A Class of Operator Algebras Induced by Probabilistic Conditional Expectations

ALAN LAMBERT & BARNET M. WEINSTOCK

Introduction

If $(X, \mathcal{A}, m)$ is a probability space and $\mathcal{B}$ is a σ-subalgebra of $\mathcal{A}$, then the conditional expectation operator $\mathcal{E} = \mathcal{E}(\ |\mathcal{B})$ acts on $L^2(X, \mathcal{A}, m)$ as the orthogonal projection onto $L^2(X, \mathcal{B}, m)$. In this note we introduce an (associative) algebra $\mathfrak{D}$ of bounded operators on $L^2(X, \mathcal{A}, m)$ such that each of the operators in $\mathfrak{D}$ has its range contained in the kernel of $\mathcal{E}$. $\mathfrak{D}$ is in fact the set of operators adjoint to a Banach–Lie algebra $\mathcal{L} = \mathcal{L}(\mathcal{A}\,|\,\mathcal{B})$ which is defined in terms of $\mathcal{E}$ and whose members form a dense linear manifold in $L^2(X, \mathcal{A}, m)$.

In Section 1 we introduce the basic notation and terminology employed in this article, review the properties of conditional expectation relevant to our investigation, and define the operator algebra $\mathfrak{D}$ to be studied. We also establish the existence of a set B_0 in $\mathcal{B}$ which is maximal with respect to a condition equivalent to carrying no information about $\mathcal{B}$. Examples are presented of sigma algebras where B_0 has positive measure and sigma algebras where this set has measure zero. As these examples indicate, the case of B_0 having measure zero appears to be the more natural of the two and, in any case, it is shown that the operator algebras under consideration and the sigma algebras can always be decomposed into direct sums along these lines. Moreover, $\mathfrak{D}$ restricted to $L^2(B_0)$ *is* the zero operator algebra. In Lemma 7 we show that when B_0 has measure zero there is an almost everywhere nonzero bounded function whose conditional expectation is zero. This fact turns out to be especially useful.

Section 2 deals with the algebra $\mathfrak{D}$ of extensions to all of L^2 of the operators adjoint to $\mathcal{L}$. It is shown that when B_0 has measure zero, $\mathfrak{D}$ is closed in the weak operator topology. The range of each operator in $\mathfrak{D}$ is carefully analyzed, and its spectrum is shown to consist of zero and the essential range of the expectation of its defining function in $\mathcal{L}$. It is hoped that the detailed information about ranges will prove useful in later investigations of the fine structure of the kernel of $\mathcal{E}$. This seems to be a difficult matter in general. The commutant of $\mathfrak{D}$ is studied in Section 2, and is shown to be the von Neumann algebra of multiplications by $L^\infty(\mathcal{B})$ functions, which also turns out to be the diagonal $\mathfrak{D} \cap \mathfrak{D}^*$ of $\mathfrak{D}$.

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Section 3 is devoted to a brief discussion of $\mathcal{L}$ as a Lie algebra. It is shown that $\mathcal{L}$ has trivial center precisely when B_0 has measure zero, and that in this setting $\mathcal{L}$ is in fact a Banach–Lie algebra. When B_0 has measure zero, so that by the Corollary to Lemma 3 the adjoint representation is faithful, the operator norm on $\mathfrak{D}$ is equivalent to the norm induced by the Banach–Lie algebra norm of $\mathcal{L}$.

The authors, in conjunction with J. Daughtry, have begun the examination of algebras analogous to $\mathfrak{D}$ in the case that $\mathcal{E}$ is a conditional expectation on a von Neumann algebra or, more generally, a C^*-algebra. The results of this investigation will appear elsewhere. The present measure-theoretic setting is of special interest because of its relation to and potential applications in probability and measure theories, as well as its wealth of examples. Moreover, certain constructions, such as the function of full support in the kernel of $\mathcal{E}$ when B_0 has measure zero, do not seem to have natural analogues in the general setting.

1

Let X be a set, $\mathcal{A}$ a σ-algebra of subsets of X, and m a complete probability measure on the measurable space $(X, \mathcal{A})$. In this paper all algebras of sets are assumed to be complete σ-subalgebras of $\mathcal{A}$. Also, equalities of sets or functions, set inclusions, and inequalities between functions are interpreted as being valid up to an m-null set. In describing sets, we shall use abbreviated notation such as $\{g > h\}$ for $\{x \in X: g(x) > h(x)\}$. We shall denote the characteristic function of a set by χ_A. The reader will note that each argument concerning supports of elements of L^2 involves only countably many such elements. For each function f chosen in such an argument, we assign a measurable set S_f, referred to as the *support of f*, such that S_f and $\{f \neq 0\}$ are equal modulo the null sets.

Suppose that $\mathcal{B}$ is a subalgebra of $\mathcal{A}$. For $f \in L^1(X, \mathcal{A}, m)$ we denote by $\mathcal{E}^{\mathcal{B}\mathcal{A}}(f)$ the unique $\mathcal{B}$-measurable function with the property that

$$\int_B f\, dm = \int_B \mathcal{E}^{\mathcal{B}\mathcal{A}}(f)\, dm$$

for all $B \in \mathcal{B}$. The function $\mathcal{E}^{\mathcal{B}\mathcal{A}}(f)$ is called *conditional expectation of f with respect to* $\mathcal{B}$. If there is no possibility of confusion we write $\mathcal{E}^{\mathcal{B}}(f)$ or simply $\mathcal{E}(f)$ in place of $\mathcal{E}^{\mathcal{B}\mathcal{A}}(f)$.

The operator $\mathcal{E}^{\mathcal{B}\mathcal{A}}$ is called the *conditional expectation operator.* Its restriction to $L^2(X, \mathcal{A}, m)$, which is of primary interest in this paper, is the orthogonal projection onto $L^2(X, \mathcal{B}, m)$. We will need the following standard facts concerning conditional expectation operators.

- $\mathcal{E}^{\mathcal{B}} g = g$ if and only if g is $\mathcal{B}$-measurable.
- If f is $\mathcal{B}$-measurable then $\mathcal{E}^{\mathcal{B}}(fg) = f\mathcal{E}^{\mathcal{B}}(g)$.
- $|\mathcal{E}^{\mathcal{B}}(fg)|^2 \leq \mathcal{E}^{\mathcal{B}}(|f|^2) \cdot \mathcal{E}^{\mathcal{B}}(|g|^2)$ (conditional Cauchy–Schwarz inequality).

- If $f \in L^\infty(X, \mathcal{A}, m)$ then $\mathcal{E}^{\mathcal{B}}(f) \in L^\infty(X, \mathcal{B}, m)$.
- If $f \geq 0$ then $\mathcal{E}(f) \geq 0$; if $f > 0$ then $\mathcal{E}(f) > 0$.
- If $f_n \uparrow f$ a.e. then $\mathcal{E}f_n \uparrow \mathcal{E}f$ a.e. (conditional monotone convergence property).

It is known [3] that various subsets of this list of properties actually characterize conditional expectation operators.

The following proposition lists some elementary properties of conditional expectation which will be used in this investigation.

PROPOSITION. *Let $\mathcal{A} \supset \mathcal{B}$ and let $\mathcal{E} = \mathcal{E}^{\mathcal{B}\mathcal{A}}$. Then*

(i) *For each nonnegative $\mathcal{A}$-measurable function f, $S_f \subset S_{\mathcal{E}f}$.*
(ii) *For each $\mathcal{A}$-measurable set $\mathcal{A}$, $\{\mathcal{E}\chi_A = 1\}$ is a subset of $\mathcal{A}$ (modulo null sets), and is the largest $\mathcal{B}$-set contained in $\mathcal{A}$.*
(iii) *If A is an $\mathcal{A}$-set that contains no $\mathcal{B}$-set of positive measure, then $\mathcal{E}\chi_A$ and $\chi_A - \mathcal{E}\chi_A$ have identical supports.*
(iv) *For each $\mathcal{A}$-measurable set A, $S_{\mathcal{E}\chi_A}$ is the smallest $\mathcal{B}$-measurable set containing A.*

Proof. (i) For $f \geq 0$,

$$0 = \int_{X - S_{\mathcal{E}f}} \mathcal{E}f \, dm = \int_{X - S_{\mathcal{E}f}} f \, dm,$$

so that $f = 0$ a.e. off $S_{\mathcal{E}f}$.

(ii) We have

$$m(\{\mathcal{E}\chi_A = 1\}) = \int_{\{\mathcal{E}\chi_A = 1\}} \mathcal{E}\chi_A \, dm = \int_{\{\mathcal{E}\chi_A = 1\}} \chi_A \, dm = m(A \cap \{\mathcal{E}\chi_A = 1\}).$$

Thus $\{\mathcal{E}\chi_A = 1\} \subset A$. If B is a $\mathcal{B}$-subset of A then $m(B) = \int_B \chi_A \, dm = \int_B \mathcal{E}\chi_A \, dm$. Since $\mathcal{E}\chi_A \leq 1$ a.e., $\mathcal{E}\chi_A = 1$ a.e. on B.

