The Inner Carathéodory Distance for the Annulus II

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Let

$$A=\{\lambda\in\mathbf{C}: 1/R<|\lambda|<R\} \quad (R>1),$$

and let c_A, c_A^i denote the *Carathéodory distance* and the *inner Carathéodory distance* for the annulus A, respectively (cf. [3]). It is known that $c_A\neq c_A^i$ (cf. [2; 4])—more precisely, for any $\lambda',\lambda''\in A$, the following equivalence is true:

$$c_A(\lambda',\lambda'')=c_A^i(\lambda',\lambda'') \quad \text{if and only if } \lambda' \text{ and } \lambda'' \text{ lie on the same radius, i.e., } \arg\lambda'=\arg\lambda'' \quad (\text{cf. [4]}). \tag{1}$$

Recall (cf. [1]) that

$$c_A^i(\lambda',\lambda'')=\inf\{L_{\gamma_A}(\alpha):\alpha:[0,1]\to A \text{ is a piecewise } C^1\text{-curve with } \alpha(0)=\lambda',\ \alpha(1)=\lambda''\}, \tag{2}$$

where $L_{\gamma_A}(\alpha)$ denotes the γ_A-length of α given by the formula

$$L_{\gamma_A}(\alpha)=\int_0^1\gamma_A(\alpha(\vartheta);\alpha'(\vartheta))\,d\vartheta. \tag{3}$$

In (3), $\gamma_A: A\times\mathbf{C}\to\mathbf{R}_+$ denotes the *Carathéodory-Reiffen metric* for A.

It is known (cf. [6]) that

$$\gamma_A(\lambda;X)=\frac{1}{R|\lambda|^2}\cdot f\left(\frac{1}{|\lambda|},-|\lambda|\right)\cdot\Pi(|\lambda|,|\lambda|)\cdot|X| \tag{4}$$

for λ in A and X in $\mathbf{C}$, where

$$f(s,\lambda)=\left(1-\frac{\lambda}{s}\right)\cdot\Pi(s,\lambda) \tag{5}$$

and

$$\Pi(s,\lambda)=\frac{\Pi_{n=1}^{\infty}(1-(\lambda/s)R^{-4n})(1-(s/\lambda)R^{-4n})}{\Pi_{n=1}^{\infty}(1-\lambda sR^{-4n+2})(1-(1/\lambda s)R^{-4n+2})} \tag{6}$$

for $1/R<s<R$ and $\lambda\in A$.

The aim of this note is to provide effective formulas for c_A^i—more precisely, for any $\lambda',\lambda''\in A$, we will find an effective description of the *shortest*

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curve for (λ', λ''), that is, a curve $\alpha\colon [0,1] \to A$ with $\alpha(0) = \lambda'$, $\alpha(1) = \lambda''$, and $L_{\gamma_A}(\alpha) = c_A^i(\lambda', \lambda'')$.

REMARK. Since c_A^i is invariant under conformal automorphisms of A, we need only consider $c_A^i(a, be^{i\beta})$ for $1/R < a \le 1$, $1/R < b < R$, and $0 < \beta \le \pi$ (the case $\beta = 0$ is covered by (1)). Moreover, if $a = 1$ then one can assume that $1 \le b < R$.

Define $\delta(s)$ by

$$\delta(s) = \gamma_A(s; s) \quad \text{when } 1/R < s < R.$$

Using (4), (5), and (6), one can prove that

$$\delta(1/s) = \delta(s) \quad \text{and} \quad \delta'(s) > 0 \quad \text{when } 1 < s < R. \tag{7}$$

In particular, the function δ has the global minimum at $s = 1$. Define

$$\delta^{-1} = (\delta|_{(1/R, 1]})^{-1};$$

$$B_+ = \{(C, u) \in (0, \delta(a)) \times (a, R) \colon 1^\circ \text{ if } C = \delta(1) \text{ then } u < 1,$$
$$2^\circ \text{ if } \delta(1) < C < \delta(a) \text{ then } u \le \delta^{-1}(C)\},$$

$$\Psi_+(C, u) = C \int_a^u \frac{ds}{s\sqrt{\delta^2(s) - C^2}} \quad \text{for } (C, u) \text{ in } B_+;$$

$$B_- = \{(C, u) \in (0, \delta(a)] \times (1/R, a) \colon \text{if } a = 1 \text{ then } C < \delta(1)\},$$

$$\Psi_-(C, u) = C \int_u^a \frac{ds}{s\sqrt{\delta^2(s) - C^2}} \quad \text{for } (C, u) \text{ in } B_-.$$

Note that if $a = 1$ then $B_+ = (0, \delta(1)) \times (1, R)$ and $B_- = (0, \delta(1)) \times (1/R, 1)$.

