The Infinite Nielsen Kernels of Some Bordered Riemann Surfaces

CLIFFORD J. EARLE

To Lipman Bers

1. Introduction

Let X be a Riemann surface of finite type (p, n, m), with $m \geq 1$. That means X can be conformally embedded in a closed Riemann surface Y of genus p so that $Y \setminus X$ consists of $m \geq 1$ disjoint closed disks and $n \geq 0$ additional points, called the *punctures* of X. There is a unique bordered Riemann surface $\bar{X}$ whose interior is X and whose border B is the union of m disjoint simple loops C_j, called the *boundary loops* of X.

Suppose X has negative Euler characteristic:

$$e(X) = 2 - (2p + n + m) < 0. \tag{1.1}$$

Then each boundary loop C_j is freely homotopic in $\bar{X}$ to a unique simple closed Poincaré geodesic C_j' in X, and C_j' and C_k' are disjoint if $j \neq k$. (These geodesics are defined using the complete Poincaré metric of curvature -1 on X, which puts the boundary loops C_j at infinite distance.) The Nielsen kernel of X is the interior $N(X)$ of the bordered Riemann surface obtained from X by removing the m annuli bounded by the pairs of freely homotopic loops C_j and C_j', $1 \leq j \leq m$.

Viewed as a Riemann surface in its own right, $N(X)$ has the same finite type (p, n, m) as X. We can therefore iterate the construction above, forming the nested sequence of Riemann surfaces

$$N^{k+1}(X) = N(N^k(X)) \subset N^k(X) \subset N^1(X) = N(X) \subset X, \quad k \geq 1.$$

Bers [2] suggested an investigation of the set

$$N^\infty(X) = \bigcap_{k=1}^{\infty} N^k(X),$$

called the *infinite Nielsen kernel* of X. That is a hard problem. The first progress was made by Wason [11] and Halpern [5; 6]. They compared lengths and distances of certain closed geodesics on X and $N(X)$, and studied the effect of iteration. Their results suggest that $N^\infty(X)$ is a rather thin set. That

Received December 9, 1991.

This research was supported in part by the Institut Mittag-Leffler and by NSF grant DMS-8901729.

Michigan Math. J. 40 (1993).

is indeed the case. Cao [3] has recently proved that $N^{\infty}(X)$ has no interior points. However, more precise information remains elusive.

In this paper we compute $N^{\infty}(X)$ explicitly in a number of cases. In these cases $N^{\infty}(X)$ turns out to be a 1-dimensional spine consisting of geodesic arcs. Our method is to use two standard tools that have not previously been applied to $N(X)$. One is the annular covering surface (see [8] or [9]) associated to a given boundary loop of X. The other is Hadamard's three-circle theorem. These tools provide a new characterization of $N^{\infty}(X)$, stated in Section 3 as Theorem 1. They also provide new information about the size of $N(X)$, stated and proved in Section 8 as Theorem 2. Theorem 1 can be applied directly to find $N^{\infty}(X)$ in some cases. We state the simplest applications as corollaries of Theorem 1 in Section 3. Proposition 1 in the same section allows us to determine many additional infinite Nielsen kernels by judicious use of branched covering maps between Riemann surfaces. The examples in Section 4 illustrate how this technique works.

To make our methods as flexible as possible, we have used Riemann surfaces with orbifold structures. The facts we need about finite bordered orbifolds and their Nielsen kernels are reviewed in Sections 2 and 5. Section 7 introduces the annular covering surfaces, and the remainder of the paper contains the proofs of our results.

We discovered the Riemann surface version of Theorem 1 in 1990 and presented some applications in an Institut Mittag–Leffler technical report [4], which is superseded by this paper. A number of people have helped us make our applications of the theorem more systematic and effective. We are especially grateful to John Smillie for suggesting that more examples could be obtained by using Proposition 1 and to Mika Seppälä for helping to determine the form of the orbifolds in Corollaries 1 and 2. These special orbifolds are the basic building blocks for all our examples.

2. Finite Bordered Orbifolds and Their Nielsen Kernels

In this section we shall define the infinite Nielsen kernel of a finite bordered Riemann surface that has an appropriate orbifold structure. When there are no orbifold points our definition will agree with the one given in the introduction. The general theory of orbifolds is developed in Chapter 13 of Thurston's notes [10]. We shall review some basic facts and definitions, confining ourselves to the case when the underlying topological space is a Riemann surface.

For our purposes an orbifold structure on the Riemann surface X is a function μ on X such that $\mu(x)$ is a positive integer for every x in X and the set $\{x \in X;\ \mu(x) > 1\}$ is discrete. The points where $\mu(x) > 1$ are called the *orbifold points* of X. An orbifold is a Riemann surface with an orbifold structure. The orbifold consisting of the Riemann surface X with orbifold structure μ will be denoted by (X, μ), or simply by X if it is clear what orbifold structure we have in mind.

If (X, μ) is an orbifold and Y is a subregion of X, the open suborbifold Y is defined in the obvious way: the Riemann surface structure on Y is the one it inherits from X, and the orbifold structure is the restriction of μ to Y.

By definition, an orbifold covering map $f: (X_1, \mu_1) \to (X_2, \mu_2)$ is a holomorphic map f of X_1 onto X_2 such that f is topologically a branched covering map and, for each x in X_1, the multiplicity of f at x is $\mu_2(f(x))/\mu_1(x)$. Thus f has no branch points if $\mu_2(f(x)) = \mu_1(x)$ for all x. The maximum cardinality of a fiber $f^{-1}(x)$ for x in X_2 is attained at every non-orbifold point of X_2 and is called the *degree* of f.