(iii) Suppose that A contains no $\mathcal{B}$-set of positive measure. It follows from (ii) that $\mathcal{E}\chi_A < 1$ a.e. Thus $\{\chi_A - \mathcal{E}\chi_A = 0\} = \{\chi_A = 0 \text{ and } \mathcal{E}\chi_A = 0\}$. By (i), this set is the complement of the support of $\mathcal{E}\chi_A$.

(iv) Let B be a $\mathcal{B}$-set containing A. Then $\int_{X-B} \mathcal{E}\chi_A \, dm = \int_{X-B} \chi_A \, dm = 0$. □

We now introduce the basic construction which motivated this study.

Let $\mathcal{B}$ be a subalgebra of $\mathcal{A}$. For each $g, f \in L^2(\mathcal{A})$ let

$$D_f(g) = (\mathcal{E}^{\mathcal{B}} f) g - (\mathcal{E}^{\mathcal{B}} g) f.$$

(Where necessary for clarity we will write $D_f^{\mathcal{B}}$ or $D_f^{\mathcal{B}\mathcal{A}}$ in place of D_f.) We define $\mathcal{L}(\mathcal{A} \mid \mathcal{B})$ to be the set of $f \in L^2(\mathcal{A})$ for which D_f is a bounded operator on $L^2(\mathcal{A})$. It is clear that $L^\infty(\mathcal{A}) \subset \mathcal{L}$. We also denote by $\mathfrak{D}(\mathcal{A} \mid \mathcal{B})$ (or simply $\mathfrak{D}$ if there is no possibility of confusion) the set of all operators D_f with f in $\mathcal{L}(\mathcal{A} \mid \mathcal{B})$. A straightforward calculation shows that $\mathfrak{D}$ is an (associative) algebra of bounded linear operators on $L^2(\mathcal{A})$. Indeed, $D_f D_g = D_{(\mathcal{E}f)g}$. As we note in Section 3, $\mathcal{L}$ is a Lie algebra with respect to the multiplication

$[f, g] = D_f g$, so that D_f acting on $\mathcal{L}$ is just ad f. The following example completely describes the algebras $\mathfrak{D}(\mathcal{A} | \mathcal{B})$ in the finite-dimensional case.

EXAMPLE. When $X = \{1, \ldots, n\}$, $\mathcal{A} = 2^X$, and m = normalized counting measure, we identify $L^2(X, \mathcal{A}, m)$ with $\mathbf{C}^n$ so that the algebras $\mathfrak{D}$ are algebras of $n \times n$ matrices. If P is a partition of $\{1, \ldots, n\}$, let $\mathcal{B}_P$ denote the σ-subalgebra of $\mathcal{A}$ generated by P. Without loss of generality we can assume that the elements of P are of the form $\{(\sum_{j<k} n_j)+1, \ldots, n_k\}$, $1 \le k \le d$, with $\sum n_k = n$. We use the notation $P = [n_1 : n_2 : \cdots : n_d]$. When $P = [n]$, so that $\mathcal{B}_P = \{\emptyset, X\}$, the conditional expectation operator $\mathcal{E}^{\mathcal{B}_P \mathcal{A}}$ is defined by the matrix $E_n = (a_{ij})$, where $a_{ij} = n^{-1}$ for all i, j. The algebra $\mathfrak{D}$ in this case has dimension n (unless $n = 1$) and consists of all operators D_a, $a \in \mathbf{C}^n$, defined by

$$D_a x = (a, \epsilon_n)x - (x, \epsilon_n)a,$$

where $x \in \mathbf{C}^n$, $(\cdot, \cdot)$ is the standard hermitian inner product on $\mathbf{C}^n$, and $\epsilon_n = (n^{-1}, \ldots, n^{-1})$. (When $n = 1$, $\mathfrak{D} = \{0\}$.) In the general case, where $P = [n_1 : n_2 : \cdots : n_d]$, we decompose $\mathbf{C}^n$ as $\mathbf{C}^{n_1} \oplus \cdots \oplus \mathbf{C}^{n_d}$, and if $a \in \mathbf{C}^n$ we write $a = a_1 \oplus \cdots \oplus a_d$. Then $\mathcal{E}^{\mathcal{B}_P \mathcal{A}} = E_{n_1} \oplus \cdots \oplus E_{n_d}$ and $\mathfrak{D}$ consists of all operators of the form $D_{a_1} \oplus \cdots \oplus D_{a_d}$, so that $\mathfrak{D}$ has dimension $n - p$, where p is the number of singletons in P.

The algebra $\mathfrak{D}(\mathcal{A} | \mathcal{B})$ should prove to be a useful tool in studying the relation of the σ-subalgebra $\mathcal{B}$ to its parent algebra $\mathcal{A}$. In general, the conditional expectation $\mathcal{E}$ is used to relate and connect $\mathcal{A}$-measurable functions with $\mathcal{B}$-measurable functions. However, the kernel of $\mathcal{E}$ is usually far less tractable. For example, there are no positive functions in $\ker \mathcal{E}$, and it is difficult to analyze $\ker \mathcal{E}$ effectively. Since each operator in $\mathfrak{D}(\mathcal{A} | \mathcal{B})$ has its range contained in $\ker \mathcal{E}$, $\mathfrak{D}(\mathcal{A} | \mathcal{B})$ opens a window through which one may observe $\ker \mathcal{E}$. The fact that $\mathfrak{D}(\mathcal{A} | \mathcal{B})$ is an operator algebra gives us more than just the vector space structure of $\ker \mathcal{E}$ to work with.

If $\mathcal{B}$ is a subalgebra of $\mathcal{A}$, we denote by $\mathcal{B}_0$ the family of sets

$$\{S \in \mathcal{A} : \mathcal{A} \cap S \subset \mathcal{B}\},$$

where $\mathcal{A} \cap S$ denotes $\{T \cap S : T \in \mathcal{A}\}$.

LEMMA 1. *For each subalgebra $\mathcal{B}$, the family $\mathcal{B}_0$ contains a maximum set B_0; that is, there exists $B_0 \in \mathcal{B}_0$ such that each set $S \in \mathcal{A}$ of positive measure which is disjoint from B_0 contains an $\mathcal{A}$-measurable set which is not $\mathcal{B}$-measurable.*

Proof. Let $r = \sup\{m(S) : \mathcal{A} \cap S \subset \mathcal{B}\}$. If $r = 0$ there is nothing to prove, so suppose $r > 0$. Let $\{S_n\}$ be a sequence of sets such that

$$\mathcal{A} \cap S_n \subset \mathcal{B} \quad \text{and} \quad m(S_n) \to r.$$

Let $B_0 = \bigcup S_n$. If $B \subset \bigcup S_n$ then $B = \bigcup (B \cap S_n) \in \mathcal{B}$ so that $\mathcal{A} \cap B_0 \subset \mathcal{B}$. Thus $m(B_0) = r$, so that every set of positive measure disjoint from B_0 contains a set of positive measure which is not $\mathcal{B}$-measurable. □

EXAMPLES. (i) Let $X=[-1,1]$ with normalized Lebesgue measure. Let $\mathcal{B}$ be the subalgebra consisting of sets symmetric about the origin. If S is any set of positive Lebesgue measure then either $S\cap[-1,0]$ or $S\cap[0,1]$ has positive measure; hence, in this example, B_0 has measure zero.

(ii) Let $(X,\mathcal{A},m)$ be as in the previous example. Define $T: X\to X$ by

$$T(x)=\begin{cases} \frac{1}{2}(x+1), & -1\le x\le 0, \\ 1-|x-\frac{1}{2}|, & 0\le x\le 1, \end{cases}$$

and let $\mathcal{B}=T^{-1}(\mathcal{A})$. Then $B_0=[-1,0]$.

Given a subalgebra $\mathcal{B}\subset\mathcal{A}$, the set B_0 of Lemma 1 has a number of interesting properties.

LEMMA 2. *$D_f=0$ if and only if $\mathcal{A}\cap S_f\subset\mathcal{B}$.*

Proof. If $D_f=0$ then $(\mathcal{E}f)g=(\mathcal{E}g)f$ for every g. Choosing $g=1$, we see that $\mathcal{E}f=f$; that is, f is $\mathcal{B}$-measurable, in which case $\mathcal{E}fg=(\mathcal{E}g)f$. Let $A\subset S_f$ have positive measure. Then $S_f\cap S_{\mathcal{E}\chi_A}=S_f\cap A=A$. But $S_f\cap S_{\mathcal{E}\chi_A}\in\mathcal{B}$.

Conversely, suppose that every $\mathcal{A}$-measurable subset of S_f is in fact $\mathcal{B}$-measurable; indeed, every measurable function supported in S_f is $\mathcal{B}$-measurable. Thus, for each g, $D_fg=fg-(\mathcal{E}g)f=fg-\mathcal{E}(fg)$. But $S_{fg}\subset S_f$, so fg is $\mathcal{B}$-measurable; hence $D_f=0$. □

COROLLARY. *The mapping $f\to D_f$ is one-to-one if and only if B_0 has measure zero.*

LEMMA 3. $L^2(B_0)=\bigcap_{f\in\mathcal{L}}\ker D_f$.

