Invariant Subspaces in Bergman Spaces and the Biharmonic Equation

P. Duren, D. Khavinson,
H. S. Shapiro, & C. Sundberg

1. Introduction

Let $\Omega \subset \mathbf{C}$ be a bounded domain in the complex plane. For $0 < p < \infty$, the *Bergman space* $A^p(\Omega)$ consists of all functions f analytic in Ω for which

$$\|f\|_p^p = \int_\Omega |f(z)|^p \, d\sigma < \infty.$$

Here $d\sigma$ denotes the normalized element of area, so that $\|1\|_p = 1$.

In our previous paper [1] we developed a theory of contractive zero-divisors in $A^p = A^p(\mathbf{D})$ for $1 \le p \le \infty$, where $\mathbf{D}$ is the unit disk. The general approach through an extremal problem had been introduced by Hedenmalm [3] for the case $p = 2$, but new methods were needed for other values of p. We exploited the positivity of the biharmonic Green function in the disk, and we proved the regularity of the canonical divisors by representing them in terms of the reproducing kernels of certain weighted A^2 spaces.

Our purpose is now to generalize the theory in various directions. It turns out that the arguments in [1], if suitably arranged, actually give a theory of contractive zero-divisors in A^p spaces with $0 < p < 1$. Furthermore, some of the theory applies to arbitrary invariant subspaces (under multiplication by polynomials) and is not restricted to the special invariant subspaces defined by zero-sets. Hedenmalm [3] has already pointed this out for the case $p = 2$. The key to our more general results is an integral formula involving the biharmonic Green function, which has the advantage of circumventing Hedenmalm's boundary-value problem and the consequent need for smooth boundary values. Finally, we show that most of the results extend to Bergman spaces $A^p(\Omega)$ over *simply connected* Jordan domains with analytic boundary. The paper concludes with some special observations and remarks.

2. Background

Again let $A^p(\Omega)$ be the Bergman space over a bounded domain Ω, with $0 < p < \infty$. It is easy to see that even for $p < 1$, each point-evaluation is a bounded

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linear functional. Indeed, because $|f|^p$ is subharmonic and so has the local sub-mean-value property, it is easily shown that

$$|f(\zeta)| \le C d(\zeta, \partial\Omega)^{-2/p} \|f\|_p, \quad \zeta \in \Omega,$$

where C is a constant depending only on p and $d(\zeta, \partial\Omega)$ denotes the distance from the point ζ to the boundary of Ω.

An *invariant subspace* of $A^p(\Omega)$ is a closed subspace I such that $zI \subset I$; in other words, $zf(z)$ is in I whenever f is in I. It follows that $Qf \in I$ for every polynomial Q. The *common zero-set* $Z(I)$ is the set of points z in Ω such that $f(z) = 0$ for all $f \in I$. One example of an invariant subspace is the set of all functions $f \in A^p(\Omega)$ that vanish at least on a given set of points with prescribed multiplicity or higher. Another example is the closure of the set of all polynomial multiples of a given function in $A^p(\Omega)$.

The theory of contractive zero-divisors in $A^p = A^p(\mathbf{D})$, as developed in [3] and [1] for $1 \le p < \infty$, will now be briefly described. Given an A^p zero-set $\{\zeta_j\}$ in $\mathbf{D} \setminus \{0\}$, let N^p be the set of functions in A^p that vanish on $\{\zeta_j\}$. Consider the extremal problem of maximizing $|f(0)|$ among all $f \in N^p$ with $\|f\|_p = 1$. There is a unique extremal function G with $G(0) > 0$, called the *canonical divisor* of the zero-set $\{\zeta_j\}$.

The canonical divisor has some remarkable properties. It is an expansive multiplier: $\|Gf\|_p \ge \|f\|_p$ for all $f \in A^p$. It has no extraneous zeros: it vanishes on $\{\zeta_j\}$ to exactly the prescribed multiplicity and has no additional zeros. It is a contractive zero-divisor: $\|f/G\|_p \le \|f\|_p$ for all $f \in N^p$. The canonical divisor G_α of a single point $\alpha \in \mathbf{D}$ is

$$G_\alpha(z) = C \frac{z - \alpha}{1 - \bar{\alpha} z} \left\{ 1 + \frac{p}{2} \left(1 + \bar{\alpha} \frac{z - \alpha}{1 - \bar{\alpha} z} \right) \right\}^{2/p}, \tag{1}$$

where

$$C = -\frac{\bar{\alpha}}{|\alpha|} \left[1 + \frac{p}{2} (1 - |\alpha|^2) \right]^{-1/p}. \tag{2}$$

For a *finite* zero-set $\{\zeta_j\}$, the canonical divisor is

$$G(z) = J(0,0)^{-1/p} B(z) J(z,0)^{2/p}, \tag{3}$$

where $J(z, \zeta)$ is the kernel function of the Bergman space A^2_w with weight $w = |B|^p$, and B is the finite Blaschke product associated with $\{\zeta_j\}$. In particular, $J(z, 0) \ne 0$ in $\bar{\mathbf{D}}$. It follows that the canonical divisor of a finite zero-set has an analytic continuation to a larger disk.

