

PROPOSITION A. *For each real number α, $(A^{\alpha-1})^* = \mathcal{G}_\alpha$. More explicitly, if $g(z) = \sum_{n=0}^{\infty} b_n z^n \in A^{\alpha-1}$ and $f(z) = \sum_{n=0}^{\infty} c_n z^n \in \mathcal{G}_\alpha$, then*

$$L_f(g) = \lim_{r\to 1^-} \sum_{n=0}^{\infty} b_n c_n r^n$$

defines a continuous linear functional on $A^{\alpha-1}$. Conversely, given any $L \in (A^{\alpha-1})^*$, there exists $f \in \mathcal{G}_\alpha$ such that $L = L_f$. Moreover, the norm of L_f in $(A^{\alpha-1})^*$ agrees with the norm of f in $\mathcal{G}_\alpha$.

Sketch of Proof. By definition, the map $T_\alpha\colon \mathcal{G}_1 \to \mathcal{G}_\alpha$ given by

$$T_\alpha\left(\sum_{n=0}^{\infty} a_n z^n\right) = \sum_{n=0}^{\infty} (n+1)^{\alpha-1} a_n z^n$$

is an isometric isomorphism of $\mathcal{G}_1$ onto $\mathcal{G}_\alpha$. Similarly, the map $S_\beta\colon A^0 \to A^\beta$ given by

$$S_\beta\left(\sum_{n=0}^{\infty} a_n z^n\right) = \sum_{n=0}^{\infty} \frac{a_n}{(n+1)^\beta} z^n$$

is an isometric isomorphism of A^0 onto A^β.

Thus, if one can establish the proposition for the case $\alpha = 1$ then the general case is an easy consequence. The identification $A_0^* = \mathcal{G}_1$ comes about as follows. Let $f \in \mathcal{G}_1$ be represented by the measure μ, and associate with f the linear functional L_f defined for $h(z) = \sum_{n=0}^{\infty} a_n z^n$ in A^0 by

$$\begin{aligned} L_f(h) = \int_\Gamma h(x)\, d\mu(x) &= \lim_{r\to 1^-} \int_\Gamma h(rx)\, d\mu(x) \\ &= \lim_{r\to 1^-} \int_\Gamma \sum_{n=0}^{\infty} a_n r^n x^n\, d\mu(x) \\ &= \lim_{r\to 1^-} \sum_{n=0}^{\infty} a_n \int x^n\, d\mu(x) r^n = \lim_{r\to 1^-} \sum_{n=0}^{\infty} a_n b_n r^n. \end{aligned}$$

Note that f as an element of $\mathcal{G}_1$ has the form

$$f(z) = \int \sum_{n=0}^{\infty} x^n z^n\, d\mu(x) = \sum_{n=0}^{\infty} \int x^n\, d\mu(x) z^n = \sum_{n=0}^{\infty} b_n z^n.$$

All measures representing f yield the same functional, so L_f is well-defined.

Conversely, given $L \in A_0^*$, extend L by the Hahn–Banach theorem to the space of all continuous functions on Γ, represent the extension by a measure μ, and thereby produce a function f in $\mathcal{G}_1$ with the property that $L = L_f$, as defined above. Two measures representing extensions of L yield the same f in $\mathcal{G}_1$ and hence L is identified with a unique element of $\mathcal{G}_1$. Standard Banach-space arguments together with the definition of the norm in $\mathcal{G}_1$ yield the norm isometry.

Our next step is to show that the topology S inherits from $\mathcal{G}_\alpha$ when $\alpha > 2$ coincides with the usual topology on S. Once this is done we have $A^{\alpha-1} \subset (A^{\alpha-1})^{**} = \mathcal{G}_\alpha^* \subset S^*$, the last containment arising by restriction. Since it is an elementary fact that convergence in $\mathcal{G}_\alpha$ implies uniform convergence on compact subsets of Δ, it suffices to establish the following theorem.