Proof. Let $g\in L^2(B_0)$. Then g is in fact $\mathcal{B}$-measurable as a consequence of Lemma 2. Thus, if $f\in\mathcal{L}$, we have $D_fg=(f-\mathcal{E}f)g$; that is, D_fg is supported on B_0 and hence is $\mathcal{B}$-measurable. But $\mathcal{E}D_fg=0$, which shows that $D_f=0$ on $L^2(B_0)$.

Conversely, let $g\in\bigcap\ker D_f$. Then for each $f\in\mathcal{L}$, $(\mathcal{E}f)g=(\mathcal{E}g)f$. Choosing $f=1$ shows that g is $\mathcal{B}$-measurable. It then follows that for each $f\in\mathcal{L}$, $(f-\mathcal{E}f)g=0$. Let $B\subset(X-B_0)\cap S_g$. If $m(B)>0$ then B contains a set C of positive measure which is not $\mathcal{B}$-measurable, so that $\mathcal{E}\chi_C\ne\chi_C$. But by part (iv) of the Proposition, the support of $\mathcal{E}\chi_C$ is the smallest $\mathcal{B}$-measurable set containing C and $(X-B_0)\cap S_g$ is a $\mathcal{B}$-measurable set containing C. Thus

$$\operatorname{supp}(\chi_C-\mathcal{E}\chi_C)\subset(X-B_0)\cap S_g.$$

But $(\chi_C-\mathcal{E}\chi_C)g=0$, so $\chi_C=\mathcal{E}\chi_C$. This contradiction shows that $m(B)=0$, so that B_0 contains S_g. □

LEMMA 4.

(a) *If $f\in\mathcal{L}$ then D_f^* is given by $D_f^*g=(\overline{\mathcal{E}f})g-\mathcal{E}(\bar{f}g)$.*

(b) *$D_f^*\in\mathfrak{D}$ if and only if f is $\mathcal{B}$-measurable. In this case $D_f=M_f(I-\mathcal{E})=D_{\bar{f}}^*$.*

Proof. (a) We have

$$\begin{aligned}(D_f h, g) &= (h\mathcal{E}f - f\mathcal{E}h, g)\\ &= (h, \overline{\mathcal{E}f}g) - (\mathcal{E}h, \bar{f}g)\\ &= (h, \overline{\mathcal{E}f}g) - (h, \mathcal{E}(\bar{f}g))\\ &= (h, (\overline{\mathcal{E}f})g - \mathcal{E}(\bar{f}g)).\end{aligned}$$

(b) Suppose $D_f^* = D_u$. Then, for each g,

$$(\overline{\mathcal{E}f})g - \mathcal{E}(\bar{f}g) = (\mathcal{E}u)g - u\mathcal{E}g.$$

Thus for each $\mathcal{B}$-measurable function g, $(\mathcal{E}u)g - u\mathcal{E}g = 0$. Choosing $g = 1$, we conclude that $\mathcal{E}u = u$; that is, u is $\mathcal{B}$-measurable. Then $D_u = M_u(I - \mathcal{E})$ and $D_f = D_u^* = (I - \mathcal{E})M_{\bar{u}} = M_{\bar{u}}(I - \mathcal{E}) = D_{\bar{u}}$. By Lemma 2, $f - \bar{u}$, and hence f, is $\mathcal{B}$-measurable. □

COROLLARY. *$L^2(B_0)$ (and consequently $L^2(X - B_0)$) reduces $\mathfrak{D}$.*

Proof. By Lemma 3, for each $f \in \mathcal{L}$, $D_f = 0$ on $L^2(B_0)$. By Lemma 4,

$$D_f^* g = (\overline{\mathcal{E}f})g - \mathcal{E}(\bar{f}g).$$

But $g \in L^2(B_0)$ implies that g is $\mathcal{B}$-measurable, so

$$D_f^* g = (\overline{\mathcal{E}f} + \mathcal{E}\bar{f})g = 0. \qquad \square$$

NOTATION. For each subset L of $L^2(\mathcal{A})$ and each subspace $\mathcal{S}$ of $L^2(\mathcal{A})$, let $\mathfrak{M}(L, \mathcal{S}) = \{u \in L^2(\mathcal{A}) : uL \subset \mathcal{S}\}$. When $\mathcal{S} = L^2(\mathcal{A})$ we write $\mathfrak{M}(L)$ for $\mathfrak{M}(L, \mathcal{S})$.

LEMMA 5. $\mathfrak{M}(L^2(\mathcal{B})) = \{f : \mathcal{E}|f|^2 \in L^\infty\}$.

Proof. Let $f \in L^2(\mathcal{A})$, $g \in L^2(\mathcal{B})$. Then

$$\int |f|^2 |g|^2 \, dm = \int (\mathcal{E}|f|^2)|g|^2 \, dm.$$

Therefore $f \in \mathfrak{M}(L^2(\mathcal{B}))$ if and only if $\sqrt{\mathcal{E}|f|^2} \in \mathfrak{M}(L^2(\mathcal{B}), L^2(\mathcal{B})) = L^\infty(\mathcal{B})$. □

DEFINITION.

(a) $\mathcal{B}$ is a type-0 subalgebra of $\mathcal{A}$ if $m(B_0) = 0$.
(b) $\mathcal{B}$ is a type-1 subalgebra of $\mathcal{A}$ if every $\mathcal{B}$-measurable function in $\mathfrak{M}(\ker \mathcal{E})$ is in $L^\infty(\mathcal{B})$.

REMARKS. (1) In light of the preceding Corollary, we concentrate our attention on the study of $\mathfrak{D}(\mathcal{A} | \mathcal{B})$ in the case where $\mathcal{B}$ is a type-0 subalgebra.
(2) The following result will be strengthened in Theorem 2.

LEMMA 6. *If $\mathcal{B}$ is type 0 then $\mathcal{B}$ is type 1. Conversely, $\mathcal{B}$ is type 1 but not type 0 if and only if B_0 consists precisely of a finite number of atoms.*

Proof. Suppose $\mathcal{B}$ is type 0. Then every set of positive measure contains a set of positive measure which is not $\mathcal{B}$-measurable. Let φ be a $\mathcal{B}$-measurable function in $\mathfrak{M}(\ker \mathcal{E})$, and let L denote the linear transformation of multiplication by φ on $\ker \mathcal{E}$. Then L maps $\ker \mathcal{E}$ into $\ker \mathcal{E}$, and it follows easily from the closed graph theorem that L is bounded. Let $c = \|L\|$ as an operator on $\ker \mathcal{E}$. We show that c is the essential supremum of φ.

Let $\epsilon > 0$ and let $S = \{|\varphi| \geq c + \epsilon\}$. If S has positive measure then there exists $T \subset S$ which has positive measure and is not $\mathcal{B}$-measurable. Let $g = \chi_T - \mathcal{E}\chi_T$. Then g is a nonzero vector in $\ker \mathcal{E}$. But $S \in \mathcal{B}$, so S contains the support of $\mathcal{E}\chi_T$, and consequently g is supported in S. Therefore

$$(c+\epsilon)^2\|g\|^2 \leq \int_S |\varphi|^2 |g|^2\, dm \leq c^2 \|g\|^2,$$

which is impossible. Thus S has measure zero.

Suppose $\mathcal{B}$ is type 1. If $g \in \ker \mathcal{E}$ then $g = 0$ a.e. on B_0. If $m(B_0) \neq 0$ and B_0 does not consist of a finite number of atoms, then there exists an unbounded measurable function φ supported on B_0. This φ would be $\mathcal{B}$-measurable and belong to $\mathfrak{M}(\ker \mathcal{E})$. Conversely, if B_0 does consist of a finite number of atoms, then $\mathcal{B}$-measurable multipliers of $\ker \mathcal{E}$ are bounded on B_0. But the argument in the previous paragraph shows that $\mathcal{B}$-measurable multipliers on $\ker \mathcal{E}$ are always essentially bounded on $X - B_0$. □

THEOREM 1. *Let $\mathcal{B}$ be a type-0 σ-subalgebra of $\mathcal{A}$. Then*

$$\mathcal{L} = \{f \in L^2(\mathcal{A}) : \mathcal{E}|f|^2 \in L^\infty(\mathcal{B})\}.$$

Proof. By Lemma 5, if $f \in L^2(\mathcal{A})$ and $\mathcal{E}|f|^2 \in L^\infty(\mathcal{B})$ then $f \in \mathfrak{M}(L^2(\mathcal{B}))$. Also, $\mathcal{E}f \in L^\infty(\mathcal{B})$. Thus $f \in \mathcal{L}$. Conversely, let $u \in \mathcal{L}$. For $f \in L^2(\mathcal{A})$ we have

$$[(I-\mathcal{E})f][\mathcal{E}u] - [(I-\mathcal{E})u][\mathcal{E}f] \in L^2.$$

Choosing $f \in \ker \mathcal{E}$ shows that $\mathcal{E}u \in \mathfrak{M}(\ker \mathcal{E})$.