The proof of the expansive property begins with a variational argument to show that $|G|^p - 1$ is orthogonal to all bounded harmonic functions in the disk. For a finite zero-set, the regularity of G at the boundary allows the definition of a function ϕ satisfying $\Delta\phi = |G|^p - 1$ in $\mathbf{D}$ and $\phi = 0$ on $\mathbf{T} = \partial\mathbf{D}$, where Δ denotes the Laplacian. It is then shown that $\partial\phi/\partial n = 0$ on $\mathbf{T}$ and $\phi > 0$ in $\mathbf{D}$. As a consequence, the integral formula

$$\int_{\mathbf{D}} (|G|^p - 1)|f|^p \, d\sigma = \pi \int_{\mathbf{D}} \int_{\mathbf{D}} \Gamma(z, \zeta) \Delta(|G(z)|^p) \Delta(|f(\zeta)|^p) \, d\sigma(z) \, d\sigma(\zeta) \tag{4}$$

holds for every function $f \in A^p$, where

$$\Gamma(z,\zeta)=\frac{1}{16\pi}\left\{|z-\zeta|^2\log\left|\frac{z-\zeta}{1-\bar{\zeta}z}\right|^2+(1-|z|^2)(1-|\zeta|^2)\right\} \tag{5}$$

is the biharmonic Green function of $\mathbf{D}$. The expansive property is an immediate consequence of (4), since $\Gamma(z,\zeta)>0$ in $\mathbf{D}$.

3. The Integral Formula

Our first objective is to prove the integral formula (4) in a more general setting which may not permit the intervention of Hedenmalm's boundary-value problem because there will be no guarantee that the relevant extremal function has a smooth extension to the boundary.

Let I be an invariant subspace of A^p, $0<p<\infty$, and suppose that 0 is not in the common zero-set $Z(I)$. Consider the extremal problem

$$\max_{f\in I,\,\|f\|_p=1} |f(0)|. \tag{6}$$

A passage to the equivalent extremal problem

$$\min_{f\in I,\, f(0)=1} \|f\|_p \tag{7}$$

shows for $1<p<\infty$ that an extremal function of (6) exists and is unique up to rotation. This is seen from the general fact that a closed convex set in a uniformly convex Banach space has a unique element of minimum norm. The existence of an extremal function is proved for $p=1$ by exploiting the duality of A^1 with the Bloch spaces. Uniqueness follows from the fact that A^1 is strictly convex: $\|f\|=\|g\|=1$ implies $\|f+g\|<2$ unless $f=g$. Let G be an extremal function for the problem (6), normalized by $G(0)>0$. This normalized extremal function exists and is unique for $1\le p<\infty$, but in an arbitrary invariant subspace neither existence nor uniqueness is clear for $0<p<1$. Existence can be proved by a normal family argument if the subspace is weakly closed, containing locally uniform limits of sequences bounded in norm.

The variational argument of [1] still applies. If f is an extremal function for the problem (7), then $\|f^*\|_p\ge\|f\|_p$ for each function $f^*=f+\lambda Qf$, where $\lambda\in\mathbf{C}$ and Q is a polynomial with $Q(0)=0$. Note that $f^*\in I$ and $f^*(0)=1$. An easy calculation (cf. [1]) leads to the property

$$\int_{\mathbf{D}} |G|^p Q\, d\sigma=0, \quad Q(0)=0.$$

For arbitrary polynomials Q this implies

$$\int_{\mathbf{D}} (|G|^p-1)Q\, d\sigma=0.$$

It follows that

$$\int_{\mathbf{D}} (|G|^p - 1) u \, d\sigma = 0 \quad \text{for all } u \in h^\infty, \tag{8}$$

the space of bounded harmonic functions on **D**, since h^∞ is the weak-star closure of the harmonic polynomials.

The space of bounded analytic functions in **D** is called H^∞.

THEOREM 1. *Let I be an invariant subspace of A^p for some p, $0 < p < \infty$, and let G be an extremal function for the problem* (6). *Then the integral formula* (4) *holds for every polynomial f. Also, $\|Gf\|_p \geq \|f\|_p$ for every function $f \in H^\infty$.*

The proof relies on Green's formula

$$\int_\Omega (u \Delta v - v \Delta u) \, dA = \int_{\partial\Omega} \left(u \frac{\partial v}{\partial n} - v \frac{\partial u}{\partial n} \right) ds$$

and makes use of certain mollifiers. Let $C_0^\infty(\mathbf{D})$ be the set of infinitely differentiable functions with compact support in **D**. Three lemmas will be needed.