If $1/R < a < 1$ then we put

$$\sigma_+(u) = \Psi_+(\delta(u), u) \quad \text{for } a < u < 1,$$

$$\sigma_-(u) = \Psi_-(\delta(a), u) \quad \text{for } 1/R < u < a;$$

$$B_0 = \{(C, u) \in (\delta(1), \delta(a)] \times (1/R, 1) \colon u \le \delta^{-1}(C)\};$$

$$\Psi_0(C, u) = C\left(\int_a^{\delta^{-1}(C)} + \int_u^{\delta^{-1}(C)}\right) \frac{ds}{s\sqrt{\delta^2(s) - C^2}} \quad \text{for } (C, u) \text{ in } B_0.$$

Note that $\sigma_+(u) = \Psi_0(\delta(u), u)$ when $a < u < 1$ and that $\sigma_-(u) = \Psi_0(\delta(a), u)$ when $1/R < u < a$. Also,

$$\lim_{u \to a+} \sigma_+(u) = 0, \qquad \lim_{u \to 1-} \sigma_+(u) = +\infty,$$

$$\lim_{u \to a-} \sigma_-(u) = 0, \qquad \lim_{u \to 1/R+} \sigma_-(u) < +\infty.$$

Let $\gamma_\pm$ denote the graph of $\sigma_\pm$. Let D_+ denote the part of the domain

$$D = (0, +\infty) \times (1/R, R)$$

that lies over γ_+, let D_- denote the part under γ_-, and let D_0 denote the middle part.

If $a=1$ then we set $\gamma_\pm=(0,+\infty)\times\{1\}$, $D_+=(0,+\infty)\times(1,1/R)$, $D_-=(0,+\infty)\times(1/R,1)$, and $D_0=\emptyset$. Now we can formulate our main result.

THEOREM 1. (a) *For any $0<\beta\le\pi$, the shortest curve for $(1,e^{i\beta})$ is the curve $t\to e^{it}$ for $0\le t\le\beta$ and, consequently,*

$$c_A^i(1,e^{i\beta})=\gamma_A(1;\beta).$$

(b) *For $1<b<R$ and $0<\beta\le\pi$, the shortest curve for $(1,be^{i\beta})$ is the curve*

$$t\to u(t)e^{it} \quad \text{for } 0\le t\le\beta, \tag{8}$$

where the function $u=u(t)$ $(u(0)=a, u(\beta)=b)$ and the constant $C=C_+(\beta,b)\in(0,\delta(1))$ are uniquely determined by the equations

$$\Psi_+(C,u)=t \quad \text{and} \quad \Psi_+(C,b)=\beta.$$

Moreover,

$$c_A^i(1,be^{i\beta})=\int_1^b\frac{\delta^2(s)\,ds}{s\sqrt{\delta^2(s)-C^2}} \quad \text{where } C=C_+(\beta,b).$$

(c) *For $1/R<a<1$, $1/R<b<R$, and $0<\beta\le\pi$, if $be^{i\beta}\in D_\pm\cup\gamma_\pm$ then the shortest curve for $(a,be^{i\beta})$ is of the form* (8), *where the function $u=u(t)$ and the constant $C=C_\pm(\beta,b)$ are uniquely determined by the system*

$$\Psi_\pm(C,u)=t \quad \text{and} \quad \Psi_\pm(C,b)=\beta.$$

Moreover,

$$c_A^i(a,be^{i\beta})=\pm\int_a^b\frac{\delta^2(s)\,ds}{s\sqrt{\delta^2(s)-C^2}} \quad \text{where } C=C_\pm(\beta,b).$$

(d) *For $1/R<a<1$, $1/R<b<R$, and $0<\beta\le\pi$, if $be^{i\beta}\in\gamma_+\cup D_0\cup\gamma_-$ then the shortest curve for $(a,be^{i\beta})$ is of the form* (8), *where the function $u=u(t)$ and the constant $C=C_0(\beta,b)$ are uniquely determined by the system*

$$\Psi_0(C,u)=t \quad \text{and} \quad \Psi_0(C,b)=\beta.$$

Moreover,

$$c_A^i(a,be^{i\beta})=\left(\int_a^{\delta^{-1}(C)}+\int_b^{\delta^{-1}(C)}\right)\frac{\delta^2(s)\,ds}{s\sqrt{\delta^2(s)-C^2}} \quad \text{where } C=C_0(\beta,b).$$

Proof of Theorem 1. The proof will be divided into two steps.