The orbifolds (X_1, μ_1) and (X_2, μ_2) are isomorphic if and only if there is an orbifold covering map $f: (X_1, \mu_1) \to (X_2, \mu_2)$ of degree 1; such an f is called an *isomorphism*. Two orbifold covering maps $f_i: (X_i, \mu_i) \to (X, \mu)$, $i = 1, 2$, are equivalent if and only if there is an isomorphism $\varphi: (X_1, \mu_1) \to (X_2, \mu_2)$ such that $f_2 \circ \varphi = f_1$.

Let G be a group of holomorphic automorphisms of a Riemann surface Y. If G acts properly discontinuously on Y, the quotient space Y/G has a unique Riemann surface structure that makes the quotient map $f: Y \to Y/G$ holomorphic. The standard orbifold structure on Y/G is defined by putting $\mu(f(y))$ equal to the order of the group $\{g \in G;\ g(y) = y\}$ for every y in Y. By definition the quotient orbifold Y/G is the Riemann surface Y/G with this standard orbifold structure.

If W is a G-invariant subregion of Y, then the quotient orbifold W/G is an open suborbifold of Y/G. Moreover, if H is any subgroup of G, the natural map of Y/H onto Y/G is an orbifold covering map. If Y is simply connected, then every orbifold covering of Y/G is equivalent to one of the above type. In particular, up to equivalence, a given orbifold (X, μ) has at most one orbifold covering by a simply connected Riemann surface with no orbifold points (see Chapter 13 of [10]).

By definition, a *finite bordered orbifold* is one that is isomorphic to a quotient orbifold Δ/Γ, where Δ is the open unit disk in $\mathbf{C}$, Γ is a finitely generated Fuchsian group acting on Δ, and the limit set $L(\Gamma)$ is a Cantor set in the unit circle S^1. Equivalently (see Theorem 10.4.2 of [1] or Chapter 13 of [10]), (X, μ) is a finite bordered orbifold if and only if the Riemann surface X has finite type (p, n, m), $m \geq 1$, the number of orbifold points is finite, and

$$2p - 2 + n + m + \sum_{x \in X} \left(1 - \frac{1}{\mu(x)}\right) > 0. \tag{2.1}$$

Observe that (2.1) reduces to (1.1) if there are no orbifold points.

The Nielsen kernel of the finite bordered orbifold (X, μ) is the open suborbifold $N(X, \mu)$ defined as follows. Let f be an isomorphism of a quotient orbifold Δ/Γ onto (X, μ). Since $L(\Gamma)$ is a Cantor set, Δ has a unique smallest nonempty Γ-invariant open subset that is convex with respect to Poincaré geodesics (see Theorem 8.5.2 of [1]). That set is called the Nielsen region $N(\Gamma)$ of Γ. The quotient orbifold $N(\Gamma)/\Gamma$ is an open suborbifold of Δ/Γ. Its image under the isomorphism f is the Nielsen kernel $N(X, \mu)$.

The Riemann surface $N(X, \mu)$ has the same finite type as X and contains all the orbifold points of (X, μ) (see Section 5), so we can form the Nielsen kernel of the orbifold $N(X, \mu)$ and go on to form the nested sequence of open suborbifolds

$$N^{k+1}(X, \mu) = N(N^k(X, \mu)) \subset N^k(X, \mu) \subset N^1(X, \mu) = N(X, \mu) \subset (X, \mu),$$

$k \geq 1$, each of which is an open subset of X (with the orbifold structure defined by restricting μ). The intersection of these subsets is the infinite Nielsen kernel

$$N^\infty(X, \mu) = \bigcap_{k=1}^{\infty} N^k(X, \mu).$$

REMARK. According to our definition, some finite bordered Riemann surfaces with no orbifold points (the unit disk, the punctured unit disk, and the annuli) do not qualify as finite bordered orbifolds since they do not satisfy (2.1). Our finite bordered orbifolds are the ones that carry a hyperbolic structure such that the boundary loops are geodesics. It would be more precise to call them hyperbolic finite bordered orbifolds, but we preferred a shorter name.

3. The Main Results

Our main theorem characterizes $N^\infty(X, \mu)$ by a certain minimality condition. The set $N^\infty(X, \mu)$ has properties (3.1) through (3.4) listed below, and any proper closed subset must fail to have property (3.2), (3.3), or (3.4).

Recall that if (X, μ) is a finite bordered orbifold, the Riemann surface X has finite type (p, n, m) and a border B consisting of the disjoint simple boundary loops C_j, $1 \leq j \leq m$.

THEOREM 1. *The infinite Nielsen kernel of the finite bordered orbifold (X, μ) is the unique closed subset $F \subset X$ that has the following four properties:*

- (3.1) $F \subset N^k(X, \mu)$ *for all* $k \geq 1$;
- (3.2) *F contains every orbifold point;*
- (3.3) *for each boundary loop C_j, there is a unique doubly connected component $V(C_j)$ of $X \setminus F$ that has C_j as one of its boundary components;*
- (3.4) *the map $C_j \mapsto V(C_j)$ from the boundary loops of X to the components of $X \setminus F$ is bijective.*

Theorem 1 has the following easy corollaries, which we shall prove in Section 10.

