Covolume Estimates for Discrete Groups of Hyperbolic Isometries Having Parabolic Elements

SA'AR HERSONSKY

1. Introduction

In this paper we address the problem of estimating the covolume of a discrete group of orientation-preserving isometries of hyperbolic space in any dimension. We provide a general method for obtaining a lower bound for the covolume in the case that the group has parabolic elements. For a torsion-free discrete group of orientation-preserving isometries of hyperbolic 4-space having parabolic elements, we give an explicit lower bound.

By the Bieberbach theorem (see [13, Thms. 3.2.1 and 3.2.2]), for any given dimension n there are only finitely many cocompact discrete groups of isometries of $\mathbf{R}^n$, up to affine equivalence. These are called the n-dimensional Bieberbach groups. Each n-dimensional Bieberbach group G contains a finite-index subgroup $A = A_G$ isomorphic to Z^n. The subgroup A_G consists of all translations in G. Let us denote by I_n the maximum of the indices $|G: A_G|$, where G ranges over all n-dimensional orientation-preserving Bieberbach groups.

If Λ is a lattice in $\mathbf{R}^n$, we denote by $|\Lambda|$ the Euclidean volume of $\mathbf{R}^n/\Lambda$ and by β the nonzero vector of smallest length in Λ. It is well known that for every $n > 0$ there exists a positive constant δ_n such that for any lattice Λ in $\mathbf{R}^n$ we have $|\Lambda| \geq \delta_n |\beta|^n$. (See [3] for the values of δ_n when $n \leq 8$.)

Our first main result is the following theorem.

THEOREM 1. *Let Δ be a discrete subgroup of orientation-preserving isometries of hyperbolic $(n+1)$-space $\mathbf{H}^{n+1}$. Suppose that Δ has finite covolume. Let m denote the number of orbits of points in $\hat{\mathbf{R}}^n$ which are fixed by parabolic elements of Δ. Then $m < \infty$ and*

$$\mathrm{vol}\left(\frac{\mathbf{H}^{n+1}}{\Delta}\right) \geq \frac{\delta_n m}{n I_n}.$$

Section 3 of this paper is devoted to the proof of this theorem.

Our second main result deals with the 4-dimensional torsion-free case, and will be stated in terms of hyperbolic manifolds. If Δ is a discrete, torsion-free

Received January 14, 1992. Revision received February 15, 1993.
Michigan Math. J. 40 (1993).

group of isometries of $\mathbf{H}^{n+1}$ then the quotient $M=\mathbf{H}^{n+1}/\Delta$ is a hyperbolic $(n+1)$-manifold. Each orbit of parabolic fixed points of Δ in $\hat{\mathbf{R}}^n$ is said to define a *cusp* of M.

We will obtain the following result.

THEOREM 2. *Let M be a hyperbolic oriented 4-manifold of finite covolume. Let m denote the number of cusps of M. Then $m<\infty$ and*

$$\operatorname{vol}(M) \geq \frac{\sqrt{3}m}{36}.$$

This is a 4-dimensional analogue of [8, Sec. 5] and [4], which give a lower bound for the volume of a hyperbolic 3-manifold with cusps. We will derive Theorem 2 from Theorem 1 by an analysis of the torsion-free orientation-preserving Bieberbach groups in $\mathbf{R}^3$. The proof occupies Section 4.

In Section 2 of this paper we use the Vahlen group (see [1]) and geometric facts about Möbius transformations in $\hat{\mathbf{R}}^n$ to prove a result (Theorem 2.3) about the action of elements of a discrete group on horoballs based at parabolic fixed points. In Section 3 we reinterpret the latter result in the light of the Bieberbach theorem and use it to prove Theorem 1.

In [12], Wielenberg implicitly obtained results equivalent to some of those in Section 2 from an entirely different point of view.

A number of arguments in the present paper generalize to n dimensions arguments that appear in the literature in the 3-dimensional case. In some instances we have been able to do this in such a way that the proofs become more geometric and, even when specialized to dimension 3, simpler than those appearing elsewhere. In this connection we would like to call particular attention to the use of isometric spheres in the proof of Proposition 2.1, and to the apparently new notion of "canonical horoball" used in Section 3.