Step 1°: Reduction to a variational problem. Fix $1/R<a\le1$, $1/R<b<R$, and $0<\beta\le\pi$ (if $a=1$ then we take $1\le b<R$). It is clear that in (2) the infimum may be taken only over the class of all curves α of the form $\alpha(\vartheta)=r(\vartheta)e^{i\mu(\vartheta)}$ for $0\le\vartheta\le1$, where $r:[0,1]\to(1/R,R)$ and $\mu:[0,1]\to[0,\pi]$ are C^1-functions with $r(0)=a$, $r(1)=b$, $\mu(0)=0$, and $\mu(1)=\beta$. In view of (1), if $\mu(\vartheta_1)=\mu(\vartheta_2)$ for some $0\le\vartheta_1<\vartheta_2\le1$ then the γ_A-length of the segment $[\alpha(\vartheta_1),\alpha(\vartheta_2)]$ is not larger than the γ_A-length of the curve $\alpha|_{[\vartheta_1,\vartheta_2]}$. This implies that the class of "admissible" curves may be reduced to the class of all curves of the form $\alpha(t)=u(t)e^{it}$ $(0\le t\le\beta)$, where $u:[0,\beta]\to(1/R,R)$ is a C^1-function with $u(0)=a$ and $u(\beta)=b$.

Thus, in order to characterize the shortest curve and to calculate $c_A^i(a, be^{i\beta})$, it suffices to minimize the following functional (cf. (3)):

$$u \to \int_0^\beta \gamma_A(u(t); \sqrt{u^2(t)+u'^2(t)})\,dt \tag{9}$$

when u is in

$$\mathfrak{D} = \{u: [0,\beta] \to (1/R, R): u \in C^1, u(0)=a, u(\beta)=b\}.$$

In view of (7), if $u(t) \geq u_0 = u(t_1) = u(t_2) \geq 1$ for $t_1 \leq t \leq t_2$, then the γ_A-length of the curve $\alpha|_{[t_1,t_2]}$ is not larger than the γ_A-length of the arc $t \to u_0 e^{it}$ when $t_1 \leq t \leq t_2$. The same is true if $u(t) \leq u_0 = u(t_1) = u(t_2) \leq 1$ for $t_1 \leq t \leq t_2$. As a direct consequence of these remarks, we obtain statement (a) of the theorem.

Step 2°: *Solution of the variational problem.* We are going to minimize (9) using a modification of the classical Weierstrass method (cf. [7]). Let

$$F(u,v) = \gamma_A(u; \sqrt{u^2+v^2}) \quad \text{for } 1/R < u < R \text{ and } v \in \mathbf{R}$$

and let

$$\mathcal{E}(u, v_1, v_2) = F(u, v_2) - F(u, v_1) - \frac{\partial F}{\partial v}(u, v_1)\cdot(v_2 - v_1)$$

for $1/R < u < R$ with v_1 and v_2 in $\mathbf{R}$ be the *Weierstrass function* for F. First observe that $\mathcal{E}(u, v_1, v_2) > 0$ when $v_1 \neq v_2$. Thus, the main problem is to cover the domain D by a "sufficiently" regular family of *stationary curves* for (9). More precisely, it suffices to find for each pair (t_0, u_0) in D a unique solution $u(t) = u(t_0, u_0; t)$ of the Euler-Lagrange equation

$$\frac{\partial F}{\partial u}(u(t), u'(t)) = \frac{d}{dt}\frac{\partial F}{\partial v}(u(t), u'(t)) \tag{10}$$

with $u(0) = a$ and $u(t_0) = u_0$ in such a way that the following function on D

$$(t_0, u_0) \to \Phi(t_0, u_0) = \frac{\partial u}{\partial t}(t_0, u_0; t_0)$$

is globally continuous and of class C^1 in each of the domains D_+, D_0, and D_- separately. In view of (7), the only constant solution of (10) is $u \equiv 1$. Moreover, one can easily prove that in the class of nonconstant solutions, equation (10) is equivalent to the following 1-parameter family of equations:

$$\delta(u(t))\,u(t) = C\sqrt{u^2(t)+u'^2(t)}, \quad C > 0. \tag{11}$$

Note that there are constant solutions of (11) (e.g., $u \equiv \delta^{-1}(C) \neq 1$) that are not solutions of (10).