COROLLARY 1. *Let $X = \Delta \setminus \{z_1, \ldots, z_n\}$, $n \geq 0$. Suppose the orbifold points and the points z_j, $1 \leq j \leq n$, all lie on the real axis. Let E be the smallest closed interval that contains all these points. Then $N^\infty(X, \mu) = X \cap E$.*

Note that condition (2.1) insures that E is a nontrivial closed interval. If a and b are real numbers with $0 < a < b$, we put

$$A(a, b) = \{z \in \mathbf{C}; a < |z| < b\}.$$

Corollary 2. *Choose any number $r > 1$ and let $X = A(r^{-1}, r) \backslash \{z_1, \dots, z_n\}$, $n \geq 0$. Suppose the orbifold points and the points z_j all lie on the unit circle S^1. Then $N^\infty(X, \mu) = X \cap S^1$.*

Our further applications of Theorem 1 rely on the following proposition, which is essentially a restatement of the well known fact that if a subgroup Γ_1 has finite index in a Fuchsian group Γ_2, then Γ_1 and Γ_2 have the same limit sets and Nielsen regions. Its relevance to the problem of finding infinite Nielsen kernels was brought to our attention by John Smillie.

Proposition 1. *Let $f: (X, \mu) \to (Y, \nu)$ be an orbifold covering map between the finite bordered orbifolds (X, μ) and (Y, ν). Suppose the degree of f is finite. Then for all x in X we have*

(3.5) *$f(x) \in N(Y, \nu)$ if and only if $x \in N(X, \mu)$, and*
(3.6) *$f(x) \in N^\infty(Y, \nu)$ if and only if $x \in N^\infty(X, \mu)$.*

The examples in the next section illustrate how to use Proposition 1 and Corollary 1 to find a number of infinite Nielsen kernels.

Remark. Two Fuchsian groups Γ_1 and Γ_2 are said to be *commensurable* if the intersection $\Gamma_1 \cap \Gamma_2$ has finite index in both of them. Let the finite bordered orbifolds Δ/Γ_1 and Δ/Γ_2 be defined by commensurable groups. If we know $N^\infty(\Delta/\Gamma_1)$ we can find $N^\infty(\Delta/\Gamma_2)$ by applying Proposition 1 to the natural maps $\Delta/(\Gamma_1 \cap \Gamma_2) \to \Delta/\Gamma_i$, $i = 1, 2$. Corollaries 1 and 2 provide us with groups Γ for which $N^\infty(\Delta/\Gamma)$ is known, so we can find $N^\infty(\Delta/\Gamma')$ whenever Γ' is commensurable with one of these groups Γ. These orbifolds Δ/Γ' are the only ones whose infinite Nielsen kernels we have been able to compute.

4. Some Examples

We offer four examples of Riemann surfaces X whose infinite Nielsen kernels are found by using the results from Section 3. In each example we regard X as a finite bordered orbifold with the trivial orbifold structure $\mu(x) \equiv 1$, and we find an orbifold covering map between X and an orbifold whose infinite Nielsen kernel we already know.

Example 1. Let X be a pair of pants whose three boundary loops are interchangeable. To be concrete, put $\omega = \exp(2\pi i/3)$, choose r in the open interval $(1, 2)$, and let X be the Riemann sphere with the three disjoint closed disks

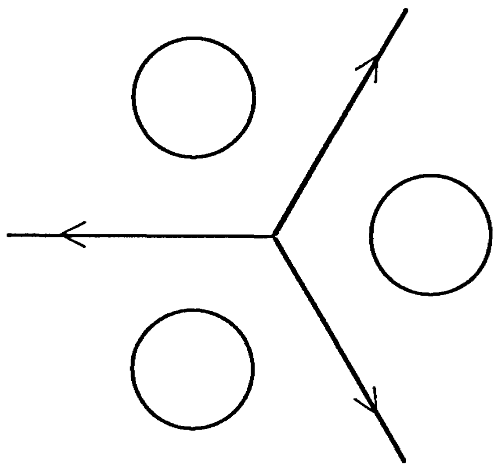

Figure 1

$$\{z \in \mathbf{C}; |z - r\omega^j|^2 \le r^2 - 1\}, \quad j = 1, 2, 3,$$

deleted (see Figure 1). Let G be the cyclic group of order 3 generated by the rotation $z \mapsto \omega z$. The quotient orbifold X/G is a simply connected bordered Riemann surface with two orbifold points (corresponding to the points 0 and ∞ in X). The involution $z \mapsto \bar{z}$ of X induces a conjugate holomorphic involution φ of X/G. The fixed point set of φ is a connected 1-dimensional submanifold M of X/G, containing both orbifold points. Corollary 1 of Theorem 1 tells us that $N^\infty(X/G)$ is the closed interval I in M joining the two orbifold points, and Proposition 1 tells us that $N^\infty(X)$ is the inverse image of I under the quotient map $X \to X/G$. That inverse image consists of the points 0 and ∞, the negative real axis, and its images under G, as shown in Figure 1.