2. Isometric Spheres and Actions on a Horoball

We shall use the following conventions. We shall write $\hat{\mathbf{R}}^n=\mathbf{R}^n\cup\{\infty\}$. We shall denote by $GM(\hat{\mathbf{R}}^n)$ the group of all Möbius transformations of $\hat{\mathbf{R}}^n$. The Euclidean distance from x to y in $\mathbf{R}^n$ will be denoted $|x-y|$. The Euclidean sphere of radius r about a point $x\in\mathbf{R}^n$ will be denoted $S(A,r)$. The Poincaré extension of $\phi\in GM(\hat{\mathbf{R}}^n)$ will be denoted $\tilde{\phi}$.

The isometric sphere of a Möbius transformation in $\hat{\mathbf{R}}^n$ will play a key role in our proofs. Recall that any Möbius transformation in $\hat{\mathbf{R}}^n$ which does not fix ∞ restricts to a Euclidean isometry on a unique Euclidean sphere in $\mathbf{R}^n$. (See [2, Sec. 3.5].) The following proposition is crucial in all that follows.

PROPOSITION 2.1. *Suppose that $\phi\in GM(\hat{\mathbf{R}}^n)$ and that $\phi(\infty)\neq\infty$. Let $S(A,r)$ be the isometric sphere of ϕ. Set*

$$H_r=\{x\in\mathbf{H}^{n+1}\mid x=x_1e_1+\cdots+x_ne_n+x_{n+1}e_{n+1}: x_{n+1}>r\},$$

where $e_1,\dots,e_{n+1}$ are the usual basis vectors in $\mathbf{R}^{n+1}$. Then the following conclusions hold.

(a) *For each P in $\mathbf{H}^{n+1}$ we have*

$$|\tilde{\phi}(P)-\tilde{\phi}(\infty)||P-\tilde{\phi}^{-1}(\infty)|=r^2.$$

(b) *For each P in H_r we have*

$$|\tilde{\phi}(P)-\tilde{\phi}(\infty)|<r.$$

(c) *We have $\tilde{\phi}(H_r)\cap H_r=\emptyset$.*

Proof. Let σ denote the inversion in $S(A,r)$. In [2, Sec. 3.5] it is shown that

$$A=\phi^{-1}(\infty), \tag{1}$$

and

there is a unique Euclidean isometry ψ of $\mathbf{R}^n$ such that $\phi=\psi\sigma$. (2)

It follows that

$$\tilde{\phi}=\tilde{\psi}\tilde{\sigma}. \tag{3}$$

The Poincaré extension $\tilde{\sigma}$ is the inversion in the sphere $S(\tilde{A},r)$, where $\tilde{A}=(A,0)$ is a point on the hyperplane $x_{n+1}=0$. Furthermore, $\tilde{\psi}$ is a Euclidean isometry of $\mathbf{R}^{n+1}$. For every point P in $\mathbf{H}^{n+1}$ we therefore have

$$\begin{aligned}|\tilde{\phi}(P)-\tilde{\phi}(\infty)||P-\tilde{\phi}^{-1}(\infty)|&=|\tilde{\psi}\tilde{\sigma}(P)-\tilde{\psi}\tilde{\sigma}(\infty)||P-\tilde{\sigma}^{-1}\tilde{\psi}^{-1}(\infty)|\\&=|\tilde{\sigma}(P)-\tilde{\sigma}(\infty)||P-\tilde{\sigma}^{-1}(\infty)|\\&=|\tilde{\sigma}(P)-\tilde{A}||P-\tilde{A}|=r^2,\end{aligned} \tag{4}$$

where in the last equality we have used the basic geometric property of inversion in a sphere. This proves assertion (a) of the proposition.

For every point P in H_r, by definition we have

$$|P-\tilde{A}|>x_{n+1}>r. \tag{5}$$

We combine (4) and (5) and the fact that $\tilde{A}=\tilde{\phi}^{-1}(\infty)$ to deduce that for every P in H_r we have

$$|\tilde{\phi}(P)-\tilde{\phi}(\infty)|=\frac{r^2}{|P-\tilde{A}|}<\frac{r^2}{r}=r. \tag{6}$$

This proves assertion (b).

