Inextendible Conformal Realizations of Lorentz Surfaces in Minkowski 3-Space

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A simply connected surface which supports an indefinite metric must be diffeomorphic to the plane. Yet there are infinitely many distinct conformal classes of indefinite metrics available on a simply connected surface (see [2]). This contrasts sharply with the two distinct conformal classes of definite metrics available on a surface diffeomorphic to the plane.

It is natural to ask how many distinct conformal types are represented by timelike surfaces in Minkowski 3-space E_1^3. Of course, all subsets of the Minkowski 2-plane E_1^2 are isometrically realized as subsets of any timelike plane in E_1^3. The question is whether such sets can be conformally realized by timelike surfaces which are in some sense *inextendibly* imbedded (or immersed) in E_1^3. Moreover, one seeks particularly nice inextendible imbeddings (or immersions), say, with constant Gauss curvature K or mean curvature H.

This paper presents methods for generating such conformal realizations. The examples produced display considerable variety in conformal type, but all have $K \equiv 0$ or $H \equiv 0$. To aid in distinguishing between conformally inequivalent surfaces, we develop a number of indices whose values are preserved by conformal diffeomorphisms.

The conformal indices defined in Section 2 are based on properties of the conformal boundary $\partial_0(S, h)$ constructed by Kulkarni in [2] for any simply connected surface S with an indefinite metric h. The conformal indices defined in Section 3 are based on simple properties of the null lines for h. Two lemmas in Section 3 allow use of such properties to characterize the conformal type of E_1^2, and of subsets of E_1^2.

In Section 4 we determine the conformal type of a few familiar surfaces in E_1^3 by solving for global null coordinates u, v. The domain of such coordinates in the (u, v)-plane with metric $du\,dv$ then displays the conformal type of the surface.

In Section 5 we use the Weierstrass representation (see [4]) to generate timelike minimal surfaces in E_1^3 of various conformal types. The imbeddings

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and immersions obtained are inextendible in the sense that, at any finite boundary point in E_1^3, either the tangent plane becomes null or the Gauss curvature K becomes infinite.

In Section 6 we use a complex variable $z=x+iy$, and restrict attention to those indefinite metrics h that are proportional to the pullback $g^*(du\,dv)$ by an analytic function $w=g(z)=u+iv$ over the domain on which $g'(z)\neq 0$. In particular, we find a conformal immersion of the z-plane with metric

$$h=\sin x(dx^2-dy^2)+2\cos x\,dx\,dy \tag{1}$$

in E_1^3 so that the image is an inextendibly imbedded, timelike minimal surface on which K changes sign. The z-plane with the metric h in (1) is *not* conformally equivalent to any subset of E_1^2, and has the pattern of null lines shown in Diagram 1. It remains to be seen whether the z-plane with this metric h can be conformally imbedded in E_1^3.

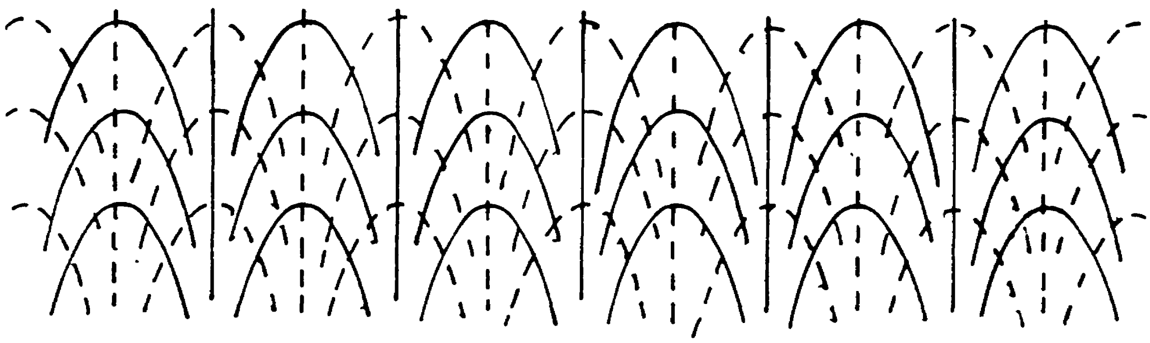

Diagram 1

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All surfaces in this paper are assumed to be C^∞, connected and oriented. For E_1^3 we take (ξ,η,ζ)-space with metric $d\xi^2+d\eta^2-d\zeta^2$. Any immersion $Z\colon S\to E_1^3$ of a surface S in E_1^3 is assumed to be C^∞, with I the metric induced on S by Z. If $\det \mathrm{I}<0$ on S, Z is called *timelike.* An immersion Z is called an *imbedding* when it is one-one.

Any metric h on S is assumed to be C^∞, and is *definite* (resp. indefinite) when $\det h>0$ (resp. <0). An immersion $Z\colon(S,h)\to E_1^3$ with $\det h\neq 0$ is called *conformal* if and only if $h=\lambda \mathrm{I}$ for a function $\lambda\neq 0$.

On a surface S with indefinite metric h, two naturally ordered *null directions* in which $h=0$ are distinguished at each point (see [6]). Integral curves of the null-direction fields are called *null lines.* Together, the two naturally ordered families of null lines comprise the *null net* on (S,h). In the neighborhood of any point, the null lines can be made level lines for *null coordinates* u, v on S in terms of which $h=\lambda\,du\,dv$ for some function $\lambda\neq 0$. By the definition that follows (using $-v$, u in place of u, v if $\lambda<0$), a surface S with indefinite metric h is always *locally* conformally equivalent to E_1^2.