LEMMA 1. *If $F \in L^1$ and $F \perp h^\infty$, then there is a sequence of functions $\varphi_j \in C_0^\infty(\mathbf{D})$ such that*

(a) $\|\Delta \varphi_j - F\|_1 \to 0$ *and*

(b) $\|\varphi_j - \Phi\|_1 \to 0$ *as* $j \to \infty$,

where

$$\Phi(z) = \int_{\mathbf{D}} F(\zeta) \Delta_\zeta \Gamma(z, \zeta) \, dA. \tag{9}$$

LEMMA 2. *For each positive $\epsilon < 1$, there is a function $\psi_\epsilon \in C_0^\infty(\mathbf{D})$ with the properties $0 \leq \psi_\epsilon(z) \leq 1$, $\psi_\epsilon(z) = 1$ for $|z| \leq 1 - \epsilon$, $|\nabla \psi_\epsilon(z)| \leq c/\epsilon$, and $|\Delta \psi_\epsilon(z)| \leq c/\epsilon^2$, where c is an absolute constant.* (Here ∇ denotes the gradient.)

LEMMA 3. *If $F \in L^1$, then for each point $z \in \mathbf{D}$,*

$$\int_{\mathbf{D}} F(\zeta) \Delta_\zeta (\psi_\epsilon(\zeta) \Gamma(z, \zeta)) \, dA \to \Phi(z) \quad \text{as } \epsilon \to 0.$$

The notation $F \perp h^\infty$ means that F is orthogonal to every function $u \in h^\infty$: $\int F u \, d\sigma = 0$. Lemma 2 is a standard fact about mollifiers, so the proof will be omitted (see e.g. [4, p. 25]). Deferring the proofs of Lemmas 1 and 3, let us show first how they imply the theorem.

Proof of theorem. Let $F = |G|^p - 1$ and use Green's formula to write

$$\int_{\mathbf{D}} F(\zeta) \Delta_\zeta (\psi_\epsilon(\zeta) \Gamma(z, \zeta)) \, dA(\zeta) = \int_{\mathbf{D}} \psi_\epsilon(\zeta) \Gamma(z, \zeta) \Delta F(\zeta) \, dA(\zeta). \tag{10}$$

Here it is important to observe that $\Delta(|G|^p) = p^2 |G|^{p-2} |G'|^2$ is locally integrable even for $0 < p < 2$, although it has singularities at the zeros of G. In using Green's formula, one must place small "safety circles" around the (finitely many) zeros of G that lie in the support of ψ_ϵ, and then observe that the boundary contribution of each circle tends to zero as its radius shrinks to zero. The details will be omitted. Now let $\epsilon \to 0$ in (10) and use Lemma 3 to conclude that

$$\Phi(z)=\int_{\mathbf{D}} \Gamma(z,\zeta)\Delta F(\zeta)\,dA, \tag{11}$$

where Φ is defined by (9). This justifies a formal application of Green's formula to (9). Next let $s=|f|^p$, where f is a polynomial. Then by property (8) of the extremal function G, Lemma 1, and the alternate form (11) of Φ,

$$\begin{aligned}\int_{\mathbf{D}} Fs\,dA &= \lim_{j\to\infty}\int_{\mathbf{D}} (\Delta\varphi_j)s\,dA=\lim_{j\to\infty}\int_{\mathbf{D}} \varphi_j\Delta s\,dA\\ &=\int_{\mathbf{D}} \Phi\Delta s\,dA=\int_{\mathbf{D}} \Delta s(z)\int_{\mathbf{D}} \Delta F(\zeta)\Gamma(z,\zeta)\,dA(\zeta)\,dA(z),\end{aligned}$$

which establishes the integral formula (4). It follows that $\|Gf\|_p \geq \|f\|_p$ for every polynomial f, and an approximation extends the inequality to all bounded analytic functions. This concludes the proof of Theorem 1. □

Proof of Lemma 1. If $u \in h^\infty$, an application of Green's formula shows that $\int u\Delta\varphi\,d\sigma=0$ for all $\varphi \in C_0^\infty$. The converse follows from Weyl's lemma (see [2, p. 103]). Thus h^∞ is precisely the set of functions $u \in L^\infty$ such that $\int u\Delta\varphi\,d\sigma=0$ for all $\varphi\in C_0^\infty$. Consequently, the hypothesis of the lemma asserts that $F \perp u$ whenever $u \in L^\infty$ and $u \perp \Delta\varphi$ for all $\varphi \in C_0^\infty$. In other words, if Λ is a bounded linear functional on L^1 and $\Lambda(\Delta\varphi)=0$ for all $\varphi\in C_0^\infty$, then $\Lambda(F)=0$. By the Hahn–Banach theorem, this says that F is in the L^1-closure of $\{\Delta\varphi : \varphi \in C_0^\infty\}$, as asserted.