The set H_r is a horoball with center at ∞, whereas $\tilde{\phi}(H_r)$ is a Euclidean $(n+1)$-sphere. The hyperplane $x_{n+1}=0$, which we identify with $\hat{\mathbf{R}}^n$, is tangent to $\tilde{\phi}(H_r)$ at the point $\tilde{\phi}(\infty)$. Since $\tilde{\phi}(\infty)$ is the center of the isometric sphere of ϕ^{-1}, it belongs to $\mathbf{R}^n$. We now conclude from (6) that $\tilde{\phi}(H_r)\cap H_r=\emptyset$. This proves assertion (c) and completes the proof of Proposition 2.1. □

It was shown by Vahlen that any Möbius transformation in $\hat{\mathbf{R}}^n$ can be represented as a 2×2 matrix which has entries in the n-dimensional Clifford algebra and is subject to certain conditions. Such a matrix is called a *Vahlen matrix*. Vahlen showed that the set of all such matrices is a group under matrix multiplication. This group is denoted by $PSL_+(\Gamma_n)$ and acts on $\hat{\mathbf{R}}^n$ according to the formula $x\to(ax+b)(cx+d)^{-1}$. This action can be used to define an isomorphism between $PSL_+(\Gamma_n)$ and $SO^+(1,n)$.

If

$$M=\begin{pmatrix} a & b \\ c & d \end{pmatrix}\in PSL_+(\Gamma_n),$$

the Möbius transformation corresponding to M fixes ∞ if and only if $c=0$. When $c\neq 0$ the isometric sphere of the Möbius transformation corresponding to M is the set $\{x\in \mathbf{R}^n \mid |cx+d|^{-2}=1\}$. With the notation of Proposition 2.1, we have $A=-c^{-1}d$ and $r=|c|^{-1}$. For a detailed exposition of Vahlen matrices see [1].

In [5] we used this representation to generalize the Shimizu–Leutbecher inequality (see [11] and [7]) to Möbius transformations in $\hat{\mathbf{R}}^n$. The result we obtained was the following.

THEOREM 2.2. *Let*

$$U=\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \quad \textit{and} \quad M=\begin{pmatrix} a & b \\ c & d \end{pmatrix},$$

with $c\neq 0$, *be elements of* $SL_+(\Gamma_n)$. *Suppose that* $\langle M, U\rangle$ *is discrete. Then* $|c|\geq 1$.

Using Proposition 2.1 and Theorem 2.2, we shall now describe the action of elements of a discrete group on horoballs based at parabolic fixed points. The following theorem will be crucial in the proof of Theorem 1. This description generalizes a classical result for the case $n=3$. The argument given below, when specialized to the 3-dimensional case, proves the classical result without complicated computations involving quaternions.

THEOREM 2.3. *Let* Δ *be a discrete subgroup of* $PSL_+(\Gamma_n)$. *Let*

$$U=\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\in \Delta.$$

Set

$$H_1=\{x\in \mathbf{H}^{n+1} \mid x=x_1e_1+\cdots+x_ne_n+x_{n+1}e_{n+1}: x_{n+1}>1\}.$$

Let

$$M=\begin{pmatrix} a & b \\ c & d \end{pmatrix}\in PSL_+(\Gamma_n)$$

such that $M(\infty)\neq\infty$ *and* $T=MUM^{-1}\in\Delta$. *Then* $H_1\cap \tilde{M}(H_1)=\emptyset$.

Proof. The set H_1 is a horoball with center at ∞. The Euclidean $(n+1)$-sphere $\tilde{M}(H_1)$ is tangent at the point $\tilde{M}(\infty)$ to the hyperplane $x_{n+1}=0$ (which we identify with $\hat{\mathbf{R}}^n$). By the first assumption on M we have that $c\neq 0$. A simple calculation based on the properties of the Vahlen matrices (see [1] or [5, Thm. A]) shows that the lower left entry of T equals $-cc^*$. (There is a unique linear involution $a\mapsto a^*$ on the Clifford algebra in dimension n which is the identity on the n-dimensional vector space spanned by the basis elements of the algebra.) It is also shown in [1] that the norm is multiplicative on the entries of any Vahlen matrix and that $|c|=|c^*|$. Hence,

$$|-cc^*|=|c|^2. \tag{1}$$

By Theorem 2.2 we have

$$|c|^{-2}<1. \tag{2}$$

This implies that

$$|c|^{-1}<1. \tag{3}$$

Proposition 2.1 with $r=|c|^{-1}$ implies that, for each P in H_1, we have

$$|\tilde{M}(P)-\tilde{M}(\infty)|<|c|^{-1}. \tag{4}$$

We use (3) to deduce that for each P in H_1 we have

$$|\tilde{M}(P)-\tilde{M}(\infty)|<1. \tag{5}$$

This completes the proof. □

REMARKS.