EXAMPLE 2. Let X be the Riemann surface of type $(2, 0, 1)$ shown in Figure 2. We assume that the reflection φ_1 in the plane of the figure and the reflection φ_2 in the orthogonal plane through the horizontal axis are conjugate holomorphic involutions of X. The composite $\varphi_1 \circ \varphi_2$ is the rotation of order 2 about the horizontal axis; let G be the group of order 2 that it generates. The orbifold X/G is again a simply connected bordered Riemann surface, with five orbifold points corresponding to the fixed points of $\varphi_1 \circ \varphi_2$. The reflections φ_1 and φ_2 of X cover the same conjugate holomorphic involution φ

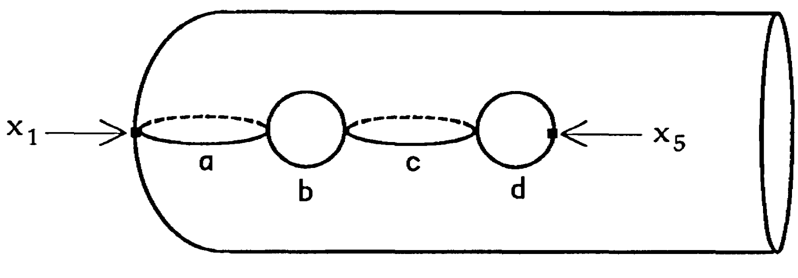

Figure 2

of X/G. Its fixed point set is again a 1-dimensional submanifold M containing all the orbifold points. Let I be the closed interval in M that joins the orbifold points that correspond to the points x_1 and x_5 in Figure 2. Since I contains all the orbifold points, Corollary 1 again tells us that $N^\infty(X/G)$ equals I, so $N^\infty(X)$ is again the inverse image of I under the quotient map $X \to X/G$. This time that inverse image is the union of the four simple loops labelled a, b, c, and d in Figure 2.

EXAMPLE 3. Next we find $N^\infty(Y)$ for a Riemann surface Y that is a 2-fold unbranched covering of the pairs of pants X in Example 1. Explicitly, we set $Y = \psi^{-1}(X)$, where ψ is the rational function

$$\psi(\zeta) = \frac{\zeta^2 + \omega}{\omega\zeta^2 + 1}, \quad \zeta \in \mathbf{C} \cup \{\infty\}.$$

Since $\psi: Y \to X$ has no branch points it is an orbifold covering map (with respect to the trivial orbifold structures on X and Y), so Proposition 1 tells us that $N^\infty(Y) = \psi^{-1}(N^\infty(X))$. The boundary of the plane region Y consists of four disjoint ovals, and the subset $N^\infty(Y)$ is shown in Figure 3.

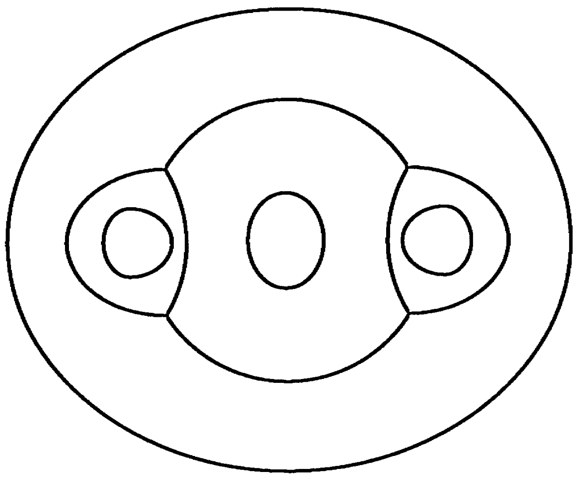

Figure 3

EXAMPLE 4. The Riemann surface X in Example 1 is mapped onto itself by $f(z) = 1/z$. The rational function $\psi(\zeta)$ in Example 3 satisfies the identity

$$\psi(1/\zeta) = f(\psi(\zeta)).$$

Therefore the transformation $g(\zeta) = 1/\zeta$ is a holomorphic automorphism of $Y = \psi^{-1}(X)$. Let G be the group of order two that it generates. The quotient $W = Y/G$ is a pair of pants, and there are no orbifold points since $g(\zeta)$ has no fixed points in Y. By Proposition 1, $N^\infty(W)$ is the image of $N^\infty(Y)$ under the quotient map. Figure 4 shows W and $N^\infty(W)$. It is interesting to observe that the infinite Nielsen kernels of the pairs of pants in Figures 1 and 4 are nonisomorphic graphs, each with two vertices and three edges.

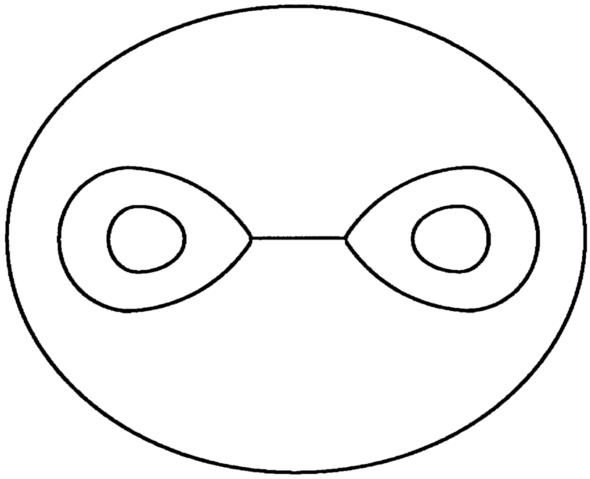

Figure 4

REMARK. For the simplest applications of Proposition 1, we choose a Riemann surface X and a subgroup G of its group of holomorphic automorphisms so that Corollary 1 or 2 can be used to find $N^\infty(X/G)$. Examples 1 and 2 are both of that form. Example 3 is more sophisticated. The orbifold covering maps $Y \to X$ in Example 3 and $X \to X/G$ in Example 1 can be composed to produce an orbifold covering map $Y \to X/G$ of degree 4. That covering map can be used to obtain $N^\infty(Y)$ directly from $N^\infty(X/G)$, but it is not defined by a group of automorphisms of Y. Example 4 carries us still further. The pair of pants W is a quotient orbifold of Y, so we can find $N^\infty(W)$, but W itself does not cover any orbifolds to which Corollary 1 or 2 can be applied. We are forced to apply Proposition 1 twice, first to find $N^\infty(Y)$, then $N^\infty(W)$.

