Embedding Theorems for Spaces of Analytic Functions via Khinchine's Inequality

DANIEL H. LUECKING

1. Introduction

We show how known embeddings of sequence spaces into spaces of analytic functions on the unit disk **D** (and other domains) can be combined with Khinchine's inequality to obtain lower estimates on expressions like

$$\int_{\mathbf{D}} |f^{(n)}|^s \, d\mu.$$

These lower estimates allow us to obtain necessary conditions on the measure μ on **D** in order that there exists a constant C with

$$\left(\int_{\mathbf{D}} |f^{(n)}|^s \, d\mu\right)^{1/s} \le C\|f\|.$$

(Here $\|f\|$ denotes the norm appropriate for the space of analytic functions.) In each of the cases presented here, the necessary condition coincides with a sufficient condition obtained by straightforward estimates. Moreover, this necessary and sufficient condition reduces to the characterization of the multipliers between certain related sequence spaces. The spaces of analytic functions considered include the Bergman spaces, the mixed norm spaces, and a new class of spaces: the analytic functions belonging to certain weighted tent spaces. There is essentially no restriction on the exponent s and the exponents defining the spaces of analytic functions. Moreover, more general expressions than the $L^s(\mu)$ norm may appear on the left-hand side of the above inequality.

Let **D** be the open unit disk in the complex plane **C** and let A^p denote the Bergman space of analytic functions f on **D** whose L^p norm $\|f\|_p \overset{\text{def}}{=} (\int_{\mathbf{D}} |f|^p \, dA)^{1/p}$ is finite (dA is area measure). Let ρ denote the *pseudohyperbolic metric*

$$\rho(z,w) \overset{\text{def}}{=} \left|\frac{z-w}{1-\bar{z}w}\right|$$

on **D** and

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$$h \overset{\text{def}}{=} \log \frac{1+\rho}{1-\rho}$$

the hyperbolic metric. Let $\{z_n\}$ be a *separated sequence* in $\mathbf{D}$. This means the *separation constant* $\gamma = \inf\{h(z_k, z_m): m \neq k\}$ is positive. In [1], Amar proved a theorem of which the following is a special case.

AMAR'S THEOREM. *If the separation constant γ of $\{z_n\}$ is sufficiently large, then the operator T defined on A^p by $T(f)_k = f(z_k)(1-|z_k|^2)^{2/p}$ maps A^p onto l^p.*

(Amar proved the corresponding theorem for weighted Bergman spaces in the ball and the polydisk in $\mathbf{C}^n$, $n \geq 1$.) Amar's proof may be divided into two parts, given by the following two lemmas.

LEMMA 1. *Define the operator S on sequences by*

$$S((a_n)) = \sum_n a_n \frac{(1-|z_n|^2)^{M-2/p}}{(1-\bar{z}_n z)^M},$$

where $M > 1 + 1/p$ if $p > 1$ and $M > 2/p$ if $p \leq 1$. Then S is bounded from l^p to A^p whenever $\{z_n\}$ is separated.

LEMMA 2. *Let γ be the separation constant of $\{z_n\}$; then $\|I - TS\|_{l^p \to l^p}$ converges to zero as γ tends to ∞. That is, TS is invertible for γ sufficiently large, whence T is onto.*

The proofs of these two lemmas are remarkably similar (but it does not seem feasible to derive one from the other). Both lemmas have been extended to other contexts by Coifman and Rochberg [3] and Rochberg [16]. In [11] this author used Amar's theorem to characterize those measures $\mu \geq 0$ on $\mathbf{D}$ satisfying for $0 < q < p$: There exists a constant $C > 0$ such that, for all $f \in A^p$,

$$\left(\int |f|^q \, d\mu\right)^{1/q} \leq C\|f\|_p. \tag{1.1}$$

Amar's proof is easy to extend to the operators $T^{(n)}$ defined by

$$T^{(n)}(f)_k = f^{(n)}(z_k)(1-|z_k|^2)^{n+2/p},$$

and with it the characterization of measures μ satisfying the following is also possible: There exists a constant $C > 0$ such that, for all $f \in A^p$,

$$\left(\int |f^{(n)}|^q \, d\mu\right)^{1/q} \leq C\|f\|_p. \tag{1.2}$$

There now exist two further results analogous to these, which characterize measures satisfying Carleson-like inequalities. The spaces of analytic functions involved are, respectively, the so-called mixed norm spaces and the Hardy spaces. These results, however, use the same method of proof, which may be outlined as follows:

(1) Prove the analogue of Lemma 1.
(2) Prove the analogue of Lemma 2, and deduce the analogous interpolation result (the surjectivity of the appropriate mapping T).
(3) Prove that small perturbations in $\{z_n\}$ (in the hyperbolic metric) yield small perturbations in T and combine this with the interpolation result of item 2 to obtain a necessary condition for the analogues of (1.1) and (1.2). (The sufficiency of the condition is usually a straightforward estimation.)