THEOREM 1. *Let f_n be a sequence of functions in S such that $f_n \to f$ uniformly on compact subsets of Δ. Then $f_n \to f$ in $\mathcal{G}_\alpha$ for every $\alpha > 2$.*

For the proof we recall the following facts (see [9] and [1]).

PROPOSITION B. *Let f be a function in the Hardy class H^1. Then $f \in \mathfrak{F}_1$ and $\|f\|_{\mathfrak{F}_1} \leq \|f\|_{H^1}$.*

PROPOSITION C. *For a function f in $\mathfrak{A}$, let*

$$T_\alpha f(z) = (\alpha - 1)\int_0^1 (1-t)^{\alpha-2} f(tz)\,dt.$$

(i) *Let $\alpha > 1$. Then $f \in \mathfrak{F}_\alpha$ if and only if $T_\alpha f \in \mathfrak{F}_1$. Moreover,*

$$\|f\|_{\mathfrak{F}_\alpha} = \|T_\alpha f\|_{\mathfrak{F}_1}.$$

(ii) *Let $\alpha > 2$. If $f \in S$ then $T_\alpha f \in H^1$.*

COROLLARY. *If $\alpha > 2$ then $S \subset \mathfrak{F}_\alpha$.*

Proof of Theorem 1. For $\alpha > 2$ let

$$g_n(z) = T_\alpha f_n(z) \quad \text{and} \quad g(z) = T_\alpha f(z).$$

Then $g_n, g \in H^1$ and $\|g_n - g\|_{H^1} \geq \|g_n - g\|_{\mathfrak{F}_1} = \|f_n - f\|_{\mathfrak{F}_\alpha}$ by Propositions B and C. We will show that $\|g_n - g\|_{H^1} \to 0$ which, since $\mathfrak{F}_\alpha$ and $\mathcal{G}_\alpha$ are the same set with equivalent norms, yields the conclusion of the theorem. Now

$$\|g_n - g\|_{H^1} = \sup_{0 \leq r < 1}\left\{\frac{1}{2\pi}\int_0^{2\pi} |g_n(re^{i\theta}) - g(re^{i\theta})|\,d\theta\right\}$$

$$\leq \sup_{0 \leq r < 1}\left\{\frac{1}{2\pi}\int_0^{2\pi} (\alpha-1)\int_0^1 (1-t)^{\alpha-2}|f_n(tre^{i\theta}) - f(tre^{i\theta})|\,dt\,d\theta\right\}.$$

Let

$$I_n(r) = \frac{1}{2\pi}\int_0^{2\pi}\int_0^1 (1-t)^{\alpha-2}|f_n(tre^{i\theta}) - f(tre^{i\theta})|\,dt\,d\theta.$$

Note that $\int_0^1 (1-t)^{\alpha-3}\,dt < \infty$ since $\alpha > 2$. Hence, given $\epsilon > 0$, there exists δ with $0 < \delta < 1$ and $\int_{1-\delta}^1 (1-t)^{\alpha-3}\,dt < \epsilon/4$. Also, if $f \in S$, it follows from Prawitz's inequality [4, p. 61] and the growth theorem [4, p. 33] that

$$\frac{1}{2\pi}\int_0^{2\pi} |f(re^{i\theta})|\,d\theta \leq \frac{r}{1-r}.$$

Hence

$$\int_{1-\delta}^1 \left\{\int_0^{2\pi} |f_n(tre^{i\theta}) - f(tre^{i\theta})|\,\frac{d\theta}{2\pi}\right\}(1-t)^{\alpha-2}\,dt$$

$$\leq \int_{1-\delta}^{1} \frac{2tr}{1-tr}(1-t)^{\alpha-2}\,dt$$

$$\leq \int_{1-\delta}^{1} 2(1-t)^{\alpha-3}\,dt < \frac{\epsilon}{2}$$

for all $n=1,2,3,\ldots$ and all r with $0 \leq r < 1$. Also, since $f_n \to f$ uniformly on compact subsets of Δ, we can choose a positive integer N such that, for $n \geq N$,

$$\frac{1}{2\pi}\int_0^{2\pi}\int_0^{1-\delta}(1-t)^{\alpha-2}|f_n(tre^{i\theta})-f(tre^{i\theta})|\,dt\,d\theta < \frac{\epsilon}{2}.$$