Let Γ be a Fuchsian group such that the quotient orbifold Δ/Γ is isomorphic to the orbifold X/G in Example 1. Then Γ has a normal subgroup Γ_1 such that $X = \Delta/\Gamma_1$ and a subgroup Γ_2, not normal, such that $Y = \Delta/\Gamma_2$. But the best we can do for W is to find a Fuchsian group Γ_3 so that $W = \Delta/\Gamma_3$ and $\Gamma \cap \Gamma_3 = \Gamma_2$.

5. The Structure of $N(X, \mu)$

We shall need more precise information about the Nielsen kernel $N(X, \mu)$ and Nielsen region $N(\Gamma)$. The facts we need are summarized in this section, which draws together some remarks from the opening paragraphs of Sections 8.5 and 10.4 of [1].

PROPOSITION 2 (see [1]). *The Nielsen kernel of the finite bordered orbifold* (X, μ) *contains every orbifold point. The complement of* $N(X, \mu)$ *in* X *has finitely many components, one for each boundary loop. The interior of the component* $W(C_j)$ *corresponding to* C_j *is an annulus. The boundary of* $W(C_j)$ *in* X *is an analytic simple loop* C_j'*, and its boundary in* $X \cup B$

is $C_j \cup C_j'$. Moreover, C_j and C_j' are freely isotopic in the closed annulus $W(C_j) \cup C_j$, and they each generate its fundamental group.

Here is a sketch of the proof (following [1]). We may assume that (X, μ) is a quotient orbifold Δ/Γ, so that $N(X, \mu)$ is the quotient orbifold $N(\Gamma)/\Gamma$. We need a more explicit description of the Nielsen region $N(\Gamma)$, or rather of its complement in Δ.

The limit set $L(\Gamma)$ is a Cantor set in S^1, so its complement in S^1 is the union of countably many mutually disjoint open arcs α_i. Let α_i' be the Poincaré geodesic with the same endpoints as α_i, and let H_i be the closed Poincaré half-plane bounded by α_i and α_i'. (Since by definition H_i is a closed subset of Δ, it contains α_i' but not α_i or its endpoints.) The sets H_i are mutually disjoint; their union is a Γ-invariant subset $F(\Gamma)$ of Δ. Any compact set in Δ intersects only a finite number of the H_i, so $F(\Gamma)$ is closed in Δ and its connected components are the sets H_i (since each H_i is both open and closed in $F(\Gamma)$). The complementary set $\Delta \backslash F(\Gamma)$ is the Nielsen region $N(\Gamma)$.

The components of the complement of $N(\Gamma)/\Gamma$ in Δ/Γ are the images of the sets H_i under the quotient map

$$\pi : \Delta \cup (S^1 \backslash L(\Gamma)) \to (\Delta \cup (S^1 \backslash L(\Gamma)))/\Gamma.$$

Now $\pi(H_i) = H_i/\Gamma_i$, where Γ_i is the group

$$\Gamma_i = \{\gamma \in \Gamma; \gamma(H_i) = H_i\} = \{\gamma \in \Gamma; \gamma(\alpha_i) = \alpha_i\}.$$

Each group Γ_i is infinite cyclic because the sets $\pi(\alpha_i) = \alpha_i/\Gamma_i$ are the boundary loops of X. It follows easily that the complement of $N(\Gamma)/\Gamma$ has the required form.

As for the orbifold points, they are the images under π of the elliptic fixed points of Γ. No elliptic fixed point can belong to a set H_i, for if γ in Γ fixes z in H_i then γ belongs to Γ_i, so γ is the identity. Thus, every elliptic fixed point belongs to $N(\Gamma)$, as required.

Remark. It is obvious that the Riemann surface obtained from X by removing the sets $W(C_j)$ has the same finite type as X.

6. Proof of Proposition 1

The description of $N(\Gamma)$ in Section 5 reduces Proposition 1 to a triviality. Choose isomorphisms $\varphi : \Delta/\Gamma \to (X, \mu)$ and $\psi : \Delta/\Gamma' \to (Y, \nu)$ so that Γ is a subgroup of Γ' and $\psi^{-1} \circ f \circ \varphi$ is the natural map of Δ/Γ onto Δ/Γ'. Since f has finite degree, the index of Γ in Γ' is finite. Therefore the limit sets $L(\Gamma)$ and $L(\Gamma')$ are equal, so $F(\Gamma)$ and $F(\Gamma')$ consist of exactly the same closed half-planes H_i, and $N(\Gamma) = N(\Gamma')$.

To deduce (3.5) we observe that $N(Y, \nu) = \psi(N(\Gamma')/\Gamma')$, $N(X, \mu) = \varphi(N(\Gamma)/\Gamma)$, and $N(\Gamma) = N(\Gamma')$, which is Γ'-invariant.