If $\det h_1 \neq 0$ on S_1 and $\det h_2 \neq 0$ on S_2, we call (S_1, h_1) and (S_2, h_2) *conformally equivalent* if and only if there is an orientation-preserving diffeomorphism f of S_1 onto S_2 so that $f^*(h_2) = \lambda h_1$ for some function $\lambda > 0$ on S_1. When $\det h_1 < 0$, $f^*(h_2) = \lambda h_1$ for some $\lambda > 0$ if and only if the null nets for h_1 and $f^*(h_2)$ coincide, so that f takes the h_1-null net on S_1 to the h_2-null net on S_2.

Diagram 2 shows the null lines for the metric $du\,dv$ on six open subsets of the (u, v)-plane. *Throughout this paper, we use* $\{(u, v)\text{-plane}, du\,dv\}$ *as the conformal model for* E_1^2. Note that the map $(u, v) \to (\tan(\pi u/2R), \tan(\pi v/2R))$ takes any square $-R < u < R$, $-R < v < R$ onto the whole (u, v)-plane, and any triangle $-R < u < R$, $u < v < R$ onto the half-plane $v > u$ so as to preserve the null net for $du\,dv$. Thus 2a is conformally equivalent to E_1^2, and 2b to the half-plane $v > u$ in E_1^2.

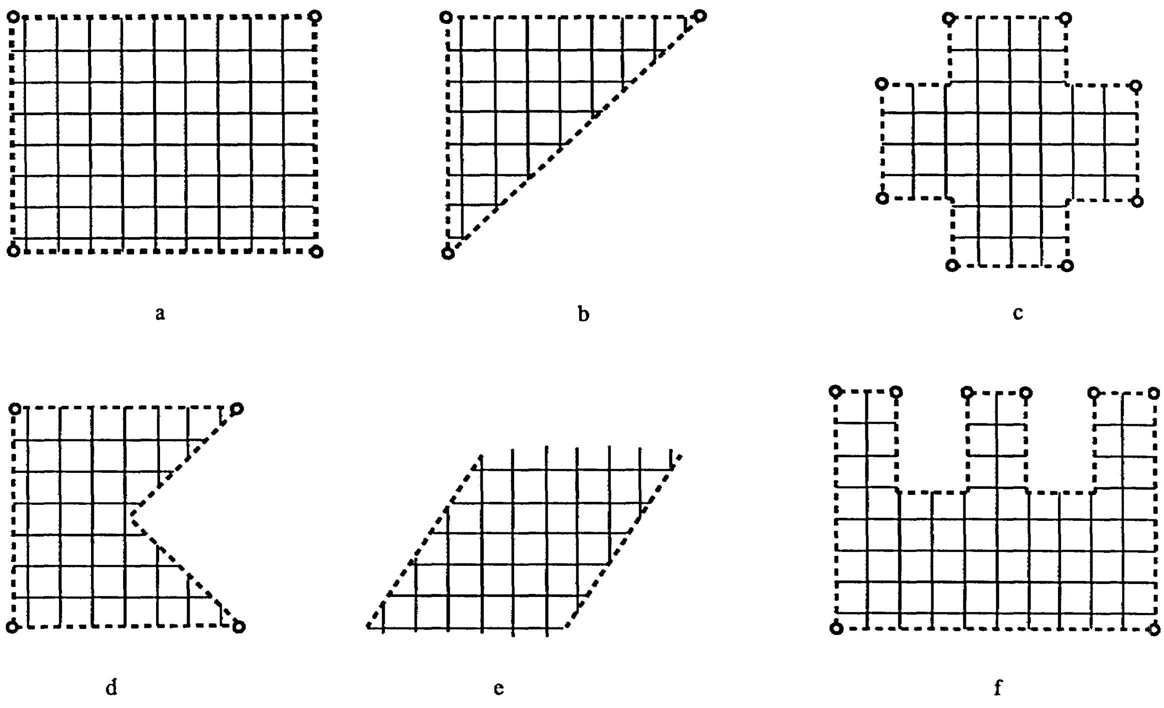

Diagram 2

Two conformal indices derived from Kulkarni's conformal boundary $\partial_0(S, h)$ nearly suffice to distinguish among the conformal types in Diagram 2. To define them, suppose S is simply connected with indefinite metric h. Then the null direction fields can be globally oriented, thereby orienting all null lines on S. Let Σ be the set of ideal end points n^-, n^+ for all inextendible oriented null lines n on S. Kulkarni establishes an equivalence relation on Σ whose equivalence classes are the points of the conformal boundary $\partial_0(S, h)$. A topology is then defined on $S \cup \partial_0(S, h)$ which induces a topology on $\partial_0(S, h)$. (See [2] for details.)

One conformal invariant of (S, h) is the number of entries $i(P)$ in any equivalence class P on $\partial_0(S, h)$. Another conformal invariant $j(S, h)$ is the number of connected components in $\partial_0(S, h)$.

To picture $\partial_0(S, h)$, let S be an open simply connected subset of E_1^2. Then any point p on the *topological* boundary ∂S which is the end point of a null line in S represents at least one P on $\partial_0(S, h)$. If S is bounded, then every point of $\partial_0(S, h)$ is represented by some p on ∂S. In many cases (if ∂S is a Jordan curve, or a line, or a pair of parallel lines, or a pair of distinct rays from a common point), any p on ∂S represents at most one P on $\partial_0(S, h)$; then $i(P)$ is given by the number of distinct inextendible null lines in S which end at p. In the examples of Diagram 2, $\partial_0(S, h)$ can be identified with the subset of ∂S containing all end points of all inextendible null lines in S.

Thus, in 2a, $j(S, h) = 4$ and $i(P) \equiv 1$ on $\partial_0(S, h)$. In 2b, $j(S, h) = 3$, $i(P) \equiv 1$ on two components of $\partial_0(S, h)$, and $i(P) \equiv 2$ on the third component. In both 2c and 2f, all but the eight outer corners of ∂S represent points of $\partial_0(S, h)$, $j(S, h) = 8$, and $i(P) \equiv 1$ except at the four inner corners on ∂S where $i(P) = 2$. In 2d, $j(S, h) = 4$, $i(P) = 1$ on three components of $\partial_0(S, h)$, and $i(P) \equiv 2$ on the fourth component except at one inner corner point of ∂S where $i(P) = 3$. In 2e, $j(S, h) = 2$ and $i(P) \equiv 2$ on $\partial_0(S, h)$. The conformal indices defined in Section 3 will distinguish between examples 2c and 2f.