Thus, given $\epsilon > 0$, there exists a positive integer N such that $I_n(r) < \epsilon$ for all $n \geq N$ and each r in $[0,1)$; that is, $\|g_n - g\|_{H^1} \to 0$, completing the proof of the theorem. □

COROLLARY. *If* $\alpha > 2$, *then* $A^{\alpha-1} \subset (A^{\alpha-1})^{**} = \mathcal{G}_\alpha^* \subset \mathcal{S}^*$.

It is now immediate from the definition of $A^{\alpha-1}$ and Proposition A that, if $\alpha > 2$, $f(z) = \sum_{n=0}^{\infty} c_n z^n \in A^0$, and $g(z) = \sum_{n=1}^{\infty} a_n z^n \in S$, then

$$J_f(g) = \lim_{r \to 1^-} \sum_{n=1}^{\infty} \frac{a_n c_n}{(n+1)^{\alpha-1}} r^n$$

is a continuous linear functional on S. We thus have a large pool of such functionals from which we will now construct an example that is not of integral type. The essential idea is to exhibit a function in the disk algebra whose coefficients grow sufficiently slowly. We require a useful fact from [10, p. 197].

PROPOSITION D. *Let* $0 < \delta < 1$. *Then the power series* $\sum_{n=1}^{\infty}(e^{in\log n}/n^{1/2+\delta})e^{int}$ *converges uniformly to a function* $\varphi_\delta(t)$ *belonging to the Lipschitz class of order* δ *on* $[0, 2\pi]$.

REMARK. The conclusion of Proposition D also holds for the power series $\sum_{n=1}^{\infty}(e^{in\log n}/(n+1)^{1/2+\delta})e^{int}$.

COROLLARY. *Given* γ *with* $0 < \gamma < \frac{1}{2}$, *there exists a function* $f(z) = \sum_{n=1}^{\infty} c_n z^n$ *in* A^0 *such that* $\sum_{n=1}^{\infty}(|c_n|/n^\gamma) = +\infty$.

Proof. Let $\delta = \frac{1}{2} - \gamma$. Let $c_n = e^{in\log n}/(n+1)^{1/2+\delta}$ for $n = 1,2,3,\ldots$. By Proposition D, $f(z) = \sum_{n=1}^{\infty} c_n z^n \in A^0$. Also,

$$\sum_{n=1}^{\infty} \frac{|c_n|}{n^\gamma} = \sum_{n=1}^{\infty} \frac{|e^{in\log n}|}{n^{1/2+\delta} n^\gamma} \cdot \frac{n^{1/2+\delta}}{(n+1)^{1/2+\delta}}$$

$$\geq C \sum_{n=1}^{\infty} \frac{1}{n^{1-\gamma} n^\gamma},$$

where $C = 2^{-1/2-\delta}$. Thus

$$\sum_{n=1}^{\infty} \frac{|c_n|}{n^\gamma} \geq C \sum_{n=1}^{\infty} \frac{1}{n} = +\infty. \qquad \square$$

We state one final proposition [10, p. 87].

PROPOSITION E (Littlewood's Tauberian Theorem). *Suppose*

$$\lim_{r \to 1^-} \sum_{n=0}^{\infty} b_n r^n$$

exists and equals l. Suppose also that $b_n = O(1/n)$. Then $\sum_{n=0}^{\infty} b_n$ converges and $\sum_{n=0}^{\infty} b_n = l$.

THEOREM 2. *There exists a functional $L \in S^*$ and a sequence λ_n of complex numbers such that*

$$L(f) = \sum_{n=1}^{\infty} a_n \lambda_n$$

for $f(z) = \sum_{n=1}^{\infty} a_n z^n$ in S, but $\sum_{n=1}^{\infty} n|\lambda_n| = +\infty$. In particular, L is not of integral type.