Finally, (3.6) is a simple consequence of (3.5), for (3.5) implies that the restriction of f defines an orbifold covering map $f: N(X, \mu) \to N(Y, \nu)$, still of finite degree. We can therefore proceed inductively to prove that for every x in X and every positive integer k,

$$f(x) \in N^k(Y, \nu) \quad \text{if and only if} \quad x \in N^k(X, \mu).$$

Statement (3.6) follows at once. □

7. The Annular Covering Associated with a Boundary Loop

Given a finite bordered orbifold (X, μ) and a boundary loop C_j of X, we shall define an orbifold covering of (X, μ) by an annulus $\mathfrak{A}(C_j)$. (If there are no orbifold points, $\mathfrak{A}(C_j)$ is one of the annular covering surfaces considered in [8] and [9].) Our definition will involve some choices, but different choices will obviously produce an equivalent covering. First we choose an isomorphism $\varphi: \Delta/\Gamma \to (X, \mu)$. According to Section 5, the component of $X \setminus N(X, \mu)$ that corresponds to C_j has the form $W(C_j) = \varphi(H_i/\Gamma_i)$, where H_i is a closed Poincaré half-plane and Γ_i is the subgroup of Γ that stabilizes a component α_i of $S^1 \setminus L(\Gamma)$. In addition, the boundary loop α_i/Γ_i of Δ/Γ corresponds under φ to the loop C_j.

By definition $\mathfrak{A}(C_j) = \Delta/\Gamma_i$, and the orbifold covering $f: \mathfrak{A}(C_j) \to (X, \mu)$ is $\varphi \circ \pi_i$, where $\pi_i: \Delta/\Gamma_i \to \Delta/\Gamma$ is the natural map. Observe that the restriction of f to H_i/Γ_i is a homeomorphism onto $W(C_j)$.

We shall need to represent $\mathfrak{A}(C_j)$ explicitly in the form $A(a, b)$.

PROPOSITION 3. *The covering surface $\mathfrak{A}(C_j)$ is isomorphic to an annulus $A(1, \rho^2)$. The isomorphism can be so chosen that the induced orbifold covering map $A(1, \rho^2) \to (X, \mu)$ maps $\{\zeta \in \mathbf{C}; 1 < |\zeta| \leq \rho\}$ homeomorphically onto $W(C_j)$.*

Proof. We merely need to map Δ/Γ_i isomorphically to $A(1, \rho^2)$ so that H_i/Γ_i maps onto the prescribed subset. First we map Δ conformally onto the upper half-plane U so that α_i goes to the positive real axis and H_i goes to $\{z \in U;\ \mathrm{Re}(z) \geq 0\}$. Under this conformal map Γ_i becomes the infinite cyclic group generated by the transformation $z \mapsto az$ for some $a > 1$. Put $\rho = \exp(\pi^2/\log a)$. The function

$$\zeta(z) = \exp(-2\pi i \log z/\log a),$$

with $0 < \arg z < \pi$ in U, maps U onto $A(1, \rho^2)$ and induces an isomorphism of U/Γ_i onto $A(1, \rho^2)$. This isomorphism maps H_i/Γ_i onto $\{\zeta \in \mathbf{C}; 1 < |\zeta| \leq \rho\}$, as required. □

8. Proof of a Theorem

We can obtain useful information about the size and the location of the components of $X \setminus N(X, \mu)$ by combining Proposition 3 with the classical three-circle theorem of Hadamard (see Theorems 18.3.2 and 18.3.4 in [7]). As usual, $W(C_j)$ is the component that corresponds to the boundary loop C_j.

THEOREM 2. *Let (X, μ) be a finite bordered orbifold, and let*

$$f: A(r^{-1}, r) \to X$$

be a holomorphic map with a continuous extension that maps the outer boundary loop of $A(r^{-1}, r)$ homeomorphically onto C_j. Suppose that the multiplicity of f at z is a multiple of $\mu(f(z))$ for every z in $A(r^{-1}, r)$. Then $f(A(1, r)) \subset W(C_j)$.

Proof. Let $\mathcal{Q}(C_j) = A(1, \rho^2)$ be the annular covering surface determined by the boundary loop C_j. Proposition 3 implies that the orbifold covering map $\pi: A(1, \rho^2) \to (X, \mu)$ has a continuous extension that maps the unit circle homeomorphically onto C_j.

By hypothesis, the given map $f: A(r^{-1}, r) \to X$ lifts to the covering space $\mathcal{Q}(C_j)$. This means that there is a holomorphic map $g: A(r^{-1}, r) \to A(1, \rho^2)$ such that $f = \pi \circ g$ and g maps the circle $\{z; |z| = r\}$ homeomorphically onto the unit circle. Define

$$M(t) = \max\{|g(z)|;\ |z| = t\}$$

for $r^{-1} < t < r$, and extend the function $M(t)$ to the closed interval $[r^{-1}, r]$ by continuity. Then $M(r) = 1$, $M(r^{-1}) \leq \rho^2$, and Hadamard's three-circle theorem gives the inequality

$$\log M(t) \leq \frac{1}{\log(r^2)} \left[\log\left(\frac{r}{t}\right) \log(\rho^2) + \log\left(\frac{t}{r^{-1}}\right) \log 1 \right] < \log \rho$$

if $1 < t < r$. Therefore $|g(z)| < \rho$ when $1 < |z| < r$, so $g(A(1, r)) \subset A(1, \rho)$ and

$$f(A(1, r)) \subset \pi(A(1, \rho)) \subset W(C_j)$$

as required. □

9. Proof of Theorem 1

We shall prove first that $F = N^\infty(X, \mu)$ has the stated properties. Property (3.1) is obvious, and (3.2) is an immediate consequence of Proposition 2, since an easy induction argument shows that $N^k(X, \mu)$ contains all the orbifold points for every positive integer k.