To prove (b), observe that each $\varphi \in C_0^\infty$ has the representation

$$\varphi(z)=\int_{\mathbf{D}} \Delta^2\varphi(\zeta)\Gamma(z,\zeta)\,dA=\int_{\mathbf{D}} \Delta\varphi(\zeta)\Delta_\zeta\Gamma(z,\zeta)\,dA$$

in terms of the biharmonic Green function $\Gamma(z,\zeta)$, where Green's formula has been applied. Thus, in view of (9),

$$\varphi(z)-\Phi(z)=\int_{\mathbf{D}} [\Delta\varphi(\zeta)-F(\zeta)]\Delta_\zeta\Gamma(z,\zeta)\,dA,$$

and so

$$\begin{aligned}\|\varphi-\Phi\|_1 &\leq \int_{\mathbf{D}}\int_{\mathbf{D}} |\Delta\varphi(\zeta)-F(\zeta)||\Delta_\zeta\Gamma(z,\zeta)|\,dA(\zeta)\,dA(z)\\ &\leq C\|\Delta\varphi-F\|_1\end{aligned}$$

by Fubini's theorem and a direct calculation based on expression (5) for $\Gamma(z,\zeta)$. This shows that $\|\varphi_j-\Phi\|_1\to 0$ whenever $\|\Delta\varphi_j-F\|_1\to 0$, completing the proof of Lemma 1. □

Proof of Lemma 3. The idea is simply to use the standard formula

$$\Delta(uv) = u\Delta v + v\Delta u + 2\nabla u \cdot \nabla v.$$

The two error terms in the corresponding integral can be shown to approach zero as $\epsilon \to 0$. Specifically, one must observe (either by direct calculation or by considering the boundary behavior of the biharmonic Green function) that for fixed $z \in \mathbf{D}$ the estimates

$$|\nabla_\zeta \Gamma(z,\zeta)| \le C(1-|\zeta|) \quad \text{and} \quad |\Gamma(z,\zeta)| \le C(1-|\zeta|)^2$$

hold as ζ tends to the boundary. Thus, in view of Lemma 2, the two error terms reduce to integrals of bounded functions over thin annuli of width less than ϵ, and so they tend to zero with ϵ. The details are straightforward and will be omitted. □

It may be remarked that the integral formula (4) actually remains valid when $|f|^p$ is replaced by any smooth function whose Laplacian is integrable and bounded away from a finite number of points. The expansive property $\int |Gf|^p \, d\sigma \ge \int |f|^p \, d\sigma$ remains valid when $|f|^p$ is replaced by an arbitrary bounded positive subharmonic function.

4. Canonical Zero-divisors for $0 < p < 1$

Our next objective is to show how the theory of canonical divisors, developed in [1] for A^p spaces with $1 \le p < \infty$, may be extended to $0 < p < 1$.

First consider a singleton zero-set $\{\alpha\}$. By a direct calculation it can be seen that the function G_α defined by the formulas (1) and (2) still has the property $(|G_\alpha|^p - 1) \perp h^\infty$ when $p < 1$. It suffices to show that $\int |G_\alpha|^p Q \, d\sigma = Q(0)$ for every polynomial Q. But the substitution

$$w = \frac{z-\alpha}{1-\bar{\alpha}z}, \quad z = \frac{w+\alpha}{1+\bar{\alpha}w},$$

with

$$d\sigma(z) = \frac{(1-|\alpha|^2)^2}{|1+\bar{\alpha}w|^4} \, d\sigma(w),$$

transforms the integral to

$$(1-|\alpha|^2)^2 |C|^p \int_{\mathbf{D}} |k(w)|^2 |w|^p Q\left(\frac{w+\alpha}{1+\bar{\alpha}w}\right) d\sigma(w),$$

where $k(w) = [1+(p/2)(1+\bar{\alpha}w)](1+\bar{\alpha}w)^{-2}$ and C is defined by (2). Now let $k(w) = \sum b_n w^n$ and

$$f(w) = k(w) Q\left(\frac{w+\alpha}{1+\bar{\alpha}w}\right) = \sum_{n=0}^{\infty} c_n w^n,$$

and integrate by polar coordinates to get

$$\int_{\mathbf{D}} |G_\alpha|^p Q \, d\sigma = (1-|\alpha|^2)^2 |C|^p \sum_{n=0}^{\infty} \frac{2}{2n+p+2} \overline{b_n} c_n = (1-|\alpha|^2)^2 |C|^p f(-\alpha),$$

since $b_n = (n+p/2+1)(-\bar{\alpha})^n$. But in view of (2) and the definition of k, this last expression reduces to $Q(0)$, as claimed above.

Because G_α has the property (8), the proof of Theorem 1 gives the inequality $\|G_\alpha Q\|_p \geq \|Q\|_p$ for all polynomials Q. Since now $G_\alpha \in L^\infty$, it follows that $\|G_\alpha f\|_p \geq \|f\|_p$ for all $f \in A^p$. In fact, the integral formula (4) applies and shows that $\|G_\alpha f\|_p > \|f\|_p$ unless f is constant. It now follows that G_α is a contractive divisor, because obviously it has an analytic extension to a larger disk without additional zeros. Specifically, if $f \in A^p$ and $f(\alpha) = 0$, then $f/G_\alpha \in A^p$ and $\|f/G_\alpha\|_p < \|f\|_p$ unless f is a constant multiple of G_α.

On the other hand, it follows from the subharmonic property of $|f|^p$ that $|f(0)| < \|f\|_p$ unless f is constant. In particular, if $f \in A^p$ and $f(\alpha) = 0$, then

$$|f(0)/G_\alpha(0)| < \|f/G_\alpha\|_p \leq \|f\|_p,$$

so $|f(0)| < |G_\alpha(0)|$ if $\|f\|_p = 1$ and $f(\alpha) = 0$, unless f is a constant multiple of G_α. This shows directly that G_α is the *unique* solution to the extremal problem (6) with $f(0) > 0$, where $I = \{f \in A^p : f(\alpha) = 0\}$.