Proof. Fix γ, $0 < \gamma < \frac{1}{2}$. Let $c_n = e^{in \log n}/(n+1)^{1-\gamma}$ as in the proof of the corollary to Proposition D. Then

$$f(z) = \sum_{n=0}^{\infty} c_n z^n$$

is in the disk algebra.

Let $\lambda_n = c_n/(n+1)^{1+\gamma}$, and let $\alpha = 2+\gamma$. Note that $|\lambda_n| = 1/(n+1)^2$, $n = 1, 2, 3, \ldots$. By the corollary to Theorem 1 we have $J_f \in S^*$, where J_f is defined by

$$J_f(g) = \lim_{r \to ^-} \sum_{n=1}^{\infty} \frac{a_n c_n r^n}{(n+1)^{\alpha-1}}$$

if $g(z) = \sum_{n=1}^{\infty} a_n z^n \in S$. Hence

$$J_f(g) = \lim_{r \to 1^-} \sum_{n=1}^{\infty} \frac{a_n c_n}{(n+1)^{1+\gamma}} r^n = \lim_{r \to 1^-} \sum_{n=1}^{\infty} \frac{a_n \lambda_n (n+1)^{1+\gamma}}{(n+1)^{1+\gamma}}$$

$$= \lim_{r \to 1^-} \sum_{n=1}^{\infty} a_n \lambda_n r^n.$$

Now $\sum_{n=1}^{\infty} a_n z^n$ in S provides $|a_n| = O(n)$, so that $|a_n \lambda_n| = O(1/n)$. By Proposition E, the functional J_f has the form $J_f(g) = \sum_{n=1}^{\infty} a_n \lambda_n$ for $g(z) = \sum_{n=1}^{\infty} a_n z^n \in \operatorname{sp} S$. Finally,

$$\sum_{n=1}^{\infty} n|\lambda_n| = \sum_{n=1}^{\infty} \frac{n}{(n+1)^2} = +\infty$$

as required. $\square$

In conclusion, we wish to give a description of S^* which "approximates" the duality between A^1, the set of analytic functions whose derivatives have continuous boundary values, and the integral family $\mathfrak{F}_2$.

THEOREM 3. *Let L be an element of S^*. Then there exists $g(z)=\sum_{n=1}^{\infty}\lambda_n z^n$ in A^1 such that*

$$L(f)=L_g(f)=\lim_{r\to 1^-}\sum_{n=1}^{\infty}\lambda_n a_n r^n$$

for each $f(z)=\sum_{n=1}^{\infty}a_n z^n$ in S.

Proof. Let $\lambda_n=L(z^n)$, which is defined since $z^n\in \operatorname{sp} S$ for $n=1,2,3,\ldots$. If $f(z)=\sum_{n=1}^{\infty}a_n z^n\in S$, then $(1/r)f(rz)=f_r(z)\in S$ for $0<r<1$. Also, $f_r\to f$ uniformly on compact subsets of Δ as $r\uparrow 1$. Hence

$$\lim_{r\to 1^-}L(f_r)=L(f). \tag{1}$$

Since f_r is analytic and univalent in a neighborhood of $\bar{\Delta}$, the partial sums of f_r are eventually univalent on $\bar{\Delta}$. That is, for large m,

$$\sum_{n=1}^{m}a_n r^{n-1}z^n\in S \quad\text{and}\quad \lim_{m\to\infty}\sum_{n=1}^{m}a_n r^{n-1}z^n=f_r(z)$$

uniformly on $\bar{\Delta}$. It follows that

$$L(f_r)=\lim_{m\to\infty}L\left(\sum_{n=1}^{m}a_n r^{n-1}z^n\right)=\lim_{m\to\infty}\sum_{n=1}^{m}a_n\lambda_n r^{n-1}=\sum_{n=1}^{\infty}a_n\lambda_n r^{n-1}. \tag{2}$$

Combine (1) and (2) and multiply by r to obtain

$$L(f)=\lim_{r\to 1^-}\sum_{n=1}^{\infty}\lambda_n a_n r^n.$$

We claim $g(z)=\sum_{n=1}^{\infty}\lambda_n z^n$ is in A^1 or, equivalently, $\sum_{n=1}^{\infty}n\lambda_n z^n$ is in the disk algebra A^0.