From now on we will assume that $1/R < a < 1$; the case $a = 1$ is analogous. Fix (t_0, u_0) in $D_\pm \cup \gamma_\pm$ (resp. in $\gamma_+ \cup D_0 \cup \gamma_-$). In view of (11), it suffices to prove that the system

$$\Psi_\pm(C, u) = t, \;\; \Psi_\pm(C, u_0) = t_0$$

has exactly one solution $u=u(t)$ and $C=C_\pm(t_0,u_0)$ such that the function $(t,u)\to C_\pm(t,u)$ is continuous on $D_\pm\cup\gamma_\pm$ and of class C^1 in $D_\pm$ (resp., the system $\Psi_0(C,u)=t$, $\Psi_0(C,u_0)=t_0$ has exactly one solution $u=u(t)$ and $C=C_0(t_0,u_0)$ such that the function $(t,u)\to C_0(t,u)$ is continuous on $\gamma_+\cup D_0\cup\gamma_-$ and of class C^1 in D_0).

In the first case the situation is simple because

$$\pm\frac{\partial\Psi_\pm}{\partial u}>0 \text{ and } \frac{\partial\Psi_\pm}{\partial C}>0 \text{ in } \operatorname{int}(B_\pm),$$

and therefore one can use the implicit function theorem. In the second case the situation is more complicated; it is clear that $\partial\Psi_0/\partial u<0$, but the proof that $\partial\Psi_0/\partial C<0$ in $\operatorname{int}(B_0)$ needs some work.

It suffices to prove that

$$\frac{\partial\Psi}{\partial C}<0, \tag{12}$$

where

$$\Psi(C,u)=C\int_u^{\delta^{-1}(C)}\frac{ds}{s\sqrt{\delta^2(s)-C^2}} \quad \text{for } 1/R<u<\delta^{-1}(C)<1.$$

Recall (cf. [4, 5]) that

$$\delta(e^{2\pi i\xi})=\frac{k}{2\pi}\operatorname{cn}(2\xi),$$

where cn denotes the *cosinus amplitudinis* with periods $\omega_1=1/2$ and $\omega_2=\tau/2$ ($e^{i\pi\tau}=1/R^2$), and k is the "Jacobi Modul" for the theta functions. Hence, from standard properties of cn, sn, and dn, we get

$$[\delta'(s)s]^2=4\left[\delta^2(s)-\left(\frac{k}{2\pi}\right)^2\right]\left[\delta^2(s)+\left(\frac{k'}{2\pi}\right)^2\right]$$

when $1/R<s<R$ and when $k'=\sqrt{1-k^2}$. Using this identity, after elementary calculations we conclude that

$$\Psi(C,u)=\frac{\pi}{\sqrt{2}}\Lambda\left(\frac{P}{\delta^2(u)}+Q,\frac{P}{C^2}+Q\right), \tag{13}$$

where

$$P=\frac{1}{2}\left(\frac{kk'}{\pi}\right)^2, \quad Q=k^2-k'^2, \quad \text{and}$$

$$\Lambda(\xi,x)=\int_\xi^x\frac{dt}{\sqrt{(1-t^2)(x-t)}} \quad \text{for } -1<\xi<x<1.$$

Observe that (the idea is due to P. Tworzewski):

$$\Lambda(\xi,x)=\int_0^1\sqrt{\frac{x-\xi}{1-[\xi+\eta(x-\xi)]^2}}\,\frac{d\eta}{\sqrt{1-\eta}}.$$

Hence

$$\frac{\partial \Lambda}{\partial x} > 0$$

and consequently, in view of (13), we get (12). □

References

[1] S. Dineen, *The Schwarz lemma,* Clarendon Press, Oxford, 1989.
[2] M. Jarnicki and P. Pflug, *The simplest example for the non-innerness of the Carathéodory distance,* Resultate Math. 18 (1990), 57-59.
[3] ———, *Invariant pseudodistances and pseudometrics—completeness and product property,* Ann. Polon. Math. 55 (1991), 169-189.
[4] ———, *The inner Carathéodory distance for the annulus,* Math. Ann. 289 (1991), 335-339.
[5] F. Rühs, *Funktionentheorie,* VEB Deutscher Verlag der Wissenschaften, Berlin, 1962.
[6] R. R. Simha, *The Carathéodory metric of the annulus,* Proc. Amer. Math. Soc. 50 (1975), 162-166.
[7] J. L. Troutman, *Variational calculus with elementary convexity,* Springer, Berlin, 1983.

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