Close reading of [2] provides the following tool for recognizing (S, h) which cannot be conformally realized as subsets of E_1^2.

REMARK 1. If $i(P) \geq 4$ for some P in $\partial_0(S, h)$, then (S, h) is not conformally equivalent to any subset of E_1^2.

Thus the universal cover (S, h) of the punctured square $|u| < 1$, $|v| < 1$, $(u, v) \neq (0, 0)$ in E_1^2 cannot be conformally realized in E_1^2, since $i(P) = \infty$ at the point of $\partial_0(S, h)$ corresponding to the origin. However, we show in Section 6 that this (S, h) is conformally equivalent to the example in Diagram 1.

Note that for any integer k there is a Jordan curve γ in E_1^2 made up of finitely many horizontal and vertical line segments and having at least k outer corner points. If S is the interior region determined by γ then $j(S, h) \geq k$, which justifies the next statement.

REMARK 2. There are infinitely many distinct conformal types of simply connected subsets of E_1^2.

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The existence of global h-null coordinates characterizes those (S, h) conformally equivalent to a subset of E_1^2. To study conformal type without reference to the conformal boundary, we prove two lemmas which give conditions necessary and/or sufficient for the existence of global h-null coordinates on S.

If S is simply connected, the inextendible, oriented null lines for h fall into two classes of mutually disjoint curves, which we call l-lines and m-lines. As noted in [2], Hopf index arguments show that every l-line or m-line is a properly imbedded smooth open curve on S. Moreover, each l-line intersects any m-line in at most one point. Let $Q(l)$ denote the open nonempty set of all points on all m-lines which intersect l, and $Q(m)$ the open nonempty set of all points on all l-lines which intersect m.

LEMMA 1. *If $Q(l)=Q(m)$ for one l-line l and one m-line m on a simply connected S with indefinite metric h, then $Q(l)=Q(m)=S$ and (S,h) is conformally equivalent to a subset of E_1^2.*

Proof. Let l_p and m_p be the uniquely determined l-line and m-line through any point p on S. Generally, m_p lies in $Q(l)$ if p does, and l_p lies in $Q(m)$ if p does. Since $Q(l)=Q(m)=Q$ here, both l_p and m_p lie in Q if p does. Thus any null line with a point in Q lies entirely in Q.

Suppose there is a point r of S *not* in Q. Since S is connected and locally conformally equivalent to E_1^2, r can be joined to a fixed point q in Q by a continuous curve made up of finitely many oriented null arcs $n_1,\dots,n_k$, with n_1 starting at q, n_j starting at the endpoint of n_{j-1} for $j=2,\dots,k$, and n_k ending at r. Since n_1 starts in Q, it must end in Q. If n_{j-1} ends in Q for $j=2,\dots,k$, then n_j starts in Q and must end in Q. Inductively, r lies in Q, a contradiction unless $S=Q$.

To find global null coordinates u, v on (S,h), note that any m-line m' on S must intersect l. Otherwise, since $Q(l)=S$, any point p on m' lies on some m-line $m''\neq m'$ which intersects l, a contradiction since disjoint m-lines cannot intersect. Similarly, any l-line l' on S must intersect m.

Fix regular C^∞ parameterizations α and β of l and m over $(-\infty,\infty)$ so zero goes to $q=l\cap m$ with $\alpha'(0),\beta'(0)$ a properly oriented basis for the tangent space S_q. Think of l and m as coordinate axes, assigning to any p on S the coordinate u at the point on l where m_p intersects l, and the coordinate v at the point on m where l_p intersects m. This map of (S,h) into the (u,v)-plane with metric $du\,dv$ is a diffeomorphism preserving the null net. But the map need not be onto. □

Diagram 2c shows that (S,h) need not be conformally equivalent to E_1^2 even if $Q(l)=Q(m)=S$ for infinitely many choices of an l-line l and an m-line m on S. The following result shows that among the examples in Diagram 2, only 2a is conformally equivalent to E_1^2.

COROLLARY TO LEMMA 1. *A simply connected surface S with indefinite metric h is conformally equivalent to E_1^2 if and only if $Q(l)=S$ for every l-line l on S.*

Proof. Argument is needed in just one direction. Suppose $Q(l)=S$ for every l-line l on S. Fix an l-line l and an m-line m on S. Since $Q(l)=S$, m intersects l. Moreover, $Q(m)=S$, since otherwise some l-line l' fails to intersect m, contradicting $Q(l')=S$. Thus Lemma 1 applies and we can think of (S,h) as a subset of E_1^2, with l along the u-axis and m along the v-axis. If $(S,h)\neq E_1^2$ then there is a point $(\hat{u},\hat{v})$ not in (S,h). But then $Q(\hat{l})$ for the l-line $\hat{l}$ through $(0, \hat{v})$ does not contain the point $(\hat{u}, 0)$ on l, contradicting the assumption that $Q(\hat{l}) = S$. □

The next result characterizes (S, h) with global null coordinates as those which have conformal self-images in arbitrarily small open neighborhoods. Note that S need not be simply connected, just as the domain of global null coordinates need not be simply connected.

LEMMA 2. *A surface S with indefinite metric h is conformally equivalent to a subset of E_1^2 if and only if any open set $W \neq \emptyset$ on S contains a subset U with (U, h) conformally equivalent to (S, h).*

Proof. On any surface S with indefinite metric h, every open set $W \neq \emptyset$ contains a null net square R, and by the Corollary to Lemma 1, (R, h) is conformally equivalent to E_1^2.