To see this, consider $k_\zeta(z)=z/(1-\zeta z)^2\in S$ for $|\zeta|\le 1$. If $\zeta\to x$, $|\zeta|<1$, and $|x|=1$, then $k_\zeta\to k_x$ uniformly on compact subsets of Δ. Therefore, $L(k_\zeta)\to L(k_x)$; that is,

$$\lim_{\substack{\zeta\to x\\ |\zeta|<1}}\lim_{r\to 1^-}\sum_{n=1}^{\infty}n\lambda_n(r\zeta)^n$$

exists for every $x\in\Gamma$. Thus $g(z)\in A^1$. □

COROLLARY. *For each element L in S^* there is a function g in A^1 such that $L=L_g$. Through this correspondence we then have*

$$S^*\subset A^1\subset (A^1)^{**}=\mathfrak{F}_2^*.$$

Moreover, the action is essentially coefficient multiplication as in the duality $(A^{\alpha-1})^*=\mathfrak{F}_\alpha$.

MacGregor [9] showed that S is not contained in $\mathfrak{F}_2$ even though every support point of S is, in fact, an element of $\mathfrak{F}_2$. This latter observation follows from the obvious fact that the functions $k_\zeta(z)=z/(1-\zeta z)^2$ with $|\zeta|\le 1$ are elements of $\mathfrak{F}_2$; from the fact (see [2]) that every support point of S, say

$h(z)$, can be expressed in the form $h(z)=q(z)k_\zeta(z)$ for some $\zeta\in\Gamma$, where $q(z)$ is analytic in a neighborhood of $\bar{\Delta}$; and from the fact (see [7]) that each such $q(z)$ is a multiplier of $\mathfrak{F}_2$. Thus Theorem 3 provides additional information on the intimate, complex relationship between S and $\mathfrak{F}_2$.

References

[1] L. Brickman, D. J. Hallenbeck, T. H. MacGregor, and D. R. Wilken, *Convex hulls and extreme points of families of starlike and convex mappings,* Trans. Amer. Math. Soc. 185 (1973), 413–428.

[2] L. Brickman and D. Wilken, *Support points of the set of univalent functions,* Proc. Amer. Math. Soc. 42 (1974), 523–528.

[3] A. Dansereau, *General integral families and multipliers,* Ph.D. dissertation, State University of New York at Albany, 1992.

[4] P. L. Duren, *Univalent functions,* Springer, New York, 1983.

[5] P. L. Duren and Y. J. Leung, *Generalized support points of the set of univalent functions,* J. Analyse Math. 46 (1986), 94–108.

[6] D. H. Hamilton, *Extremal boundary problems,* Proc. London Math. Soc. (3) 56 (1988), 101–113.

[7] R. A. Hibschweiler and T. H. MacGregor, *Multipliers of families of Cauchy-Stieltjes transforms,* Trans. Amer. Math. Soc. 331 (1992), 377–394.

[8] ———, *Bounded analytic functions of Cauchy-Stieltjes integrals,* Rocky Mountain J. Math. 23 (1993), 187–202.

[9] T. H. MacGregor, *Analytic and univalent functions with integral representations involving complex measures,* Indiana Univ. Math. J. 36 (1987), 109–129.

[10] A. Zygmund, *Trigonometric series,* vol. 1, 2nd ed., Cambridge Univ. Press, London, 1959.

I. M. Al-Grouz
College of Science and Technology
P.O. Box 20002
Abu Deis, Jerusalem
Israel

D. R. Wilken
Department of Mathematics
SUNY at Albany
Albany, NY 12222