1. For reasons of conjugacy one can replace the 1 in the top right entry of U in Theorem 2.2 by any β in $\mathbf{R}^n$. (The conclusion will then be that $|c\beta|\geq 1$.)
2. Similar adjustments can be made in Theorem 2.3.
3. For a different approach to some of the results of this section, see [12].

3. Proof of Theorem 1

The following proposition and corollary will describe the structure of Δ_∞, the stabilizer of ∞ in Δ. Proposition 3.1 generalizes [2, Thm. 5.1.2].

PROPOSITION 3.1. *Let f and g be elements of $PSL_+(\Gamma_n)$. Suppose that g is loxodromic. Suppose that f and g have exactly one fixed point in common. Then $\langle f, g\rangle$ is not discrete.*

Proof. The proof is identical to that of [2, Thm. 5.1.2] except that Vahlen matrices are used in place of elements of $PSL(2,\mathbf{C})$. It is essential to observe that the entries of the Vahlen matrices are elements of the Clifford group [1], on which the norm is multiplicative. □

COROLLARY 3.2. *Let Δ be a discrete subgroup of $PSL_+(\Gamma_n)$. Suppose that Δ_∞ contains a parabolic element. Then Δ_∞ is a discrete group of Euclidean isometries of $\mathbf{R}^n$.*

Now let Δ be a discrete subgroup of $PSL_+(\Gamma_n)$. Suppose that Δ has finite covolume. Let $p\in\hat{\mathbf{R}}^n$ be the fixed point of some parabolic element $\gamma\in\Delta$. Choose $M\in PSL_+(\Gamma_n)$ such that $M(\infty)=p$. The group $(M^{-1}\Delta M)_\infty$ contains the parabolic element $M^{-1}\gamma M$ and has finite volume. By the proof of [6, Thm. 2], $(M^{-1}\Delta M)_\infty$ contains a unipotent element. A unipotent element fixing ∞ can be identified with a Vahlen matrix

$$U=\begin{pmatrix}1 & \beta\\ 0 & 1\end{pmatrix}.$$

By Theorem 2.3 and the remarks following it, we get that $H_{|\beta|}/(M^{-1}\Delta M)_\infty$ embeds in $\mathbf{H}^{n+1}/(M^{-1}\Delta M)$ and therefore $\operatorname{vol}(H_{|\beta|}/(M^{-1}\Delta M)_\infty) < \infty$. By Corollary 3.2, the group $(M\Delta M^{-1})_\infty$ is a discrete group of Euclidean isometries of the horosphere

$$HS_{|\beta|} = \{x \in \mathbf{H}^{n+1} : x_{n+1} = |\beta|\},$$

which we identify with $\mathbf{R}^n$. Since $\operatorname{vol}(H_{|\beta|}/(M\Delta M^{-1})_\infty) < \infty$, it follows easily that $\operatorname{vol}(HS_{|\beta|}/(M\Delta M^{-1})_\infty) < \infty$. But according to [9, Cor. 8.25], any discrete group of Euclidean isometries of $\mathbf{R}^n$ having finite covolume is cocompact.

By the Bieberbach theorem (see [13, Thm. 3.2.8]), the group Λ consisting of all Euclidean translations in $(M^{-1}\Delta M)_\infty$ has finite index and rank n in $(M^{-1}\Delta M)_\infty$. We may identify Λ with a discrete group of vectors in $\mathbf{R}^n$. Let v be the shortest nonzero vector in Λ. Thus $|v|$ is the minimum length of any nontrivial translation in $(M^{-1}\Delta M)_\infty$.

We define the *canonical horoball* at the point p to be $\tilde{M}(H_{|v|})$. The canonical horoball is well defined. Indeed, if M_1 is any other element of $PSL_+(\Gamma_n)$ such that $T(\infty) = p$, then we have $M_1 = MT$ for some similarity of $\hat{\mathbf{R}}^n$. The group of Euclidean translations in $(M_1^{-1}\Delta M_1)_\infty$ is $T(\Lambda)$, and its shortest vector is $v_1 = T(v)$. It is clear that $T(H_{|v|}) = H_{|v_1|}$, and hence that $M_1(H_{|v_1|}) = M(H_{|v|})$.

Proposition 3.3. *Let Δ be a discrete subgroup of $PSL_+(\Gamma_n)$. Suppose that Δ has finite covolume. Then the canonical horoballs at any two distinct parabolic fixed points are disjoint.*

Proof. Without loss of generality we may assume that one of the given parabolic fixed points is ∞. Among all nontrivial Euclidean translations in Δ_∞, let U denote the one with minimal length. After conjugation by a similarity we may assume that

$$U = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

By the previous definition, H_1 is the canonical horoball at ∞.