If (S, h) is conformally equivalent to a subset of E_1^2, then any open subset W of S contains a null set square R which, being conformally equivalent to E_1^2, must contain a subset conformally equivalent to (S, h).

If (S, h) is conformally equivalent to (U, h) for some subset U of any open subset W on S, then using a null net rectangle R in W in place of W, (S, h) must be conformally equivalent to a subset of R, and hence to a subset of E_1^2. □

Conformal indices can be defined on a surface S with indefinite metric h without reference to the conformal boundary. One example is $A(S, h)$ given by

$$A(S,h) = \operatorname*{lub}_{p,q \in S} a(p,q) \geq 2,$$

where $a(p, q)$ is the least number of null arcs needed to form a piecewise smooth curve from p to q. As in Diagram 1, $A(S, h)$ can be infinite. Another pair of conformal indices,

$$B^-(S,h) = \operatorname*{glb}_{\text{all } n} b(n), \qquad B^+(S,h) = \operatorname*{lub}_{\text{all } n} b(n),$$

is obtained by counting the number $b(n)$ of connected components in the complement of $Q(n)$, the set of all points on all null lines which intersect an inextendible null line n of S. If S is simply connected, the split of inextendible null lines into l-lines and m-lines gives four such invariants,

$$B_L^-(S,h) = \operatorname*{glb}_{\text{all } l} b(l), \qquad B_L^+(S,h) = \operatorname*{lub}_{\text{all } l} b(l),$$

$$B_M^-(S,h) = \operatorname*{glb}_{\text{all } m} b(m), \qquad B_M^+(S,h) = \operatorname*{lub}_{\text{all } m} b(m).$$

These new indices distinguish between any two examples in Diagram 2, where we take l-lines horizontal and m-lines vertical. In 2a, $A = 2$, with $B_L^- = B_L^+ = B_M^- = B_M^+ = 0$. For 2b, $A = 2$ with $B_L^- = B_L^+ = B_M^- = B_M^+ = 1$. For

2c, $A=2$ with $B_L^-=2$, $B_M^-=0$, and $B_M^+=1$. For 2e, $A=\infty$ with $B_L^-=B_L^+=B_M^-=B_M^+=2$. Finally, for 2f, $A=3$ with $B_L^-=0$, $B_L^+=2$, $B_M^-=2$, and $B_M^+=3$. Lemma 1 and its Corollary give the following results.

THEOREM 1. *A simply connected surface S with indefinite metric h is conformally equivalent to a subset of E_1^2 in case $B_L^-(S,h)=B_M^-(S,h)=0$.*

THEOREM 2. *A simply connected surface S with indefinite metric h is conformally equivalent to E_1^2 if and only if $B_L^+(S,h)=0$.*

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If a timelike surface in E_1^3 is conformally equivalent to a subset of E_1^2, the most direct way of displaying its conformal type is to solve for global null coordinates and then describe their domain. This method will be used for a few familiar examples.

Suppose that S is a cylinder in E_1^3 given by $\eta=f(\zeta)$. If f is defined (and C^∞) for all real values of ζ, then S is *entire* over the (ξ,ζ)-plane. The induced metric I on S is given by

$$\mathrm{I}=d\xi^2-(1-f'^2)\,d\zeta^2,$$

so that S is timelike if and only if $|f'|<1$. The conformal type of S depends upon the values of the improper integrals

$$I_1=\int_0^\infty \sqrt{1-f'^2}\,d\zeta, \qquad I_2=\int_{-\infty}^0 \sqrt{1-f'^2}\,d\zeta,$$

as explained in the following result.

THEOREM 3. *Suppose a timelike cylinder S given by $\eta=f(\zeta)$ is entire over the (ξ,ζ)-plane. Then S is conformally equivalent to*

(i) *E_1^2 when I_1 and I_2 are both infinite,*
(ii) *the example in Diagram 2e when I_1 and I_2 are both finite, and*
(iii) *the example in Diagram 2b when one of the integrals I_1 or I_2 is finite and the other infinite.*

Proof. Use new global coordinates

$$\alpha=\xi, \qquad \beta=\int_0^\zeta \sqrt{1-f'^2}\,d\zeta,$$

on S in terms of which $\mathrm{I}=d\alpha^2-d\beta^2$. The image of the (ξ,ζ)-plane in the (α,β)-plane is the whole plane in (i), the strip $-\infty<-I_2<\beta<I_1<\infty$ in (ii), and a half-plane $-\infty<\beta<I_1<\infty$ or $-\infty<-I_2<\beta<\infty$ in (iii). Thus the domains of global null coordinates $u=\alpha-\beta$ and $v=\alpha+\beta$ are as claimed. □

REMARK 3. Theorem 3 provides examples of entire timelike surfaces in E_1^3 with $K \equiv 0$ which are conformally inequivalent to E_1^2. This contrasts with entire timelike surfaces in E_1^3 with $H \equiv 0$ which must all be conformally equivalent to E_1^2. (See [3].)

To study the conformal type of timelike developable helicoids, we first find null coordinates for the two developable surfaces associated with any C^∞ timelike curve $X(s)$ in E_1^3 with nonvanishing curvature $k(s)$.

Parameterize $X(s)$ by its E_1^3 arc length s, so that $X'(s)\cdot X'(s) \equiv -1$ in E_1^3. The tangential developables S^+ and S^- described by

$$Z(s,t) = X(s) + tX'(s) \tag{2}$$

for $t > 0$ and $t < 0$ (respectively) have induced metric

$$\mathrm{I} = (t^2k^2 - 1)\, ds^2 - 2\, ds\, dt - dt^2,$$

where $k = k(s) > 0$ is the length of the spacelike vector $X''(s) \neq 0$. The surfaces S^+ and S^- are timelike since $\det \mathrm{I} = -t^2k^2 < 0$ when $t \neq 0$. We use the global coordinates s, t to fix an orientation on S^+ and S^-.