Thus, by Lemma 6, $\mathcal{E}u$ is essentially bounded. Choosing $f \in L^2(\mathcal{B})$ shows that $(I-\mathcal{E})u \in \mathfrak{M}(L^2(\mathcal{B}))$, so that $\mathcal{E}(|(I-\mathcal{E})u|^2) \in L^\infty(\mathcal{B})$. But

$$\begin{aligned}\mathcal{E}(|(I-\mathcal{E})u|^2) &= \mathcal{E}(|u|^2 + |\mathcal{E}u|^2 - 2\,\mathrm{Re}(\bar{u}\mathcal{E}u)) \\ &= \mathcal{E}|u|^2 - |\mathcal{E}u|^2 \in L^\infty\end{aligned}$$

But $\mathcal{E}u \in L^\infty$, hence $|\mathcal{E}u|^2 \in L^\infty$, hence $\mathcal{E}|u|^2 \in L^\infty$. □

REMARK. As the proof shows, Theorem 1 is also valid under the hypothesis that $\mathcal{B}$ is type 1.

EXAMPLES. (i) Here is an example where $\mathcal{B}$ is type 0 and $\mathcal{L} = L^\infty$: Let $X = [-1, 1]$ with normalized Lebesgue measure. Let $\mathcal{B}$ be the subalgebra of sets symmetric about zero. Then $\mathcal{L}(\mathcal{A}\,|\,\mathcal{B}) = L^\infty(\mathcal{A})$. As was shown previously, in this case $B_0 = 0$. Also, in this case we have

$$(\mathcal{E}^{\mathcal{B}}f)(x)=\tfrac{1}{2}(f(x)+f(-x)).$$

Thus, by Theorem 1, $\mathcal{L}=L^\infty$.

(ii) Let $\mathcal{B}$ be an algebra for which B_0 has positive measure and does not consist of a finite number of atoms in $\mathcal{A}$. Then $\mathcal{L}$ contains an unbounded function. With these hypotheses, $L^2(B_0)$ contains an unbounded function f. Since $D_f=0$, $f\in\mathcal{L}$.

(iii) In this example $\mathcal{B}$ is type 0 but $\mathcal{L}$ contains an unbounded function: Let X be the unit square with Lebesgue area measure on the Lebesgue sets $\mathcal{A}$ in X, and let $\mathcal{B}$ be the σ-subalgebra of $\mathcal{A}$ generated by $\mathcal{C}\times[0,1]$, where $\mathcal{C}$ is the Lebesgue algebra in $[0,1]$. Then, for each f in $L^2(X,\mathcal{A},m)$, $(\mathcal{E}^{\mathcal{B}}f)(x,y)=\int_A f(x,t)\,dt$. It is not hard to see that $\mathcal{B}$ is type 0, but there are functions in $\mathcal{L}$ which are not bounded; for example, $g(x,y)\doteq y^{-x/4}$.

(iv) $\mathcal{L}=L^2(\mathcal{A})$ if $\mathcal{B}$ is generated by a finite partition of X. Let $\mathcal{B}=[B_1,\dots,B_n]$. Then $\mathcal{E}^{\mathcal{B}}f=\sum m(B_i)^{-1}(\int_{B_i}f\,dm)\chi_{B_i}$. Thus, for any $f,g\in L^2(\mathcal{A})$,

$$D_fg=\sum m(B_i)^{-1}\left[\left(\int_{B_i}f\,dm\right)g-\left(\int_{B_i}g\,dm\right)f\right]\chi_{B_i},$$

so $D_fg\in L^2$.

Conversely, if $\mathcal{B}$ is type 0 and contains a countable number of disjoint sets $\{B_i\}$ of positive measure, then there exists $f\in L^2(\mathcal{A})$ with $f\notin\mathcal{L}$. Indeed, if $f=\sum c_i\chi_{B_i}$ then $\mathcal{E}^{\mathcal{B}}|f|^2=\sum|c_i|^2\chi_{B_i}$. However, $f\in L^2(\mathcal{A})$ if and only if $\sum|c_i|^2m(B_i)<\infty$. Since $\sum m(B_i)<1$ we can choose an unbounded sequence $\{c_i\}$ with $f\in L^2(\mathcal{A})$. Then $\mathcal{E}^{\mathcal{B}}|f|^2$ is unbounded; hence $f\notin\mathcal{L}$.

The following result will be particularly useful throughout Section 2.

LEMMA 7. *Let $\mathcal{B}$ be type 0. Then there exists $\varphi\in\ker\mathcal{E}$ with $0<|\varphi|\le 1$ a.e.*

Proof. Let $\mathcal{R}=\{f\in\ker\mathcal{E}: S_f\in\mathcal{B}$ and $|f|\le 1$ a.e.$\}$. Since $\mathcal{B}$ is type 0, there exists a set A of positive measure that contains no $\mathcal{B}$-measurable set of positive measure. By part (iii) of the Proposition, the support of $\chi_A-\mathcal{E}\chi_A$ equals the support of $\mathcal{E}\chi_A$ and thus $\chi_A-\mathcal{E}\chi_A$ is in $\mathcal{R}$, so $\mathcal{R}$ is not empty. We order $\mathcal{R}$ by the relation $f<g$ if $S_f\subset S_g$ and $g\,|\,S_f=f$.

If $\{f_\alpha\}$ is a chain in $\mathcal{R}$ let $r=\sup m(S_{f_\alpha})$. Choose a sequence $f_n=f_{\alpha_n}$ such that $m(S_{f_n})\uparrow r$. Let $S=\bigcup S_{f_n}$ and define f on S by $f\,|_{S_{f_n}}=f_n$. By the definition of $\mathcal{R}$ and $<$, f is well-defined, has essential supremum ≤ 1, and is in $\ker\mathcal{E}$. Moreover, $S\in\mathcal{B}$. By Zorn's lemma, $\mathcal{R}$ contains a maximal element φ. We need only show that $S_\varphi=X$. Suppose to the contrary that $m(X-S_\varphi)>0$. Then $X-S_\varphi$ contains a set A containing no $\mathcal{B}$-measurable set of positive measure, so that $\chi_A-\mathcal{E}\chi_A$ is a member of $\mathcal{R}$ whose support is disjoint from S_φ. Let $G=\varphi+\chi_A-\mathcal{E}\chi_A$. Then $\|G\|_\infty\le 1$, $S_G\in\mathcal{B}$, and $\mathcal{E}G=0$. Thus $G\in\mathcal{R}$ and $\varphi<G$, which contradicts the maximality of φ. □

LEMMA 8. *Let $0<a<b<\infty$. Then there is a sequence $\{p_n\}$ of polynomials such that $p_n(t)$ converges uniformly to $\sqrt{t}$ on $[0,a]$ and $p_n(t)\to+\infty$ pointwise on $[b,\infty)$.*

Proof. By the Weierstrass approximation theorem we can choose p_n such that

$$|p_n(t)-\sqrt{t}|\le 1/n, \quad t\in[0,a],$$

and

$$|p_n(t)-n|\le 1/n, \quad t\in[b,b+n].$$

This sequence has the desired properties. □

The following theorem substantially strengthens the first half of Lemma 6.

THEOREM 2. *If $\mathcal{B}$ is a type-0 subalgebra of $\mathcal{A}$ and u is an $\mathcal{A}$-measurable function such that $ug\in\ker\mathcal{E}$ for every $g\in\ker\mathcal{E}$, then $u\in L^\infty(\mathcal{A})$.*

Proof. Since $\ker\mathcal{E}$ is closed under conjugation, $|u|^2$ is a multiplier on $\ker\mathcal{E}$. Define L on $\ker\mathcal{E}$ by $Lf=|u|^2 f$. It follows from the closed graph theorem that L is bounded. Also, $(Lf,f)\ge 0$ for $f\in\ker\mathcal{E}$. Let H be the positive square of the operator L. Fix $\epsilon>0$ and choose polynomials p_n as in Lemma 8 corresponding to $a=\|L\|$ and $b=\|L\|+\epsilon$. Then $p_n(L)$ converges to H in the uniform operator topology. Choose φ as in Lemma 7. Then

$$p_n(|u|^2)\varphi=p_n(L)\varphi\to H\varphi$$

in norm. Hence there is a subsequence $\{p_{n_i}\}$ such that $p_{n_i}(|u(x)|^2)\varphi(x)$ converges pointwise almost everywhere to $(H\varphi)(x)$, and hence $p_{n_i}(|u(x)|^2)$ converges a.e. to $\varphi(x)^{-1}(H\varphi)(x)$. But $\{p_{n_i}(t)\}$ diverges a.e. for $t\ge\|L\|+\epsilon$. Thus $\{|u|^2>\|L\|+\epsilon\}$ has measure zero. Since ϵ was chosen arbitrarily, $|u|\le\|L\|$ a.e. □

2

In this section we examine the algebra $\mathfrak{D}=\mathfrak{D}(\mathcal{A}|\mathcal{B})$ introduced in Section 1. Since the corollary to Lemma 4 implies that $L^2(X-B_0,\mathcal{A},m)$ reduces $\mathfrak{D}$ (and $\mathfrak{D}$ restricted to $L^2(B_0)$ is $\{0\}$), we assume throughout this section that $\mathcal{B}$ is type 0.