Let $x \in \hat{\mathbf{R}}^n$ be the other parabolic fixed point. Choose $M \in PSL_+(\Gamma_n)$ such that $M(\infty) = x$. Notice that we may choose M so that among all nontrivial Euclidean translations in $(M^{-1}\Delta M)_\infty$, the one with minimal translation length is

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

In particular, $MUM^{-1} \in \Delta$. The canonical horoball associated with x is $\tilde{M}(H_1)$. Upon applying Theorem 2.3, the assertion of the proposition follows. □

Proposition 3.4. *Let Δ be a discrete subgroup of orientation-preserving isometries of $\mathbf{H}^{n+1}$. Suppose that Δ has finite covolume. Let x be any para-*

bolic fixed point of Δ, and let H denote the canonical horoball at x. Then

$$\mathrm{vol}\left(\frac{H}{\Delta_x}\right) \geq \frac{\delta_n}{nI_n}.$$

Proof. We may assume that $x=\infty$, and that H/Δ_∞ has finite volume. By Corollary 3.2 and the discussion following it, Δ_∞ is a discrete cocompact group of isometries of $\mathbf{R}^n$, and the group Λ of all translations in Δ_∞ has finite index and rank n in Δ_∞. We identify Λ with a discrete group of vectors in $\mathbf{R}^n$.

Now let β denote the shortest nonzero vector in Λ. Then $H=H_{|\beta|}$. Let $P\subset\mathbf{R}^n$ be a fundamental parallelepiped for Λ. Then the set

$$F=\{x+r_{n+1}e_{n+1} \mid x\in P,\, r_{n+1}>|\beta|\} \tag{1}$$

is a fundamental domain for the action of Λ on $H=H_{|\beta|}$. Since H/Δ_∞ has finite volume, the hyperbolic volume of F is given by

$$\mathrm{vol}(F)=\int_{|\beta|}^{\infty} \frac{dx_1\cdots dr_{n+1}}{r_{n+1}^{n+1}}. \tag{2}$$

From (2) we obtain

$$\mathrm{vol}(F)=|\Lambda|\int_{|\beta|}^{\infty} \frac{dr_{n+1}}{r_{n+1}^{n+1}}=\frac{|\Lambda|}{n|\beta|^n}. \tag{3}$$

By the definition of δ_n we have

$$\mathrm{vol}(F)\geq\frac{\delta_n}{n}. \tag{4}$$

By the definition of I_n it follows that $|\Delta_\infty : \Lambda|\leq I_n$. Hence

$$\mathrm{vol}\left(\frac{H}{\Delta_\infty}\right)=\frac{\mathrm{vol}(F)}{|\Delta_\infty : \Lambda|}\geq\frac{\delta_n}{nI_n}. \tag{5}$$

This proves the proposition. □

Proof of Theorem 1. Let $x_1,\dots,x_m$ be inequivalent parabolic fixed points of Δ. For $i=1,\dots,m$ let H_i denote the canonical horoball at x_i. It follows from Proposition 3.3 that $H_1/\Delta_{x_1},\dots,H_m/\Delta_{x_m}$ are isometric to pairwise disjoint open subsets of $\mathbf{H}^{n+1}/\Delta$. The assertion of the theorem therefore follows from Proposition 3.4. □

REMARKS. 1. It is known (see [10]) that $I_n\leq 3^{n^2}$. It is also known (see [3]) that $\delta_n\geq(2^{-n}\pi^{n/2})/(n/2)!$ for n even, and $\delta_n\geq(\pi^{((n-1)/2)}((n-1)/2))!)/n!$ for n odd.

2. Theorem 1 provides an explicit lower bound for the covolume in terms of I_n and δ_n. This lower bound is not the best possible. The constant δ_n comes from the critical lattice in $\mathbf{R}^n$. The constant I_n comes in general from a specific Bieberbach group in $\mathbf{R}^n$ which may not have the critical lattice as its group of translations. One can get a better lower bound by an analysis of the Bieberbach groups in $\mathbf{R}^n$. This will be illustrated in the next section.