Null coordinates u, v for which $\mathrm{I} = du\, dv$ must satisfy the equations

$$u_s v_s = t^2k^2 - 1, \quad u_s v_t + u_t v_s = -2, \quad u_t v_t = -1 \tag{3}$$

for $t \neq 0$. The functions

$$u = -\left(\frac{1}{k} + t\right) \exp\left[\int k\, ds\right] \quad \text{and} \quad v = \left(\frac{1}{k} - t\right) \exp\left[-\int k\, ds\right]$$

solve (3), but we use u, v only for the timelike circular helix

$$X(s) = (\cos as, \sin as, \sqrt{a^2+1}s) \tag{4}$$

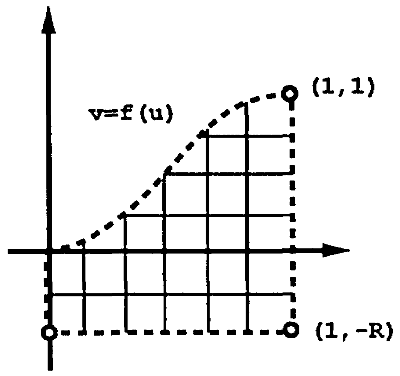
which has $k \equiv a^2 \neq 0$, so that

$$u = -\left(\frac{1}{a^2} + t\right) e^{a^2 s} \quad \text{and} \quad v = \left(\frac{1}{a^2} - t\right) e^{-a^2 s}; \tag{5}$$

$$uv = t^2 - \frac{1}{a^4}, \tag{6}$$

with $\partial(u,v)/\partial(s,t) = 2ta^2$ having the sign of t. If $u(s_1, t_1) = u(s_2, t_2)$ and $v(s_1, t_1) = v(s_2, t_2)$ with $t_1 t_2 > 0$, (6) gives $t_1 = t_2$, so that (5) gives $s_1 = s_2$. Thus the map $(s,t) \to (u,v)$ is a diffeomorphism on the half-planes $t > 0$ and $t < 0$.

Use the global null coordinates u, v on S^+ and (to preserve orientation) $-u, v$ on S^-. The domain of u, v for S^+ and (after rotation) of $-u, v$ for S^- is the region below the portion of the hyperbola $uv = -1/a^4$ with $u < 0$. The map $(u,v) \to (1 + \tanh u, \tanh v)$ gives conformal equivalence of the null coordinate domains with a region of the sort shown in Diagram 3a for $R = 1$. Finally, the following lemma (which is easily checked) justifies the claim in Theorem 4.



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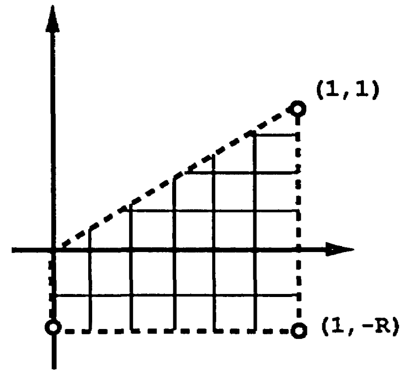

b

Diagram 3

FLATTENING LEMMA. *Suppose $v=f(u)$ is continuous on $[0,1]$ and C^∞ on $(0,1)$, with $f'(u)>0$ on $(0,1)$, $f(0)=0$, and $f(1)=1$. Then, for any $R\geq 0$, the map $(u,v)\to(f(u),v)$ takes the region in Diagram 3a conformally (for the metric $du\,dv$) onto the region in Diagram 3b.*

THEOREM 4. *The timelike developable helicoids S^+ and S^- in E_1^3 given for $t>0$ and $t<0$ respectively by* (2) *using* (4) *are both conformally equivalent to the example in Diagram 3b.*

A well-known minimal surface in E^3 is the right helicoid S given for any constant $a>0$ by

$$Z(s,t)=(s\cos t, s\sin t, at). \tag{7}$$

This surface is ruled by the lines $t\equiv$ constant, but is not developable (see [5]). The fundamental forms for S in E_1^3 are given by

$$\mathrm{I}=ds^2+(s^2-a^2)\,dt^2 \quad\text{and}\quad \mathrm{II}=-2a\,ds\,dt/\sqrt{a^2-s^2},$$

so that $H\equiv 0$, making S a minimal surface in E_1^3 as well as E^3. To find global null coordinates u, v for the timelike portion $|s|<a$ of S, set $\mathrm{I}=\lambda\,du\,dv$ so that u, v must satisfy

$$\lambda u_s v_s=1,\quad \lambda(u_s v_t+u_t v_s)=0,\quad \lambda u_t v_t=s^2-a^2.$$

Taking $\lambda=a^2-s^2$, one obtains the solutions

$$u=-t+\int_0^s\frac{ds}{\sqrt{a^2-s^2}}\quad\text{and}\quad v=t+\int_0^s\frac{ds}{\sqrt{a^2-s^2}},$$

which give a one-one orientation-preserving map of the strip $|s|<a$ in the (s,t)-plane onto the strip $|u+v|<\pi/2a$ in the (u,v)-plane. This justifies the following claim.

THEOREM 5. *The timelike portion of the right helicoid S in E_1^3 given by* (7) *is a minimal surface conformally equivalent to the example in Diagram 2e reflected in a vertical line.*

REMARK 4. On timelike surfaces that can be visualized in E_1^3 (and thus in E^3), one can often picture the null lines, which are always *helices,* since they make a constant Euclidean angle (of 45°) with a fixed oriented direction (the positive ζ-axis) while being constrained to lie on the surface. For example, one can see the null lines of the timelike equatorial band on the Euclidean sphere in E_1^3 in Figure 1-32 of Struik's classical text [5]. The conformal universal cover of this band is given by the example in Diagram 2e.