THEOREM 3. *$\mathfrak{D}$ is closed in the weak operator topology.*

Proof. (In what follows, $\to$ refers to weak convergence.) Let $\{f_\alpha\}$ be a net in $\mathcal{L}$ such that $D_{f_\alpha}\to D$ for some bounded operator D. Then $D_{f_\alpha}\mathcal{E}\to D\mathcal{E}$ and $D_{f_\alpha}(I-\mathcal{E})\to D(I-\mathcal{E})$. In particular,

$$(I-\mathcal{E})f_\alpha=D_{f_\alpha}(-1)\to f_1,$$

where $f_1\doteq D(-1)$. Then, for all g,

$$D_{f_\alpha}\mathcal{E}g=(\mathcal{E}g)(\mathcal{E}f_\alpha-f_\alpha)\to -f_1\mathcal{E}g.$$

Also, if $\mathcal{E}g=0$ then

$$(\mathcal{E}f_\alpha)g=D_{f_\alpha}g\to Dg.$$

Invoking Lemma 7, we choose an essentially bounded function φ in $\ker \mathcal{E}$ which is nonzero a.e. Then, for each g in $\ker \mathcal{E}$,

$$\varphi[(\mathcal{E}f_\alpha)g] \to \varphi Dg \quad \text{and} \quad [\varphi(\mathcal{E}f_\alpha)]g \to gD\varphi.$$

It follows that $Dg = ug$, where $u = D\varphi/\varphi$. But the range of D is contained in $\ker \mathcal{E}$, so that $u \cdot \ker \mathcal{E} \subset \ker \mathcal{E}$. But then Theorem 2 assures that u is essentially bounded.

We show next that u is $\mathcal{B}$-measurable. Let a and b be the real and imaginary parts of u, respectively. Both a and b are essentially bounded functions which multiply $\ker \mathcal{E}$ into itself. Fix $\epsilon > 0$ and let $A = \{\mathcal{E}a \geq a + \epsilon\}$. Since $\chi_A - \mathcal{E}\chi_A$ is in $\ker \mathcal{E}$, so is $a \cdot \chi_A - a \cdot \mathcal{E}\chi_A$; hence $\mathcal{E}(a\chi_A) = \mathcal{E}(a)\mathcal{E}(\chi_A)$. But

$$\mathcal{E}(a\chi_A) \leq \mathcal{E}((\mathcal{E}a - \epsilon)\chi_A) = (\mathcal{E}a - \epsilon)\mathcal{E}\chi_A,$$

so that $-\epsilon\mathcal{E}\chi_A \geq 0$ a.e. Since $\mathcal{E}\chi_A \geq 0$ a.e., it follows that $\mathcal{E}\chi_A = 0$ a.e. and consequently $m(A) = 0$. Since ϵ was chosen arbitrarily, $\mathcal{E}a \leq a$ a.e. A similar argument applies to sets of the form $\{\mathcal{E}a \leq a + \epsilon\}$ so that $\mathcal{E}a = a$ a.e. The same reasoning applies to b, so that $\mathcal{E}u = u$ a.e.

Now let $f = f_1 + u$. (Note that this exhibits the orthogonal decomposition of f induced by $I - \mathcal{E}$ and $\mathcal{E}$.) Let U be the linear transformation defined by $Ug = [f, g]$, where the domain of U is $\{g \text{ in } L^2(\mathcal{A}) : [f, g] \in L^2(\mathcal{A})\}$. Noting that sequential L^2 convergence yields subsequential a.e. convergence, it is easily verified that U is a closed linear transformation whose domain contains $L^\infty(\mathcal{A})$. Let g be an arbitrarily chosen member of $L^\infty(\mathcal{A})$. Then both $\mathcal{E}g$ and $(I - \mathcal{E})g$ are essentially bounded and

$$\begin{aligned} Ug &= [(I-\mathcal{E})g][\mathcal{E}f] - [(I-\mathcal{E})f][g] = [(I-\mathcal{E})g][u] - [f_1][\mathcal{E}g] \\ &= D(I-\mathcal{E})g - (\text{wk.})\lim[(I-\mathcal{E})f_\alpha][\mathcal{E}g] = D(I-\mathcal{E})g + D\mathcal{E}g \\ &= Dg. \end{aligned}$$

But then the closed operator U agrees with the bounded operator D on the dense set $L^\infty(\mathcal{A})$, and consequently (see [1]) $U = D$. Thus f is in $\mathcal{L}$ and $D = D_f$. □

We now examine the commutant $\mathfrak{D}'$ of $\mathfrak{D}$. Recall that the commutant of a set of operators is the collection of all operators commuting with each member of that set. We use the notation M_φ for the bounded operator of multiplication by the essentially bounded function φ, and L_ρ for the linear transformation of multiplication by the measurable function ρ. The domain of L_ρ is taken to be $\{g \text{ in } L^2 : \rho g \in L^2\}$, so that L_ρ is closed and densely defined [1].

THEOREM 4. $\mathfrak{D}' = \{M_\varphi : \varphi \text{ in } L^\infty(\mathcal{B})\}$.

Proof. A routine (and omitted) calculation shows that $\mathfrak{D}'$ contains the indicated set of multiplication operators. Let T be a member of $\mathfrak{D}'$. Then for each f in $\mathcal{L}$ and g in $L^2(\mathcal{A})$,

$$[\mathcal{E}f][Tg] - [f][\mathcal{E}Tg] = T([\mathcal{E}f][g] - [f][\mathcal{E}g]). \tag{1}$$

Letting $f=1$ in (1), we have $(I-\mathcal{E})Tg=T(I-\mathcal{E})g$. This shows that T commutes with $\mathcal{E}$. Let $\varphi=T1$. Then $\mathcal{E}\varphi=\mathcal{E}T1=T\mathcal{E}1=T1=\varphi$, showing that φ is $\mathcal{B}$-measurable. Now fixing f in $\mathcal{L}$ and taking $g=1$ in (1), we see that

$$T(I-\mathcal{E})f=L_\varphi(I-\mathcal{E})f \quad \text{for all } f \text{ in } \mathcal{L}. \tag{2}$$

By Lemma 7, we now choose an essentially bounded function f in $\ker\mathcal{E}$ which is nonzero a.e. Let g be an arbitrary member of $\mathcal{L}$. Then $D_f\mathcal{E}g=-D_{\mathcal{E}g}f$, so that

$$\begin{aligned} TD_f\mathcal{E}g &= -TD_{\mathcal{E}g}f=-D_{\mathcal{E}g}Tf \\ &= -D_{\mathcal{E}g}\varphi f=-\varphi\cdot D_{\mathcal{E}g}f \\ &= [-\varphi][\mathcal{E}g][f]. \end{aligned}$$

On the other hand,

$$TD_f\mathcal{E}g=D_fT\mathcal{E}g=D_f\mathcal{E}Tg=-[f][\mathcal{E}Tg]=-[f][T\mathcal{E}g]. \tag{3}$$

Since $f\neq 0$ a.e., $T\mathcal{E}g=\varphi\cdot\mathcal{E}g$. It then follows from (2) and (3) that for every g in $\mathcal{L}$,

$$Tg=T(I-\mathcal{E})g+T\mathcal{E}g=\varphi g=L_\varphi g.$$

Since $\mathcal{L}$ is dense and L_φ is closed, $L_\varphi=T=M_\varphi$. □

REMARK. In the general case (i.e., without the type-0 assumption), $L^2(B_0)$ is an invariant subspace for every operator T commuting with $\mathfrak{D}$ because $L^2(B_0)$ is contained in the kernel of every operator in $\mathfrak{D}$. However, this subspace need not be reducing for $\mathfrak{D}'$. Consider the following special case of the example discussed prior to Lemma 1: Let $X=\{1,2,3\}$ with uniform probability distribution, $\mathcal{A}=2^X$, and let $\mathcal{B}$ be the σ-algebra generated by the partition $\{\{1\},\{2,3\}\}$. The L^2 space under consideration is $\mathbf{C}^3$ and relative to the standard orthonormal basis,

$$\mathcal{E}=\begin{bmatrix}1&0&0\\0&1/2&1/2\\0&1/2&1/2\end{bmatrix};$$

and for f corresponding to $\langle a,b,c\rangle$,

$$D_f=\begin{bmatrix}0&0&0\\0&c/2&-b/2\\0&-c/2&b/2\end{bmatrix}.$$

It follows that

$$\mathfrak{D}'=\left\{\begin{bmatrix}x&y&y\\0&z&0\\0&0&z\end{bmatrix}:\langle x,y,z\rangle\in\mathbf{C}^3\right\}.$$

Since $B_0=\{1\}$, $L^2(B_0)$ does not reduce $\mathfrak{D}'$.