Given any boundary loop C_j and positive integer k, let W_j^k be the component of $X \setminus N^k(X, \mu)$ that contains $W(C_j)$, and let V_j^k be its interior. For fixed j, induction on k shows that each V_j^k is an annulus bounded by C_j and a simple loop C_j^k in X, that $W_j^k = V_j^k \cup C_j^k \subset V_j^{k+1}$, and that the fundamental group of V_j^k is generated by C_j^1 if $k \geq 2$. For fixed k, the sets W_j^k, $1 \leq j \leq m$, are disjoint, and their union is $X \setminus N^k(X, \mu)$.

Now put

$$V^\infty(C_j) = \bigcup_{k=1}^{\infty} V_j^k = \bigcup_{k=1}^{\infty} W_j^k, \quad 1 \leq j \leq m.$$

The sets $V^{\infty}(C_j)$ are clearly disjoint, open, and connected. Their union is $X \setminus N^{\infty}(X, \mu)$. In addition $V^{\infty}(C_j)$ contains $W(C_j)$, and C_j^1 generates its fundamental group, so (3.3) and (3.4) hold. Finally, since its complement is open, $N^{\infty}(X, \mu)$ is closed.

Now suppose, conversely, that the closed set F has properties (3.1) through (3.4). Let $V_j = V(C_j)$ be the component of $X \setminus F$ that corresponds to the boundary loop C_j. Since $F \subset N^{\infty}(X, \mu)$, the set $V^{\infty}(C_j)$ defined above is contained in V_j. We must prove that $V_j = V^{\infty}(C_j)$ or, equivalently, that $V_j \cap N^{\infty}(X, \mu)$ is empty.

By hypothesis, there is a (bijective) conformal map ψ from a plane region

$$\Omega = \{\zeta \in \mathbf{C}; \alpha < |\zeta| < \beta\}, \quad 0 \le \alpha < \beta \le \infty,$$

onto V_j. Our first objective is to show that $\alpha > 0$ and $\beta < \infty$. Once again let $\mathfrak{A}(C_j) = A(1, \rho^2)$ be the annular covering surface determined by C_j. Since V_j contains no orbifold points and $C_j^1 \subset V_j$, we can lift ψ to a holomorphic map $\varphi: \Omega \to A(1, \rho^2)$ that contains the circle $\{w \in \mathbf{C};\ |w| = \rho\}$ in its image. Choose any positive numbers a and b so that $\alpha < a < b < \beta$. The restricted map $\varphi: A(a, b) \to A(1, \rho^2)$ is homotopically nontrivial, so the annulus theorem (see Corollary 1 of Lemma 3 in [8]) implies that $a^{-1}b \le \rho^2$. We conclude that $\alpha^{-1}\beta \le \rho^2$, so $0 < \alpha$ and $\beta < \infty$. We may therefore take Ω to be an annulus $A(1, R)$, and we may assume that the outer boundary loop of $A(1, R)$ corresponds under ψ to the loop C_j. Put

$$R_k = \sup\{|\zeta|;\ \beta \in \psi^{-1}(N^k(X, \mu))\}, \quad k \ge 1.$$

Clearly, $1 < R_{k+1} < R_k < R$ if $k \ge 1$, and it suffices to prove that $R_k \to 1$ as $k \to \infty$.

Fix $k \ge 1$. As above, let $W_j^k = V_j^k \cup C_j^k$ be the component of $X \setminus N^k(X, \mu)$ that contains $W(C_j)$. Observe that $W_j^k \subset V_j$ and that $V_j \setminus W_j^k = N^k(X, \mu) \cap V_j$. In particular, $N^k(X, \mu) \cap V_j$ is an annulus, one of whose boundary components is the loop C_j^k. Let f_k map the annulus $A(r_k^{-1}, r_k)$ conformally onto $N^k(X, \mu) \cap V_j$ so that the outer boundary loop corresponds to C_j^k.

Let $h_k = \psi^{-1} \circ f_k: A(r_k^{-1}, r_k) \to A(1, R)$, put

$$M(t) = \max\{|h_k(z)|;\ |z| = t\}$$

for $r_k^{-1} < t < r_k$, and again extend the function $M(t)$ to be continuous on $[r_k^{-1}, r_k]$. This time $M(r_k^{-1}) = 1$ and $M(r_k) \le R_k$, so the three-circle theorem says that $|h_k(z)| \le R_k^{1/2}$ if $|z| \le 1$.