Next observe that for an *arbitrary* A^p zero-set ($0 < p < 1$), a canonical divisor can have no extraneous zeros. The argument using G_α is the same as for $p \geq 1$; see [1] or [3]. Knowing that the canonical divisor of a *finite* A^p zero-set has no extraneous zeros, we can apply the proof of Theorem 4 in [1] to conclude that the representation formula (3) in terms of the kernel function is still valid. The argument in [1] requires only that $p > 0$.

In particular, the canonical divisor of a finite zero-set is unique even for $0 < p < 1$, and it has a nonvanishing analytic extension across the unit circle. Thus it has the contractive property $\|f/G\|_p \leq \|f\|_p$ for every f in N^p. Finally, the truncation argument (see [1, §5]) used for $1 \leq p < \infty$ applies equally well to the case $0 < p < 1$. More precisely, let G be the canonical divisor of an infinite zero-set $\{\zeta_j\}$, and let G_n be the canonical divisor for the set $\{\zeta_1, \ldots, \zeta_n\}$. Then $G_n(z) \to G(z)$ locally uniformly, and the contractive property of G_n extends to that of G. Thus the canonical divisor of an arbitrary A^p zero-set is contractive. This now implies that the canonical divisor is unique up to rotation. Indeed, if $f \in N^p$ and $\|f\|_p = 1$, then $|f(0)/G(0)| < \|f/G\|_p \leq \|f\|_p = 1$, so that $|f(0)| < |G(0)|$ unless f is a constant multiple of G.

5. General Domains

Much of the theory of canonical divisors can be extended to arbitrary simply connected domains. Let us assume for simplicity that $\Omega \subset \mathbf{C}$ is a domain bounded by a Jordan curve Γ. Fix a point $z_0 \in \Omega$, and let $z = \varphi(w)$ be a conformal mapping of $\mathbf{D}$ onto Ω with $\varphi(0) = z_0$ and $\varphi'(0) > 0$. Let $w = \psi(z)$ be its inverse. The linear operator T defined by

$$(Tf)(w) = f(\varphi(w))\varphi'(w)^{2/p}$$

is easily seen to be an isometry of $A^p(\Omega)$ onto $A^p(\mathbf{D})$.

Given an $A^p(\Omega)$ zero-set $\{\zeta_j\}$ not containing z_0, consider the extremal problem

$$\max_{f \in N^p(\Omega), \|f\|_p = 1} |f(z_0)|, \tag{12}$$

where $N^p(\Omega)$ is the set of functions in $A^p(\Omega)$ that vanish on $\{\zeta_j\}$. Let $\zeta_j = \varphi(\omega_j\}$, so that $0 \notin \{\omega_j\}$, and let $N^p(\mathbf{D})$ be the set of functions in $A^p(\mathbf{D})$ that vanish on $\{\omega_j\}$. It is easily seen that $f \in N^p(\Omega)$ if and only if $Tf \in N^p(\Omega)$. A *canonical divisor* of the zero-set $\{\zeta_j\}$ in $A^p(\Omega)$ is a solution of the extremal problem (12) with $f(z_0) > 0$. By the strict convexity of $A^p(\Omega)$, the canonical divisor is unique if $1 \le p < \infty$. It is also unique for $0 < p < 1$, as we shall see presently.

THEOREM 2. *For $0 < p < \infty$, let $\{\zeta_j\}$ be an $A^p(\Omega)$ zero-set, and let $\{\omega_j\}$ be its preimage under the conformal mapping φ of $\mathbf{D}$ onto Ω with $\varphi(0) = z_0$, where $z_0 \notin \{\zeta_j\}$. Then $\{\omega_j\}$ is an $A^p(\mathbf{D})$ zero-set. Furthermore, a function F is a canonical divisor of $\{\zeta_j\}$ in $A^p(\Omega)$ with respect to z_0 if and only if $G = TF$ is a canonical divisor of $\{\omega_j\}$ in $A^p(\mathbf{D})$ with respect to the origin.*

COROLLARY. *A canonical divisor of a zero-set in $A^p(\Omega)$ is unique for $0 < p < \infty$.*

The corollary follows from the uniqueness, already established, of a canonical divisor in $A^p(\mathbf{D})$ for $0 < p < \infty$. □

Proof of theorem. The operator T is an isometry of $N^p(\Omega)$ onto $N^p(\mathbf{D})$, so it carries the competing functions for the problem (12) onto those for the corresponding extremal problem in the disk. Since $Tf(0) = f(z_0)\varphi'(0)^{2/p}$, it is clear that T maps extremal functions to extremal functions. □

Theorem 2 allows properties of canonical divisors in $A^p(\Omega)$ to be deduced directly from corresponding properties of canonical divisors for the disk. For instance, a canonical divisor in $A^p(\Omega)$ can have no extraneous zeros. For a *finite* zero-set, it has an analytic continuation beyond the boundary, without additional zeros, provided that $\Gamma = \partial\Omega$ is an analytic Jordan curve.