4. Proof of Theorem 2

It is well known [3] that for any lattice Λ in $\mathbf{R}^3$ we have

$$|\Lambda| \geq \frac{1}{\sqrt{2}} |\beta|^3. \tag{1}$$

When $n=3$ in the torsion-free case it is also known [13, Thm. 3.5.5] that every 3-dimensional torsion-free Bieberbach group contains a translation subgroup of index at most 6. Thus it is clear from the proof of Theorem 1 that if Δ is a discrete, torsion-free subgroup of $Iso^+(\mathbf{H}^4)$ having parabolic elements then

$$\operatorname{covol}(\Delta) \geq \frac{1}{\sqrt{2}\cdot 3\cdot 6} = \frac{\sqrt{2}}{36}. \tag{2}$$

In this case Theorem 2 provides a better estimate.

Proof of Theorem 2. The group Δ_∞ is an orientation-preserving, torsion-free 3-dimensional Bieberbach group. Theorem 3.5.5 of [13] gives a complete classification of such groups; up to affine equivalence there are exactly six of them, denoted $\mathcal{G}_1, \ldots, \mathcal{G}_6$. Let us first consider the case where Δ_∞ is isomorphic to $\mathcal{G}_5$. In this case the translation subgroup of Δ_∞ is

$$\Lambda_t = \{a_1, a_2, a_3\},$$

where a_1 is orthogonal to a_2 and a_3 and $\{a_2, a_3\}$ is a hexagonal plane lattice. The index of Λ in Δ_∞ is 6 in this case. Simple geometric arguments show that the volume of the period parallelepiped for this lattice is greater than or equal to $\sqrt{3}/2$, so in this case we obtain

$$\operatorname{covol}(\Delta) \geq \frac{\sqrt{3}/2}{3\cdot 6} = \frac{\sqrt{3}}{36}.$$

In a similar manner, one obtains a lower bound for the covolume of Δ in each of the other five cases, using the index of Λ in Δ_∞ and a geometric estimate for the volume of the period parallelepiped. By inspection, these values are all seen to be strictly greater than $\sqrt{3}/36$. In particular, if Δ_∞ is isomorphic to $\mathcal{G}_1$, then the index is 1 and the volume is at least $\sqrt{2}/2$. This completes the proof of the theorem. □

Acknowledgment. The author of this paper is a graduate student at the University of Illinois at Chicago. The author wishes to express his deep thanks to his supervisor, Professor Peter Shalen. His advice, support, and inspiration made this work possible. The author also thanks Professor Alex Lubotzky for suggesting this problem and for many useful discussions, and Professor Marc Culler and Professor Shmuel Friedland for many useful discussions.

The author thanks Professor Douglas Dickson for many useful remarks that made this paper more readable.

References

[1] L. V. Ahlfors, *On the fixed points of Möbius transformations in* $\mathbf{R}^n$, Ann. Acad. Sci. Fenn. Ser. A I Math. 10 (1985), 15–27.
[2] A. F. Beardon, *The geometry of discrete groups,* Springer, New York, 1983.
[3] J. W. S. Cassels, *An introduction to the geometry of numbers,* Springer, Berlin, 1959.
[4] J. Elstrodt, F. Grunwald, and J. Mennicke, *Discrete groups on hyperbolic* 3-*space and their automorphic functions,* Preprint.
[5] S. Hersonsky, *A generalization of the Shimizu–Leutbecher lemma and the Jorgensen inequality to Möbius transformations in* $\hat{\mathbf{R}}^n$, Proc. Amer. Math. Soc. (to appear).
[6] D. A. Kazdan and G. A. Margulis, *A proof of Selberg's conjecture,* Math. USSR-Sb. 4 (1968), 147–151.
[7] A. Leutbecher, *Über Spitzen diskontinuierlicher Gruppen von lineargebrochenen Transformationen,* Math. Z. 100 (1967), 183–200.
[8] R. Meyerhoff, *A lower bound for the volume of hyperbolic 3-manifolds,* Canad. J. Math. 39 (1987), 1038–1056.
[9] M. S. Raghunathan, *Discrete subgroups of Lie groups,* Springer, Berlin, 1972.
[10] J. P. Serre, *Lie algebras and Lie groups,* 2nd ed., Springer, Berlin, 1992.
[11] H. Shimuzu, *On discontinuous groups operating on the product of the upper half planes,* Ann. of Math. (2) 77 (1963), 33–71.
[12] N. Wielenberg, *Discrete Moebius groups,* Amer. J. Math. 99 (1977), 861–877.
[13] J. A. Wolff, *Spaces of constant curvature,* McGraw-Hill, New York, 1967.

Department of Mathematics
University of Illinois at Chicago
Chicago, IL 60680-4348