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Suppose that C^∞ real-valued functions $A(u)$ and $B(v)$ are defined on the open intervals I and J, respectively. Let

$$Z(u, v) = X(u) + Y(v), \tag{8}$$

where

$$X(u) = \int_{u_0}^{u} X'(u)\, du \quad \text{and} \quad Y(v) = \int_{v_0}^{v} Y'(v)\, dv$$

for a fixed choice of u_0 in I and v_0 in J, with

$$\sqrt{2}X'(u) = (\cos A(u), \sin A(u), 1), \qquad \sqrt{2}Y'(v) = (\cos B(v), \sin B(v), 1).$$

If S is any connected subdomain of $I \times J$ on which $A(u) \neq B(v) \bmod 2\pi$, the C^∞ map $Z: I \times J \to E_1^3$ restricts to a C^∞ immersion $Z: S \to E_1^3$ with fundamental forms

$$\mathrm{I} = -2\sin^2\left(\frac{A(u) - B(v)}{2}\right) du\, dv, \qquad \sqrt{2}\mathrm{II} = -A'(u)\, du^2 + B'(v)\, dv^2. \tag{9}$$

Thus Z is timelike over S with $\operatorname{sign} K = \operatorname{sign} A'(u)B'(v)$ at any point. The immersion Z given by (8) over S is called *Weierstrass representation* (see [4]). If Z is one-one on S then u, v are global null coordinates on $Z(S)$. We continue to use the metric $du\, dv$ in the (u, v)-plane.

If Z is not one-one on S, there are distinct points (u_1, v_1) and (u_2, v_2) with $u_1 < u_2$ on S for which $Z(u_1, v_1) = Z(u_2, v_2)$, so that

$$\int_{u_2}^{u_1} \cos A(u)\, du + \int_{v_2}^{v_1} \cos B(v)\, dv = 0, \tag{10}$$

$$\int_{u_2}^{u_1} \sin A(u)\, du + \int_{v_2}^{v_1} \sin B(v)\, dv = 0, \tag{11}$$

and

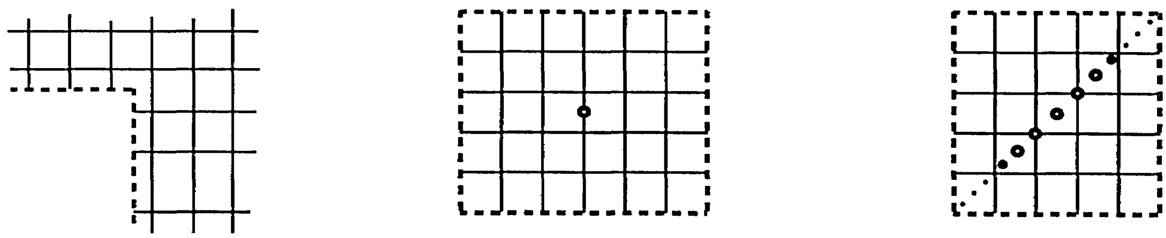
$$v_1 - v_2 = u_2 - u_1 > 0. \tag{12}$$

Thus Z is one-one on S if $\cos A(u) > 0$ almost everywhere on I, while $\cos B(v) \leq 0$ on J. Similarly, Z is one-one on S if $\sin A(u) > 0$ almost everywhere on I, while $\sin B(v) \leq 0$ on J. Though more specific arguments are often needed to show whether any particular choice of $A(u)$ and $B(v)$ leads to a one-one Z, the following observation can be useful.

REMARK 5. The map Z in (8) must be one-one on $I \times J$ if $0 < A(u) < \pi$ almost everywhere on I, while $-\pi < B(v) \leq 0$ on J.

The next result is meant to illustrate the usefulness of the Weierstrass representation in constructing inextendibly imbedded timelike minimal surfaces of differing conformal types. (It does not exhaust all the possibilities.)

THEOREM 6. *There exist inextendible conformal imbeddings as timelike minimal surfaces in E_1^3 of the examples in Diagrams 2a, 2b, 2c, 2e reflected in a vertical line, 4a, 4b and 4c.*



a b c

Diagram 4

Proof. By Remark 3, the entire timelike minimal surface $\eta = \zeta \tanh \xi$ has the conformal type of 2a. Theorem 5 provides an inextendible conformal imbedding of the example 2e reflected in a vertical line.

To produce an inextendible conformal imbedding of the example 2b, take $A(x) = B(x) = \cos^{-1}(2x-1)$ on $I = J = (0,1)$. Then $A(u) \neq B(v) \bmod 2\pi$ on the domain S in $I \times J$ on which $v > u$. Suppose the immersion Z of S given by (8) is not one-one. Then distinct points (u_1, v_1) and (u_2, v_2) exist in S with $u_1 < u_2$ for which (10) gives

$$\int_{u_2}^{u_1} (2u-1)\, du + \int_{v_2}^{v_1} (2v-1)\, dv = 0,$$

so that (12) yields

$$(u_1 - u_2)(u_1 + u_2 - v_1 - v_2) = 0.$$

With $v_1 > u_1 > 0$ and $v_2 > u_2 > 0$ on S, one must have $u_1 = u_2$ above, and hence $v_1 = v_2$, a contradiction. Thus u, v are global null coordinates and Z is an imbedding. Moreover, tangent planes to $Z(S)$ approach a null plane as (u, v) in S goes to any point on the closed segment $u = v$ in $\overline{I \times J}$, while Gauss curvature approaches infinity as (u, v) in S goes to a noncorner point on $\partial(I \times J)$. Again, $Z(S)$ is inextendible as a C^∞ timelike surface in E_1^3.