More generally, let I be an arbitrary invariant subspace of $A^p(\Omega)$ with $z_0 \notin Z(I)$, and let F be the extremal function (unique if $1 \le p < \infty$) for the problem

$$\max_{f \in I, \|f\|_p = 1} |f(z_0)| \tag{13}$$

with the normalization $F(z_0) > 0$.

THEOREM 3. *For $0 < p < \infty$, let I be an invariant subspace of $A^p(\Omega)$ with $z_0 \notin Z(I)$, and let F be a normalized extremal function with respect to the point z_0. Let $J = \{Tf : f \in I\}$ and let $G = TF$. Then J is an invariant subspace of $A^p(\mathbf{D})$ with $0 \notin Z(J)$, and G is a normalized extremal function of J with respect to the origin. Furthermore, F has the quasi-expansive property $\|Ff\|_p \ge \|\varphi'\|_\infty^{-2/p} \|f\|_p$ for $f \in H^\infty(\Omega)$, where φ is the conformal mapping of $\mathbf{D}$ onto Ω with $\varphi(0) = z_0$ and $\varphi'(0) > 0$.*

Proof. In order to prove that J is an invariant subspace, it must be shown that $h(z)=zg(z)$ belongs to J for each g in J. But $(T^{-1}h)(z)=\psi(z)(T^{-1}g)(z)$, which belongs to the invariant subspace I because $T^{-1}g\in I$ and ψ can be approximated by polynomials uniformly in $\bar{\mathbf{D}}$. Since $g(0)=\varphi'(0)^{2/p}f(z_0)$ whenever $g=Tf$, it is clear that Tf is a normalized extremal function in J with respect to the origin whenever f is a normalized solution to (13).

Recall now that $\|Gg\|_p\geq\|g\|_p$ for all $g\in H^\infty(\mathbf{D})$, by Theorem 1. Thus, for $G=TF$ and $g=Tf$, we have

$$\|f\|_p=\|g\|_p\leq\|Gg\|_p=\|T^{-1}(Gg)\|_p=\|Ff\psi'^{-2/p}\|_p\leq\|Ff\|_p\|\varphi'\|_\infty^{2/p},$$

as asserted, under the assumption that $\|\varphi'\|_\infty<\infty$. □

Note that the quasi-expansive property of Theorem 3 is meaningful only when $\|\varphi'\|_\infty<\infty$. This is the case, for instance, when Ω has analytic boundary. It is interesting to ask whether the quasi-expansive property $\|Ff\|_p\geq C\|f\|_p$ remains valid for domains Ω with $\|\varphi'\|_\infty=\infty$.

Another approach to the quasi-expansive property is through a generalized form of the integral formula (4). If F is an extremal function for the problem (13), the integral formula becomes

$$\int_\Omega(|F(z)|^p-H(z,z_0))|f(z)|^p\,d\sigma(z)$$
$$=A\int_\Omega\int_\Omega\Gamma(z,\zeta)\Delta(|F(z)|^p)\Delta(|f(\zeta)|^p)\,d\sigma(z)\,d\sigma(\zeta), \tag{14}$$

where A is the area of Ω, $\Gamma(z,\zeta)$ is the biharmonic Green function of Ω, and $H(z,\zeta)$ is its harmonic kernel function, or the reproducing kernel for harmonic functions analogous to the Bergman kernel for analytic functions. If $H(z,z_0)\geq c$ for some constant $c>0$ independent of $z\in\Omega$, and if $\Gamma(z,\zeta)\geq 0$, then it follows from (14) that F has the quasi-expansive property.

6. Concluding Remarks

The contractive property of the canonical divisor G in $A^p=A^p(\mathbf{D})$ says that $\|f/G\|_p\leq\|f\|_p$ for all $f\in N^p$. The question arises whether $\|f/G\|_p\geq c\|f\|_p$ for some constant $c>0$, or equivalently whether $\|fG\|_p\leq c\|f\|_p$ for some positive constant c and all $f\in A^p$. In other words, is G a *multiplier* in A^p?

If the zero-set $\{\zeta_j\}$ is not a Blaschke sequence, then G is not a multiplier. Indeed, G must vanish on $\{\zeta_j\}$, and it is well known that the multipliers of A^p are precisely the bounded functions. A proof that the multipliers are bounded appears for instance in [11], but the following simple argument may be of interest. Recall first that each point evaluation is a bounded linear functional:

$$|f(\zeta)|\leq\lambda(\zeta)\|f\|_p,\quad \zeta\in\mathbf{D},\ f\in A^p.$$

If $\|gf\|\leq c\|f\|$ for all $f\in A^p$, then $\|g^nf\|\leq c^n\|f\|$, $n=1,2,\ldots$. Thus

$$|g(\zeta)|^n|f(\zeta)|\le\lambda(\zeta)\|g^nf\|\le\lambda(\zeta)c^n\|f\|,\quad f\in A^p.$$

Now choose $f\in A^p$ with $f(\zeta)\ne 0$, take nth roots, and let $n\to\infty$ to conclude that $|g(\zeta)|\le c$.