Now C_j^k is a boundary loop of $N^k(X, \mu)$, and Theorem 2 (with $N^k(X, \mu)$ in the role of X) says that

$$f_k(A(1, r_k)) \subset V_j \setminus N^{k+1}(X, \mu).$$

Therefore

$$\psi(h_k(z)) = f_k(z) \notin N^{k+1}(X, \mu) \quad \text{if } |z| > 1,$$

so if $\psi(h_k(z)) \in N^{k+1}(X, \mu)$ then $|z| \leq 1$ and $|h_k(z)| \leq R_k^{1/2}$. We conclude that $R_{k+1} \leq R_k^{1/2}$, for $\psi^{-1}(N^{k+1}(X, \mu))$ is a subset of $\psi^{-1}(N^k(X, \mu))$, which is the image of h_k. It follows that $R_k \to 1$ as $k \to \infty$, as required. □

10. Proof of the Corollaries

Let (X, μ) be one of the given orbifolds, and let X' be the complement of the set of orbifold points in X. There is a conjugate holomorphic involution $\rho: X \to X$ (defined by $\rho(z) = \bar{z}$ in Corollary 1 and by $\rho(z) = 1/\bar{z}$ in Corollary 2) that fixes every orbifold point. The involution ρ maps X' onto itself, and its fixed point set in X' has finitely many components. Each of them is an open arc or interval whose endpoints are orbifold points, punctures of X, or points on the border B. Let F' be the union of the components with no endpoints on B, and let F be the union of F' and the orbifold points. We are to prove that $F = N^\infty(X, \mu)$, using Theorem 1.

It is obvious that F is closed in X and satisfies (3.2), and the given form of (X, μ) makes (3.3) and (3.4) equally obvious. We must verify condition (3.1). Since every orbifold point belongs to $N^\infty(X, \mu)$, all we need to prove is that F' is contained in $N^k(X, \mu)$ for all $k \geq 1$. We shall start by proving that $F' \subset N(X, \mu)$.

Choose an isomorphism f from a quotient orbifold Δ/Γ onto (X, μ), and compose the natural map $\Delta \to \Delta/\Gamma$ with f to obtain an orbifold covering map $\pi: \Delta \to (X, \mu)$. Let β be a component of F', and let α be a component of $\pi^{-1}(\beta)$ in Δ. Since the involution ρ fixes every orbifold point, it can be lifted to a conjugate holomorphic involution $\sigma: \Delta \to \Delta$ that fixes α pointwise. The fixed point set of σ is a Poincaré geodesic L in Δ, and α is a component of the set $\mathrm{L} \cap \Delta'$, where Δ' $(= \pi^{-1}(X'))$ is the complement of the set of elliptic fixed points of Γ in Δ.

We claim that α is contained in the Nielsen region $N(\Gamma)$. If not, then some point z_0 on α lies in one of the closed Poincaré half-planes H_i defined in Section 5. Since $H_i \subset \Delta'$, at least one of the two rays from z_0 along L is contained in α. If such a ray enters the interior of H_i, then it terminates at a point of $S^1 \setminus L(\Gamma)$, so its image in X terminates at a point of B. That is impossible since $\beta \subset F'$. We are forced to conclude that α lies along the boundary geodesic of H_i, which must therefore be the fixed point set L of the involution σ. But that too is impossible, for Δ would then be the union of H_i and $\sigma(H_i)$, and the only limit points of the group Γ would be the two endpoints of L. Thus $\alpha \subset N(\Gamma)$, as we claimed, so β (and hence F') is contained in $N(X, \mu)$.

Next we shall prove that the involution ρ maps $N(X, \mu)$ onto itself. Equivalently, σ maps $N(\Gamma)$ onto itself. To see this we observe that $\sigma \circ \gamma \circ \sigma^{-1} \in \Gamma$ for all γ in Γ. Therefore σ maps $L(\Gamma)$ onto itself. Since σ is a Poincaré isometry, the collection of half-planes H_i is mapped onto itself. Hence so is the complementary set $N(\Gamma)$.

It follows at once by induction that ρ maps $N^k(X, \mu)$ onto itself and that $F' \subset N^{k+1}(X, \mu)$ for all $k \geq 1$. This completes the proof. □

References

[1] A. F. Beardon, *The geometry of discrete groups,* Springer, Berlin, 1983.
[2] L. Bers, *Nielsen extensions of Riemann surfaces,* Ann. Acad. Sci. Fenn. Ser. A I Math. 2 (1976), 29–34.
[3] J. Cao, *The souls and the Bers-Nielsen kernels of open surfaces with negative curvature* (to appear).
[4] C. J. Earle, *The infinite Nielsen kernel of a symmetric bordered Riemann surface,* Institut Mittag-Leffler, Report No. 35 (1989/90).
[5] N. Halpern, *Some remarks on Nielsen extensions of Riemann surfaces,* Michigan Math. J. 30 (1983), 65–68.
[6] ———, *Two theorems on Nielsen cores,* Complex Variables Theory Appl. 8 (1987), 123–127.
[7] E. Hille, *Analytic function theory,* vol. II, Ginn and Co., Boston, 1962.
[8] A. Marden, I. Richards, and B. Rodin, *Analytic self-mappings of Riemann surfaces,* J. Analyse Math. 18 (1967), 197–225.
[9] K. Strebel, *Quadratic differentials,* Springer, Berlin, 1984.
[10] W. Thurston, *The geometry and topology of 3-manifolds,* Lecture Notes, Department of Mathematics, Princeton University, 1977.
[11] J. C. Wason, *On infinite Nielsen kernels,* Riemann surfaces and related topics: Proceedings of the 1978 Stony Brook Conference, pp. 497–503, Ann. of Math. Stud., 97, Princeton University Press, Princeton, NJ, 1981.

Department of Mathematics
Cornell University
Ithaca, NY 14853