We will use a detailed analysis of the ranges of operators in $\mathfrak{D}$ in characterizing the spectra of operators in $\mathfrak{D}$. For each operator T, $\operatorname{ran} T$ is the range of T, and for each set S in L^2, $\bar{S}$ denotes the norm closure of S. The following observation will prove helpful.

LEMMA 9. *Let u be in $\mathfrak{L}$. Then $u \cdot L^2(X - S_{\mathcal{E}u}, \mathcal{B}) \subset \operatorname{ran} D_u$.*

Proof. Let f be an element of $u \cdot L^2(X - S_{\mathcal{E}u}, \mathcal{B})$. Then $\mathcal{E}f = f$ and $f = 0$ on $S_{\mathcal{E}u}$, so that $D_u f = -u \cdot f$. □

THEOREM 5. *Let u be in $\mathfrak{L}$. Then*

(i) $\ker \mathcal{E} \cap L^2(S_{\mathcal{E}u}, \mathcal{A}) \subset \overline{\operatorname{ran} D_u} \subset \ker \mathcal{E} \cap L^2(S_{\mathcal{E}|u|}, \mathcal{A})$; *and*

(ii) $\overline{\operatorname{ran} D_u} = \ker \mathcal{E} \cap L^2(S_{\mathcal{E}u}, \mathcal{A}) + \overline{(u \cdot L^2(X - S_{\mathcal{E}u}, \mathcal{B}))}$.

Proof. Since both u and $\mathcal{E}u$ have their supports in the support of $\mathcal{E}|u|$, and since $\operatorname{ran} D_u \subset \ker \mathcal{E}$, (i) will follow directly from (ii). For notational convenience let

$$\mathcal{U} = \ker \mathcal{E} \cap L^2(S_{\mathcal{E}u}, \mathcal{A}) \quad \text{and} \quad \mathcal{V} = u \cdot L^2(X - S_{\mathcal{E}u}, \mathcal{B}).$$

We have already seen that $\mathcal{V} \subset \operatorname{ran} D_u$. Let $f \in \mathcal{U}$. For each number $r > 0$ define the set S_r to be $\{|\mathcal{E}u| > r\}$. Then each S_r is a $\mathcal{B}$-subset of the support of $\mathcal{E}u$. Define the function $g_r \doteq (g/\mathcal{E}u)\chi_{S_r}$. Each g_r is in $\mathcal{U}$ and $D_u g_r = [\mathcal{E}u][g_r] = f\chi_{S_r}$. Since f vanishes off the support of $\mathcal{E}u$, (wk.) $\lim_{r \downarrow 0} f\chi_{S_r} = f$. But the weak and norm closures of convex sets in L^2 agree; hence f is in $\overline{\operatorname{ran} D_u}$. This proves that $\mathcal{U} \subset \overline{\operatorname{ran} D_u}$.

We have shown that $\mathcal{U} + \mathcal{V} \subset \overline{\operatorname{ran} D_u}$. In order to establish the stated equality in (ii), let f be an arbitrary function in $L^2(\mathcal{A})$. Then

$$\begin{aligned} D_u f &= [\mathcal{E}u][f] - [u][\mathcal{E}f] = [f][\mathcal{E}u]\chi_{S_{\mathcal{E}u}} - [u][\mathcal{E}f] \\ &= [(f\chi_{S_{\mathcal{E}u}})\mathcal{E}u - u\mathcal{E}(f\chi_{S_{\mathcal{E}u}})] - [u\mathcal{E}(f\chi_{X - S_{\mathcal{E}u}})]. \end{aligned}$$

The bracketed term immediately to the right of the last inequality sign is a member of $\mathcal{U}$, while the other bracketed expression is a member of $\mathcal{V}$. In summary,

$$\mathcal{U} \subset \overline{\operatorname{ran} D_u}; \qquad \mathcal{V} \subset \operatorname{ran} D_u; \qquad \operatorname{ran} D_u \subset \mathcal{U} + \mathcal{V}.$$

Now $\mathcal{U}$ is closed and (because of the disjointness of supports) $\mathcal{U} \perp \mathcal{V}$. Consequently $\overline{\operatorname{ran} D_u} = \ker \mathcal{E} \cap L^2(S_{\mathcal{E}u}, \mathcal{A}) + \overline{(u \cdot L^2(X - S_{\mathcal{E}u}, \mathcal{B}))}$. □

REMARKS. 1. If $u \geq 0$, or more generally if $\mathcal{E}u$ and $\mathcal{E}|u|$ have equal supports, then $\overline{\operatorname{ran} D_u} = \ker \mathcal{E} \cap L^2(S_{\mathcal{E}u}, \mathcal{A})$.

2. Theorem 5 holds even if $\mathcal{B}$ is not type 0. Now, for any set A in $\mathcal{A}$, $\ker \mathcal{E} \cap L^2(A, \mathcal{A}) = \{0\}$ if and only if $\mathcal{B} \cap A = \mathcal{A} \cap A$ [4]. (Note that $\mathcal{B} \cap A$ is not a subset of $\mathcal{B}$.) For A in $\mathcal{B}$, this happens if and only if $A \subset B_0$. But u and $\mathcal{E}u$ agree on B_0. Thus $\overline{\operatorname{ran} D_u} = \overline{(u \cdot L^2(X - S_{\mathcal{E}u}, \mathcal{B}))}$ if and only if the support of u is contained in B_0.

Since $D_u u = 0$ for every u in $\mathcal{L}$, 0 is an eigenvalue for every operator in $\mathfrak{D}$. We shall prove that the spectrum of D_u is $\{0\} \cup$(essential range $\mathcal{E}u$). This will follow from the following two lemmas.

LEMMA 10. *Let f be in $\mathcal{L}$ and let λ be a nonzero complex number. Then λ is an eigenvalue for D_f if and only if $\mathcal{E}f = \lambda$ on a set of positive measure.*

Proof. Suppose that $g \neq 0$ and $D_f g = \lambda g$. Since $\lambda \neq 0$ and $\mathcal{E}D_f = 0$, $\mathcal{E}g = 0$. But then $D_f g = [\mathcal{E}f][g]$, so that $\mathcal{E}f = \lambda$ on S_g.

Conversely, suppose that L is a set of positive measure on which $\mathcal{E}f = \lambda$. Invoking the type-0 hypothesis, we may assume (by choosing a subset of L if necessary) that L is not a member of $\mathcal{B}$. Let $g = (I - \mathcal{E})\chi_L$, so that $\mathcal{E}g = 0$ and $g \neq 0$. Moreover,

$$D_f g = [\mathcal{E}f][g] = (I - \mathcal{E})((\mathcal{E}f)\chi_L) = \lambda(I - \mathcal{E})\chi_L = \lambda g. \qquad \square$$

REMARK. Lemma 10 remains valid without the type-0 hypothesis if the operative condition is weakened to "$\mathcal{E}f = \lambda$ on a set of positive measure in $X - B_0$".

Suppose now that f is in $\mathcal{L}$ and that λ is not an eigenvalue for D_f. Let $u = f - \lambda$. Then $\mathcal{E}u \neq 0$ a.e. We will make frequent use of the fact that $D_f - \lambda I = D_u - \lambda\mathcal{E}$.

LEMMA 11. *With f, λ, and u as above,*

(i) $\operatorname{ran}(D_f - \lambda I) \supset \ker \mathcal{E}$ *if and only if* $1/\mathcal{E}u$ *is a multiplier on* $\ker \mathcal{E}$;

(ii) $\operatorname{ran}(D_f - \lambda I) \supset L^2(\mathcal{B})$ *if and only if* $\mathcal{E}|u|^2/|\mathcal{E}u|^2 \in L^\infty(\mathcal{B})$.

Proof. Let g be an arbitrary member of $\ker \mathcal{E}$, and suppose that

$$g = (D_f - \lambda I)G$$

for some L^2 function G. Then, since $\lambda \neq 0$ and both g and $D_f G$ are in $\ker \mathcal{E}$, $\mathcal{E}G = 0$. But then $g = (D_f - \lambda(I - \mathcal{E}))G = D_u G = [\mathcal{E}u]G$; whence $g/\mathcal{E}u$ is in $L^2(\mathcal{A})$. This shows that if $\operatorname{ran}(D_f - \lambda I) \supset \ker \mathcal{E}$ then $1/\mathcal{E}u$ multiplies $\ker \mathcal{E}$ into itself. Conversely, suppose that $1/\mathcal{E}u$ is a multiplier of $\ker \mathcal{E}$, and let g be in $\ker \mathcal{E}$. Then

$$(D_u - \lambda\mathcal{E})\frac{g}{\mathcal{E}u} = D_u \frac{g}{\mathcal{E}u} = g,$$

showing that $\ker \mathcal{E} \subset \operatorname{ran}(D_f - \lambda I)$.