To produce an inextendible conformal imbedding of the example 4a, take any C^∞ $A(x)$ on $I = (-\infty, \infty)$ with $A(x) \equiv 0$ on $(-\infty, 0]$, $A'(x) > 0$ on $(0, \infty)$, and $A(x) \to 1$ as $x \to \infty$. Let $B(x) = -A(x)$ with $J = (-\infty, \infty)$. Then $A(u) \neq B(v) \bmod 2\pi$ on the union S of the half-planes $v > 0$ and $u > 0$. If Z is not one-one on S, distinct points (u_1, v_1) and (u_2, v_2) exist on S with (10) and (12) holding. If $u_1 \geq 0$ then $u_2 > 0$, so that

$$\int_{u_2}^{u_1} \sin A(u)\, du < 0, \qquad \int_{v_2}^{v_1} \sin B(v)\, du \leq 0, \tag{13}$$

contradicting (11). If $u_1 < 0$ then $v_1 > 0$, so that

$$\int_{u_2}^{u_1} \sin A(u)\, du \leq 0, \qquad \int_{v_2}^{v_1} \sin B(v)\, du < 0, \tag{14}$$

contradicting (11). Thus u, v are global null coordinates and Z is an imbedding. The only finite boundary points for $Z(S)$ occur as (u, v) in S goes to a finite boundary point of S, and then the tangent planes to $Z(S)$ approach a null plane. Again, $Z(S)$ is inextendible.

To produce an example with the conformal type of 2c, take any C^∞ $A(x)$ on $I = (-\infty, \infty)$ with $A(x) \equiv 0$ on $|x| \geq 1$, $A'(x) > 0$ on $(-1, 0)$, $A'(x) < 0$ on $(0, 1)$, and $A(0) = 1$. Let $B(x) = -A(x)$ with $J = (-\infty, \infty)$. Then $A(u) \neq B(v) \bmod 2\pi$ on the union S of the strips $|u| < 1$ and $|v| < 1$. If Z is not one-one on S, then distinct points (u_1, v_1) and (u_2, v_2) exist on S with (10) and (12) holding. If $u_1 \geq 1$ or $u_2 \leq -1$ then $|v_1| < 1$ *and* $|v_2| < 1$, giving (14) which contradicts (11). If $u_1 < 1$ and $u_2 > -1$ then (13) holds, contradicting (11). Thus u, v are global null coordinates and Z is an imbedding. One argues as in the previous case that $Z(S)$ is inextendible.

To produce an example with the conformal type 4b, let

$$A(x) = 1 - 2\sqrt{x - x^2}$$

and take $B(x) = -A(x)$ on $I = J = (0, 1)$. Then $A(u) \neq B(v) \bmod 2\pi$ on the domain S obtained from $I \times J$ by deleting the point $(\frac{1}{2}, \frac{1}{2})$. By Remark 5, u, v are global null coordinates on $Z(S)$ and Z is an imbedding. Since the tangent planes to $Z(S)$ approach a null plane as (u, v) in S goes to $(\frac{1}{2}, \frac{1}{2})$, while Gauss curvature approaches infinity on $Z(S)$ as (u, v) in S goes to a point on $\partial(I \times J)$, $Z(S)$ is inextendible. A slight variation on this construction is included in the next case for $k = 1$.

To produce an inextendible conformal imbedding of the example 4c (where k distinct points are removed from the diagonal $u = v$ of the square for $k = 1, 2, \ldots$), we describe (for the purposes of Section 6) a Weierstrass representation defined on the conformally equivalent domain S obtained by removing k distinct points from the line $u = v$ in the whole (u, v)-plane. Take any C^∞ $A(x)$ on $I = (-\infty, \infty)$ with $A(x) \to 1$ as $x \to \pm\infty$, $0 \leq A(x) < 1$, and $A(x) = 0$ for exactly k distinct x values. Let $B(x) = -A(x)$, so that $J = (-\infty, \infty)$. Then $A(u) \neq B(v) \bmod 2\pi$ on S. By Remark 5, u, v are global null coordinates on $Z(S)$ and Z is an imbedding. The only finite boundary points for $Z(S)$ occur as (u, v) in S approaches a deleted point, so that the tangent planes to $Z(S)$ approach a null plane. Thus $Z(S)$ is inextendible. Note that by (9) the sign of K must change on $Z(S)$. □

Remark 6. The method used to produce the last example extends easily to provide an inextendible timelike minimal surface in E_1^3 conformally imbedding the (u, v)-plane with a countably infinite set of points deleted from the line $u=v$.

The Weierstrass representation can be used to produce inextendibly *immersed* conformal representations in E_1^3 of a wide variety of subsets in E_1^2. We cite just one pair of conformally distinct examples obtained by taking $A(x)=1-2\sqrt{x-x^2}$ and $B(x)=1-\sqrt{x-x^2}$ on $I=J=(0,1)$. Then $A(u)\neq B(v) \bmod 2\pi$ on the domains S_1, S_2, and S_3 in $I\times J$ which (respectively) lie left of, right of, and between the two branches of the hyperbola $A(u)=B(v)$ in $I\times J$, as shown in Diagram 5a. One easily argues that Z is inextendible on S_1, S_2, and S_3. A suitable version of the flattening lemma then shows that S_1 and S_2 have the conformal type shown in Diagram 5b, and S_3 the type shown in Diagram 5c.