Whenever G is not a multiplier of A^p, it follows from the closed graph theorem that $Gf\notin A^p$ for some $f\in A^p$.

Suppose now that $\{\zeta_j\}$ is a Blaschke sequence with Blaschke product B, where $B(0)>0$, and let G_p be its canonical divisor in A^p. Bill Cohn raised the question whether $G_p\to B$ as $p\to\infty$. To see that this is true, write $G_p=F_pB$, so that $F_p(z)\ne 0$ in $\mathbf{D}$. The inequality $|G_p(z)|\le(1-|z|)^{-2/p}$ holds because G_p is an A^p function of unit norm. For fixed R, $0<R<1$, suppose $|z|\le R<r<1$ and express the harmonic function $\log|F_p(z)|$ by the Poisson formula over the circle $|\zeta|=r$. The inequality $|F_p(\zeta)|\le(1-|\zeta|)^{-2/p}|B(\zeta)|^{-1}$ then shows that $\log|F_p(z)|\le-(2/p)\log(1-r)+o(1)$, where "$o(1)$" indicates a term tending to zero uniformly in $|z|\le R$ as $r\to 1$. Choosing r near 1 and then letting $p\to\infty$, we conclude that $\limsup_{p\to\infty}|F_p(z)|\le 1$ uniformly in $|z|\le R$. On the other hand, the function $B/\|B\|_p$ is admissible for the extremal problem that generates the canonical divisor G_p, so $G_p(0)\ge B(0)/\|B\|_p$, and $F_p(0)\ge 1/\|B\|_p\to 1$ as $p\to\infty$. Thus $F_p(z)\to 1$, and it follows that $G_p(z)\to B(z)$ uniformly in $|z|\le R$ as $p\to\infty$.

Finally, there are a number of open questions. For instance, let I and J be general invariant subspaces of A^p with $0\notin Z(I)\cup Z(J)$, and let F and G be the respective extremal functions for the problem (6). If $I\subset J$, is it true that $\|Ff\|_p\ge\|Gf\|_p$ for all $f\in H^\infty$? This generalizes a question raised by Hedenmalm for canonical divisors of zero-sets.

A more basic problem is to describe all invariant subspaces of A^p together with their lattice structure. According to Beurling's well-known description, the invariant subspaces of the Hardy spaces H^p are all singly generated by inner functions. For the Bergman spaces it is not even known whether the invariant subspaces determined by infinite zero-sets are generated by the corresponding canonical divisors.

A related problem is to describe the invariant subspaces whose common zero-set $Z(I)$ is empty. One example is the proper subspace generated by a singular inner function with mass concentrated at one point. More generally, it is known that a singular inner function is cyclic in A^p if and only if its associated singular measure μ puts no mass on any Carleson thin set. (See [11] for a proof and further references. The necessity of the condition was found by H. S. Shapiro [9], while the sufficiency was proved independently at about the same time by Roberts [8] and Korenblum [5]. The first version of Roberts' proof was refined and circulated informally by J. H. Shapiro [10].) In other words, a singular inner function generates a proper invariant subspace I with $Z(I)=\phi$ if μ puts positive mass on some Carleson set.

The canonical extremal function, defined by the problem (6), can actually be calculated for the invariant subspace $[S]$ generated by the atomic singular inner function

$$S(z) = \exp\left\{-\frac{1+z}{1-z}\right\}.$$

As we shall see, the extremal function is unique (under the assumption that $G(0) > 0$) even for $0 < p < 1$; it is

$$G(z) = (1+p)^{-1/p} S(z)\left(1 + \frac{p}{1-z}\right)^{2/p}. \tag{15}$$

For a proof, observe first that the finite Blaschke products

$$B_n(z) = \left(\frac{(1-1/n)-z}{1-(1-1/n)z}\right)^n$$

converge to $S(z)$ as $n \to \infty$, uniformly on compact subsets of the disk. The canonical divisor of $[B_n]$ is

$$G_n(z) = (1+p(1-1/2n))^{-1/p} B_n(z)\left(1 + p\frac{1-1/2n}{1-(1-1/n)z}\right)^{2/p}. \tag{16}$$

Indeed, by equation (3) (see also [1]),

$$G_n(z) = J(0,0)^{-1/p} B_n(z)\, J(z,0)^{2/p},$$

where $J(z,\zeta)$ is the kernel function of the Bergman space A_w^2 with weight $w = |B_n|^p$. But $|B_n|^p = |b_n|^{np}$, where b_n is the simple Blaschke factor with a zero at $1-1/n$, so again by (3) the canonical divisor of $[b_n]$ in A^{np} is

$$g_n(z) = J(0,0)^{-1/p} b_n(z)\, J(z,0)^{2/p}.$$

On the other hand, the function g_n is also given by the formulas (1) and (2), with $\alpha = 1-1/n$ and p replaced by np. This allows the calculation of $J(z,0)$ and leads to the expression (16) for $G_n(z)$. Since $G_n(z) \to G(z)$ locally uniformly as $n \to \infty$, it is reasonable to expect G to be the canonical extremal function of $[S]$.