In order to establish the validity of (ii), suppose that for each function h in $L^2(\mathcal{B})$ we have a function H for which $(D_u - \lambda\mathcal{E})H = h$. For such h and H, since $\mathcal{E}D_u = 0$, we have $-\lambda\mathcal{E}H = h$. This shows that $D_u H = 0$, so that

$$[\mathcal{E}u][H] = [\mathcal{E}H][u] = -\frac{1}{\lambda}[h][u],$$

which proves that $(u/\mathcal{E}u)\cdot L^2(\mathcal{B}) \subset L^2(\mathcal{A})$. It then follows that $\mathcal{E}|u|^2/|\mathcal{E}u|^2 \in L^\infty(\mathcal{B})$.

For the converse, if $\mathcal{E}|u|^2/|\mathcal{E}u|^2$ is essentially bounded, then ($u/\mathcal{E}u$ is a multiplier of ker $\mathcal{E}$ and) for each h in $L^2(\mathcal{B})$,

$$(D_u-\lambda\mathcal{E})\left(\frac{-1}{\lambda}\frac{u}{\mathcal{E}u}h\right)=-\frac{1}{\lambda}\left([\mathcal{E}u]\left[\frac{u}{\mathcal{E}u}h\right]-[u]\left[\mathcal{E}\left(\frac{u}{\mathcal{E}u}h\right)\right]\right)$$
$$+\mathcal{E}\left(\frac{u}{\mathcal{E}u}h\right)=h. \qquad \square$$

THEOREM 6. *For each f in $\mathcal{L}$, the spectrum of D_f is $\{0\}\cup$(essential range of $\mathcal{E}f$).*

Proof. Let f be in $\mathcal{L}$ and let λ be a nonzero complex number. Then $D_f-\lambda I$ is invertible if and only if, for $u=f-\lambda$,

(i) $\mathcal{E}u\neq 0$ a.e. (i.e., $D_f-\lambda I$ is $1-1$); and
(ii) $1/\mathcal{E}u$ is a multiplier of ker $\mathcal{E}$ and $\mathcal{E}|u|^2/|\mathcal{E}u|^2$ is essentially bounded.

But since the first condition in (ii) forces $1/\mathcal{E}u$ to be essentially bounded (by Theorem 2), and since $\mathcal{E}|u|^2$ is essentially bounded because u is in $\mathcal{L}$, we have:

$$D_f-\lambda I \text{ is invertible if and only if } \mathcal{E}u \text{ is invertible in } L^\infty(\mathcal{B});$$

that is, λ is not in the essential range of $\mathcal{E}f$. Since 0 is in the spectrum of D_f, the asserted conclusion follows. $\square$

COROLLARY. *Let $\mathcal{R}=\{D-\lambda I : D\in\mathfrak{D}, \lambda\in\mathbf{C}\}$. Then $\mathcal{R}$ is a weakly closed and inverse closed algebra of operators.*

Proof. Let $\{D_{f_\alpha}-\lambda_\alpha I\}$ be a net in $\mathcal{R}$ converging weakly to the operator L. Since $\mathcal{E}D_{f_\alpha}=0$ for each α, $\{\lambda_\alpha\}$ converges to a number λ. But $\mathfrak{D}$ is weakly closed, so $\{D_{f_\alpha}\}$ converges weakly to some operator D_f in $\mathfrak{D}$, and so $L=D_f-\lambda I$. Thus $\mathcal{R}$ is weakly closed.

Now suppose that for some f in $\mathcal{L}$ and scalar λ, $D_f-\lambda I$ is invertible in the algebra of bounded operators on $L^2(\mathcal{A})$. Then $\lambda\neq 0$ and $\mathcal{E}f-\lambda$ is invertible in $L^\infty(\mathcal{B})$. Define the function

$$g=\frac{1}{\lambda}\frac{f}{\mathcal{E}f-\lambda}.$$

Noting that the denominator of the preceding rational expression is $\mathcal{B}$-measurable, we see that

$$\mathcal{E}|g|^2=\frac{1}{|\lambda|^2}\frac{\mathcal{E}|f|^2}{|\mathcal{E}f-\lambda|^2}\in L^\infty(\mathcal{B}),$$

so that g is a member of $\mathcal{L}$. Moreover,

$$\left(D_g-\frac{1}{\lambda}I\right)(D_f-\lambda I)=D_gD_f-\frac{1}{\lambda}D_f-\lambda D_g+I$$
$$=D_{[\mathcal{E}g]f}-D_{(1/\lambda)f+\lambda g}+I.$$

Now,

$$\frac{1}{\lambda}f+\lambda g=\frac{1}{\lambda}f+\frac{f}{\mathcal{E}f-\lambda}=\frac{1}{\lambda}\frac{\mathcal{E}f}{\mathcal{E}f-\lambda}f=[\mathcal{E}g][f].$$

We have established the following formula:

$$(D_f-\lambda I)^{-1}=D_{f/(\lambda(\mathcal{E}f-\lambda))}-\frac{1}{\lambda}I. \qquad \square$$

REMARK. Relative to the orthogonal decomposition of $L^2(\mathfrak{A})$ as $L^2(\mathfrak{B})\oplus L^2(\mathfrak{B})^{\perp}$, each operator D_f in $\mathfrak{D}$ has a matricial form

$$D_f=\begin{bmatrix} 0 & 0 \\ -M_{(I-\mathcal{E})f}\mathcal{E} & M_{\mathcal{E}f}(I-\mathcal{E}) \end{bmatrix},$$

where M indicates the operation of multiplication. It must be noted that although the operator in the lower left corner of this matrix is bounded, the multiplication operator $M_{(I-\mathcal{E})f}$ need not be bounded. Also, even though $M_{\mathcal{E}f}$ is a bounded operator on all of $L^2(\mathfrak{A})$ whose spectrum is the essential range of $\mathcal{E}f$, it was not clear a priori what the spectrum was for the restriction of this multiplication to the kernel of $\mathcal{E}$.

We conclude this section with the characterization of the self-adjoint, normal, seminormal, and projection operators in $\mathfrak{D}$.

THEOREM 7. *Let f be in $\mathfrak{L}$. The following are equivalent.*

(a) $f\in L^{\infty}(\mathfrak{B})$.
(b) *D_f is normal.*
(c) *D_f is seminormal (i.e., either D_f or D_f^* is hyponormal).*

Proof. (c) ⇒ (a) For any g in $L^2(\mathfrak{A})$,

$$\begin{aligned} D_f^*D_fg&=\mathcal{E}\bar{f}D_fg-\mathcal{E}(\bar{f}D_fg)=[\mathcal{E}\bar{f}][(\mathcal{E}f)g-(\mathcal{E}g)f]-\mathcal{E}(\bar{f}(\mathcal{E}f)g-\bar{f}(\mathcal{E}g)f)\\ &=|\mathcal{E}f|^2g-[f][\mathcal{E}\bar{f}][\mathcal{E}g]-[\mathcal{E}(\bar{f}g)][\mathcal{E}f]+[\mathcal{E}|f|^2][\mathcal{E}g]; \end{aligned}$$

while

$$D_fD_f^*g=\mathcal{E}fD_f^*g-f\mathcal{E}(D_f^*g)=|\mathcal{E}f|^2g-\mathcal{E}f\mathcal{E}(\bar{f}g)-f(\mathcal{E}\bar{f})\mathcal{E}g+f\mathcal{E}(\bar{f}g),$$

so that

$$(D_f^*D_f-D_fD_f^*)g=(\mathcal{E}|f|^2)\mathcal{E}g-f\mathcal{E}(\bar{f}g).$$

We have

$$\begin{aligned} ((D_f^*D_f-D_fD_f^*)g,g)&=\int(\mathcal{E}|f|^2)(\mathcal{E}g)(\bar{g})\,dm-\int(f)(\mathcal{E}(\bar{f}g))(\bar{g})\,dm\\ &=\int((\mathcal{E}|f|^2)(\mathcal{E}g|^2)-|\mathcal{E}(\bar{f}g)|^2)\,dm. \end{aligned}$$

It follows from the Cauchy-Schwarz inequality for conditional expectation that if g is $\mathfrak{B}$-measurable, then the integrand above is nonnegative. On the other hand, if $\mathcal{E}g=0$ then the integrand is nonpositive. Thus if D_f is hyponormal then for every g in $\ker\mathcal{E}$, $\int|\mathcal{E}(\bar{f}g)|^2\,dm=0$, and so $\mathcal{E}(\bar{f}g)=0$ a.e. In particular,

$$0=\mathcal{E}(\bar{f}(f-\mathcal{E}f))=\mathcal{E}|f|^2-|\mathcal{E}f|^2=\mathcal{E}|(I-\mathcal{E})f|^2.$$

(This is just the formula for conditional variance.) But then $f=\mathcal{E}f$ a.e.

Now if D_f^* is hyponormal then the above commutator equation and the Cauchy–Schwarz inequality show that for each $\mathcal{B}$-measurable g, $\mathcal{E}|f|^2|\mathcal{E}g|^2=|\mathcal{E}(\bar{f}g)|^2$. Letting $g=1$, we are led once more to the conclusion that $\mathcal{E}|f|^2=|\mathcal{E}f|^2$, which shows that $\mathcal{E}f=f$. Thus (c) $\Rightarrow$ (a). Since (b) $\Rightarrow$ (c) for any operator, we need only show that (a) $\Rightarrow$ (b). But if f is $\mathcal{B}$-measurable then, by Lemma 4, $D_f^*=D_{\bar{f}}$, and so $D_f^*D_f=D_{|f|^2}=D_fD_f^*$. □

The next corollary follows immediately from the proof of Theorem 7.