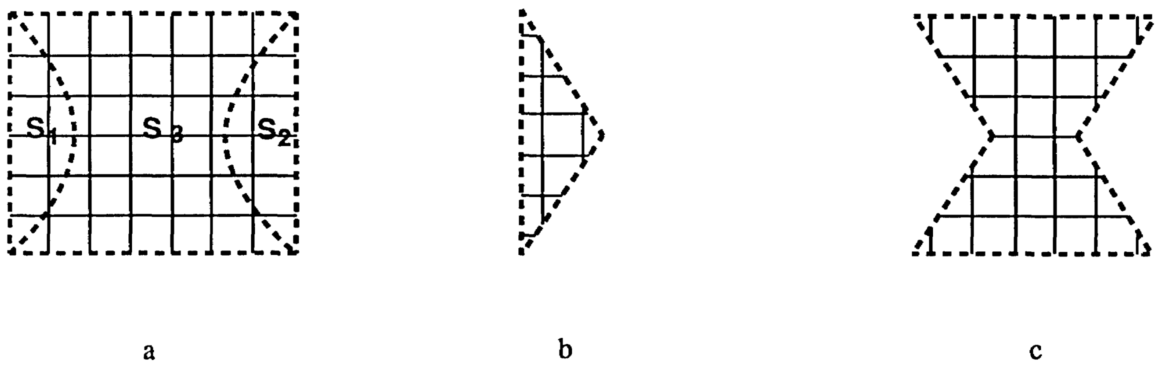


Diagram 5

6

To find a conformal immersion in E_1^3 of the (x, y)-plane with metric h given by (1), set $z=x+iy$ and use the entire function

$$w=g(z)=-2ie^{iz/2},$$

which has period 4π and assumes all values $w=u+iv\neq 0$. Here

$$u=2e^{-y/2}\sin(x/2) \quad \text{and} \quad v=-2e^{y/2}\cos(x/2),$$

so that

$$2g^*(du\,dv)=e^{-y}h.$$

Thus g composed with any conformal immersion of $(\{u+iv\neq 0\}, du\,dv)$ in E_1^3 gives a conformal immersion of $((x,y)\text{-plane}, h)$ in E_1^3.

There is an obvious imbedding given by $\xi=(u-v)/\sqrt{2}$, $\eta=0$, $\zeta=(u+v)/\sqrt{2}$, which is isometric from $((u, v)\text{-plane}, du^2+dv^2)$ into E^3 and from $((u, v)\text{-plane}, 2\,du\,dv)$ into E_1^3. This justifies the following claim.

THEOREM 7. *The (x, y)-plane with metric $h = \sin x(dx^2 - dy^2) + 2\cos x\, dx\, dy$ can be conformally immersed in E_1^3 as the universal cover of the (ξ, ζ)-plane with the origin removed.*

A more satisfying example is obtained by composing g with the imbedding of $(\{u+iv \neq 0\}, du\, dv)$ provided by Theorem 6. In particular, use the proof that the example of Diagram 4c can be realized for $k=1$. This justifies the following.

THEOREM 8. *The (x, y)-plane with metric $h = \sin x(dx^2 - dy^2) + 2\cos x\, dx\, dy$ can be conformally immersed in E_1^3 as the universal cover of an inextendibly imbedded timelike minimal surface on which K changes sign.*

REMARK 7. Given any analytic function $w = g(z)$ with $g'(z) \neq 0$ on a domain Ω in the $z = (x+iy)$-plane, composing g with a conformal imbedding or immersion of $(g(\Omega), du\, dv)$ in E_1^3 yields a conformal imbedding or immersion of $(\Omega, g^*(du\, dv))$ in E_1^3.

Use of Theorem 6 and Remark 6 gives results of the following sort.

THEOREM 9. *Let $w = g(z)$ be a polynomial of degree $n \geq 2$ so that $g'(z)$ has exactly k distinct zeros, with $1 \leq k < n$. Let Ω be the z-plane less the zeros of $g'(z)$. Then $(\Omega, g^* du\, dv)$ can be conformally immersed in E_1^3 as a covering of an inextendibly imbedded timelike minimal surface on which K changes sign.*

In the next remark, $[h]$ denotes the set of all metrics conformally equivalent to a nondegenerate metric h on S. If h is Riemannian (positive definite) then $\mathcal{R} = (S, [h])$ is a *Riemann surface.* By analogy, if h is Lorentzian (indefinite) then we call $\mathcal{L} = (S, [h])$ a *Lorentz surface.*

REMARK 8. The method in Remark 7 can be applied even when $w = g(z)$ is a "multivalued" analytic function $w = g(z)$ in a domain Ω on which $g'(z) \neq 0$. Associated with such a function g is a Riemann surface $\mathcal{R} = (S, [dx^2 + dy^2])$ defined on a naturally constructed covering surface S of Ω to which one lifts the metric $dx^2 + dy^2$. Locally, any lift of $z = x+iy$ to S gives a conformal parameter on $\mathcal{R}$, and $w = g(z)$ lifts to a function single-valued and analytic on $\mathcal{R}$ with $g'(z) \neq 0$. However, g need not be one–one on $\mathcal{R}$ (see [1] or [7]). Now use g^* to pull the metric $du\, dv$ back to S, and define the Lorentz surface $\mathcal{L} = (S, [g^* du\, dv])$. Composition of g with a conformal imbedding or immersion of $(g(\mathcal{R}), du\, dv)$ in E_1^3 yields a conformal imbedding or immersion of $\mathcal{L}$ in E_1^3. For example, if $w = g(z) = z^{1/2}$, then S is the familiar double cover of the punctured plane $\Omega = \{z \neq 0\}$, and $w = z^{1/2}$ is one–one from $\mathcal{R}$ to $\{w \neq 0\}$. Here $\mathcal{L} = (S, [g^* du\, dv])$, where

$$g^* du\, dv = y(dy^2 - dx^2) + 2x\, dx\, dy$$

on S. Composing g with the imbedding of $(\{w \neq 0\}, du\, dv)$ provided by Theorem 6, one gets an inextendible conformal imbedding of $\mathcal{L}$ in E_1^3 as a timelike minimal surface on which K changes sign. Note that the conformal structure of $(\Omega, g^* \, du \, dv)$ is more complicated than that of $\mathfrak{L}$.

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