To prove this we will show first that $[G] = [S]$. It is clear that $S \in [G]$ and hence that $[S] \subset [G]$, since the function $(1 + p/(1-z))^{-1}$ is bounded in $\mathbf{D}$. To see that $G \in [S]$, observe first that Fatou's lemma gives $\|G\|_p \le \liminf_{n\to\infty} \|G_n\|_p = 1$, which implies that

$$\int_{\mathbf{D}} |S(z)|^p |1-z|^{-2}\, d\sigma < \infty. \tag{17}$$

Now let $Q_n(z) = 1 + z + \cdots + z^n$ and observe that $1 + pQ_n(z) \ne 0$ in $\mathbf{D}$, so that $S(1+pQ_n)^{2/p} \in [S]$. Since

$$(1+p)^{-1/p} S(z)(1 + pQ_n(z))^{2/p} \to G(z)$$

pointwise in $\mathbf{D}$, the Lebesgue dominated convergence theorem and (17) allow us to conclude that the convergence holds in the A^p norm. Thus $G \in [S]$, and we have shown that $[G] = [S]$.

To conclude that G is the canonical extremal function of $[S]$, we need only show that $\|G\|_p = 1$ and $G(0) = M$, where $M = \sup_Q |Q(0)S(0)|$ and the supremum extends over all polynomials Q with $\|QS\|_p = 1$. Observe that, by definition of the canonical divisor, $G_n(0) = \sup_Q |Q(0)B_n(0)|$ for polynomials Q with $\|QB_n\|_p = 1$. Since $\|QB_n - QS\|_p \to 0$ for each polynomial Q, it is easy to see that $M \le \lim_{n\to\infty} G_n(0) = G(0)$. On the other hand, $G \in [S]$ implies $G(0) \le M\|G\|_p$. Hence $\|G\|_p = 1$ and $G(0) = M$, which completes the proof that the canonical extremal function of $[S]$ is given by (15).

For $0 < p < 1$ we do not know in general that an invariant subspace has a *unique* extremal function, but this is true for $[S]$ because it is a "limit" of the zero-subspaces $[B_n]$. In fact, for $0 < p < \infty$ the function G defined by (15) is a contractive divisor for $[S]$, meaning that $\|f/G\|_p \le \|f\|_p$ for all $f \in [S]$. To see this, suppose first that $f = SQ$ for some polynomial Q. Note that $\|B_n Q/G_n\|_p \le \|B_n Q\|_p$, since G_n is a contractive divisor for $[B_n]$, as shown in [1]. (See also Section 2 above.) But $B_n(z)Q(z)/G_n(z) \to f(z)/G(z)$ for each $z \in \mathbf{D}$, so Fatou's lemma gives

$$\|f/G\|_p \le \liminf_{n\to\infty} \|B_n Q/G_n\|_p \le \lim_{n\to\infty} \|B_n Q\|_p = \|SQ\|_p = \|f\|_p.$$

To extend this to arbitrary $f \in [S]$, let $f_n = SQ_n \to f$ in A^p norm, where $\{Q_n\}$ is some sequence of polynomials. Then

$$\|f/G\|_p \le \liminf_{n\to\infty} \|f_n/G\|_p \le \lim_{n\to\infty} \|f_n\|_p = \|f\|_p.$$

Uniqueness of the extremal function for $[S]$ is now immediate. If H is another extremal function, then $H(0) = G(0)$; while $\|H/G\|_p \le \|H\|_p = 1$, since $H \in [S]$. But $|H/G|^p$ is subharmonic, so it must be constant. Thus $H = G$.

The results we have obtained for the Bergman spaces have a curious analogy in the Dirichlet space D, consisting of all functions $f(z) = \sum a_n z^n$ analytic in the disk for which

$$\|f\|_D^2 = \sum_{n=0}^{\infty} (n+1)|a_n|^2 < \infty.$$

Let I be an invariant subspace of D, and let G be the extremal function (unique up to rotation) for the problem analogous to (6). Richter and Sundberg [6; 7] showed that G is a *contractive multiplier* in D, so that $\|Gf\|_D \le \|f\|_D$ for all $f \in D$. Their results were extended by Alexandru Aleman [Habilitationsschrift, Fernuniversität Hagen, 1993], who as a consequence proved that if $I_1 \subset I_2$ are two invariant subspaces of D with associated extremal functions G_1 and G_2, then

$$\|G_1 f\|_D \le \|G_2 f\|_D \quad \text{for all } f \in D.$$

In other words, the analog of Hedenmalm's conjecture is true in this case. The methods used to obtain these results are quite different from those in this paper. It would be interesting to find an approach that covers both the Bergman spaces and the Dirichlet space.

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P. Duren
Department of Mathematics
University of Michigan
Ann Arbor, MI 48109

D. Khavinson
Department of Mathematics
University of Arkansas
Fayetteville, AR 72701

H. S. Shapiro
Royal Institute of Technology
Department of Mathematics
S-100 44 Stockholm
Sweden

C. Sundberg
University of Tennessee
Department of Mathematics
Knoxville, TN 37916