COROLLARY.

(i) *D_f is self-adjoint if and only if f is a real-valued member of $L^\infty(\mathcal{B})$.*

(ii) *D_f is an orthogonal projection if and only if $f=\chi_B$ for some set B in $\mathcal{B}$.*

3

As we remarked previously, the set $\mathcal{L}$ also possesses a natural Lie algebra structure if, for $f,g\in\mathcal{L}$, $[f,g]$ is defined to be D_fg. Indeed, $[\cdot,\cdot]$ is clearly bilinear, while a routine computation shows that $[f,f]=0$ and that

$$[f,[g,h]]+[g,[h,f]]+[h,[f,g]]=0.$$

(To verify the latter identify it is useful to observe that for f, $g\in L^2(\mathcal{A})$, $\mathcal{E}([f,g])=0$.)

In this Lie algebra setting, D_f acting on $\mathcal{L}$ is just $\operatorname{ad} f$. Also, if B is a $\mathcal{B}$-measurable set and $\mathcal{L}(B)=\mathcal{L}\cap L^2(B)$, then $\mathcal{L}(B)$ is an ideal of the Lie algebra $\mathcal{L}$; in particular, $\mathcal{L}(B_0)$ is the center of $\mathcal{L}$. Also, $\mathcal{L}$ is a Lie module over $L^\infty(\mathcal{B})$. (That $\mathcal{L}(B)$ is an ideal follows from the fact that $S_f\subset B$ implies $S_{\mathcal{E}f}\subset B$.)

We recall from [2] that a Banach–Lie algebra is a Lie algebra $\mathcal{L}$ together with a norm $f\to\|f\|_{\mathcal{L}}$ that satisfies $\|[f,h]\|_{\mathcal{L}}\le M\|f\|_{\mathcal{L}}\|h\|_{\mathcal{L}}$ for some positive M and all f,h in $\mathcal{L}$. In the type-0 case we have the following result.

THEOREM 8. *Let $\mathcal{B}$ be type 0. Then $\mathcal{L}$ is a Banach Lie algebra with respect to the norm $\|f\|_{\mathcal{L}}=\sqrt{\|\mathcal{E}|f|^2\|_\infty}$.*

Proof. Routine use of the Cauchy–Schwarz inequality for conditional expectation and the positive definiteness of conditional expectation shows that the formula above does indeed define a norm on $\mathcal{L}$. Suppose that $\{f_n\}$ is a Cauchy sequence with regard to this norm (henceforth referred to as $\mathcal{L}$-norm, $\mathcal{L}$-Cauchy sequence, etc.). Since for all f, $\int_X|f|^2\,dm=\int_X\mathcal{E}|f|^2\,dm$, $\{f_n\}$ is a Cauchy sequence with respect to the L^2 norm and so converges in

L^2 to some function f. We shall show that f is in $\mathcal{L}$ and that $\{f_n\}$ converges to f in the $\mathcal{L}$-norm. Let $\epsilon>0$. Then there is an integer N such that, for all $n\geq N$, $\mathcal{E}|f_N-f_n|^2<\epsilon$. Let $S=\{\mathcal{E}|f_N-f|^2>2\epsilon\}$. Then S in in $\mathcal{B}$ and

$$\epsilon m(S)\geq \lim_{n\to\infty}\int_S \mathcal{E}|f_N-f_n|^2\,dm=\int_S \mathcal{E}|f_N-f|^2\,dm\geq 2\epsilon m(S).$$

Thus S has measure 0, and consequently f_N-f is in $\mathcal{L}$. This shows that f is in $\mathcal{L}$. It also shows that for each $\epsilon>0$ there is an integer N for which $\|f_N-f\|_\infty^2<\epsilon$. This establishes a subsequence of $\{f_n\}$ converging in $\mathcal{L}$-norm to f. Since $\{f_n\}$ is an $\mathcal{L}$-Cauchy sequence, it must converge in $\mathcal{L}$-norm to f.

It remains only to show that for f and h in $\mathcal{L}$, $\|[f,h]\|_{\mathcal{L}}\leq 2\|f\|_{\mathcal{L}}\|h\|_{\mathcal{L}}$. Now,

$$\begin{aligned}\mathcal{E}|[f,h]|^2&=\mathcal{E}|(\mathcal{E}f)h|^2+\mathcal{E}|(\mathcal{E}h)f|^2-2\,\mathrm{Re}\,\mathcal{E}(\overline{(\mathcal{E}f)h}(\mathcal{E}h)f)\\&=|\mathcal{E}f|^2\mathcal{E}|h|^2+\mathcal{E}|f|^2|\mathcal{E}h|^2-2\,\mathrm{Re}((\overline{\mathcal{E}f})(\mathcal{E}h)\mathcal{E}(\bar{h}f))\\&\leq 2\mathcal{E}|f|^2\mathcal{E}|h|^2+2\sqrt{\mathcal{E}|f|^2\cdot\mathcal{E}|h|^2\cdot\mathcal{E}|h|^2\cdot\mathcal{E}|f|^2}\\&=4\mathcal{E}|f|^2\mathcal{E}|h|^2.\end{aligned}$$

Consequently,

$$\|[f,h]\|_{\mathcal{L}}=\sqrt{\|\mathcal{E}|[f,h]|^2\|_\infty}\leq 2\sqrt{\|\mathcal{E}|f|^2\mathcal{E}|h|^2\|_\infty}\leq 2\|f\|_{\mathcal{L}}\cdot\|h\|_{\mathcal{L}}. \qquad \square$$

COROLLARY. *In the type-*0 *setting, the operator norm* $\|D_f\|$ *and the* $\mathcal{L}$*-norm* $\|f\|_{\mathcal{L}}$ *define equivalent norms on* $\mathfrak{D}$.

Proof. For f and g in $\mathcal{L}$,

$$\begin{aligned}\|D_fg\|&\leq\|(\mathcal{E}f)g\|+\|f\mathcal{E}g\|=\|(\mathcal{E}f)g\|+\sqrt{\textstyle\int|f|^2|\mathcal{E}g|^2\,dm}\\&=\|(\mathcal{E}f)g\|+\sqrt{\textstyle\int(\mathcal{E}|f|^2|\mathcal{E}g|^2\,dm}\leq\|\mathcal{E}f\|_\infty\|g\|+\|f\|_{\mathcal{L}}\|\mathcal{E}g\|\\&\leq 2\|f\|_{\mathcal{L}}\|g\|.\end{aligned}$$

This shows that the surjection $f\mapsto D_f$ is continuous from the Banach space $\mathcal{L}$ to the Banach space $\mathfrak{D}$. The inequality $\|f\|_{\mathcal{L}}\geq c\|D_f\|$ for some positive constant c and all f in $\mathcal{L}$ follows from the closed graph theorem. $\square$

Remarks and Questions

(1) For which σ-subalgebras $\mathcal{B}$ does $\mathcal{L}(\mathcal{A}|\mathcal{B})=L^\infty(\mathcal{A})$? Equivalently, when is $\mathcal{E}^{-1}L^\infty(\mathcal{B})=L^\infty(\mathcal{A})$?

(2) Note that for f in $L^\infty(\mathcal{A})$, $\|f\|_2\leq\|f\|_{\mathcal{L}}\leq\|f\|_\infty$ (where the subscript of 2 indicates L^2 norm). Is $L^\infty(\mathcal{A})$ $\mathcal{L}$-norm dense in $\mathcal{L}$?

(3) Suppose that $\mathcal{B}$ and $\mathcal{C}$ are σ-subalgebras of $\mathcal{A}$ and that T is a measure-preserving automorphism of $(X,\mathcal{A})$ such that $T^{-1}\mathcal{C}=\mathcal{B}$. It is known [5, p. 21] that for each f in $L^2(\mathcal{A})$, $\mathcal{E}^{T^{-1}\mathcal{C}}(f\circ T)=(\mathcal{E}^{\mathcal{C}}f)\circ T$. Letting W be the unitary operator $W\colon g\mapsto g\circ T$, it follows readily that for each f in $\mathcal{L}(\mathcal{A}|\mathcal{C})$, $WD_f^{\mathcal{C}}W^{-1}=D_{Wf}^{\mathcal{B}}$. If $\mathfrak{D}(\mathcal{A}|\mathcal{B})$ and $\mathfrak{D}(\mathcal{A}|\mathcal{C})$ are unitarily equivalent, must a measure algebra isomorphism exist between $\mathcal{B}$ and $\mathcal{C}$? If so, must there be such an isomorphism that extends to an automorphism on $\mathcal{A}$?

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Department of Mathematics
University of North Carolina at Charlotte
Charlotte, NC 28223