One of the two results alluded to above may be found in [7], where Jevtić has solved the analogue of (1.1) for mixed norm spaces. In fact he has solved a much more general problem. The other is in [13], where the author has solved the analogue of (1.2) for the Hardy spaces. For mixed norm spaces, part 1 of the proof outline was accomplished by Ricci and Taibleson in [15], part 2 is due to Jevtić in [6], and part 3 appears in [7], also by Jevtić. For the Hardy space, all three parts appear in the original version of [13]. The final version of [13], however, uses a different proof that skips from part 1 to the solution. It is the purpose of this paper to explain the method, to show that it also neatly solves both (1.1) and (1.2) in the other two settings, and to derive some new results using the method and some extensions of it.

In the next section we illustrate the method in the case of unweighted Bergman spaces. The solution to (1.1) in that case is known and, as mentioned, the solution to (1.2) can be obtained in the same way. Moreover, the Bergman space is a special case of the mixed norm spaces, which we treat in Section 4. Nevertheless I feel that it is useful to treat the simplest case separately in order to get a feel for the arguments before other complications are introduced. In Section 3, we cover some necessary background material on mixed norm spaces of analytic functions, prior to solving the mixed norm analogue of problems (1.1) and (1.2) in Section 4. A similar pair of sections follows in which a new space of analytic functions is introduced and studied; these are the tent spaces of analytic functions. The tent spaces of measurable functions were introduced in [2]. In [13], the solution of problem (1.2) for Hardy spaces required the introduction of "weighted" tent spaces: the analogue of tent spaces with more general measures in place of Lebesgue measure. Similar results were obtained independently by Harboure, Torrea, and Viviani in [5]. We will see that there are no analytic functions in the original tent spaces, but that analytic functions in weighted tent spaces arise naturally. In Section 5 we develop the background material on these spaces and in Section 6 we solve the analogues of problems (1.1) and (1.2). The final section discusses the relationships among these spaces and discusses some related questions.

2. Embedding Theorems via Khinchine's Inequality

In later sections I will obtain new results along the lines of (1.1) and (1.2), but in this section I will illustrate the method by obtaining an essentially known result in perhaps the simplest case, the unweighted Bergman spaces A^p.

Define the Rademacher functions r_k by

$$r_0(t)=\begin{cases} 1 & 0\le t-[t]<\frac{1}{2}, \\ -1 & \frac{1}{2}\le t-[t]<1; \end{cases}$$

$$r_n(t)=r_0(2^n t), \quad n>0.$$

Then *Khinchine's inequality* is the following.

KHINCHINE'S INEQUALITY. *For $0<p<\infty$ there exist constants $0<a_p\le B_p<\infty$ such that, for all natural numbers m and all complex numbers $c_1, c_2, \ldots, c_m$, we have*

$$a_p\left(\sum_{j=1}^m |c_j|^2\right)^{p/2} \le \int_0^1 \left|\sum_{j=1}^m c_j r_j(t)\right|^p dt \le B_p\left(\sum_{j=1}^m |c_j|^2\right)^{p/2}.$$

Let us establish the following theorem using only Lemma 1 and Khinchine's inequality. First, select any convenient δ_0 with $0<\delta_0<\infty$ and let $D(z)=D(z,\delta_0)\overset{\text{def}}{=}\{w: h(z,w)<\delta_0\}$.

THEOREM 1 (essentially in [12]). *Let $0<q<p<\infty$. For a measure $\mu\ge 0$ on $\mathbf{D}$, the following are equivalent.*

(a) *There exists a constant C such that, for any $f\in A^p$,*

$$\left(\int |f^{(n)}|^q\, d\mu\right)^{1/q} \le C\|f\|_p. \tag{2.1}$$

(b) *The function $z\mapsto \mu(D(z))(1-|z|^2)^{-2-nq}$ belongs to $L^{p/(p-q)}(dA)$, the dual of $L^{p/q}(dA)$.*

Note: The parameters p, q, n, δ_0, and the measure μ are regarded as fixed. The letter C will be used to stand for various constants (not necessarily the same one in every occurrence) which depend at most on these parameters and the measure μ.

Proof. We will begin with the necessity of condition (b). To this end suppose μ satisfies (2.1) with C independent of f. Put

$$f(z)=\sum_k a_k \bar{z}_k^{-n} \frac{(1-|z_k|^2)^{M-2/p}}{(1-\bar{z}_k z)^M},$$

where $\{z_k\}$ is some separated sequence with separation constant $\delta_0/2$ and $\inf_k|z_k|>0$. Then by Lemma 1 there are constants M and C such that

$$\|f\|_p \le C\left(\sum |a_k|^p\right)^{1/p},$$

whence

$$\int \left|\sum a_k \frac{(1-|z_k|^2)^{M-2/p}}{(1-\bar{z}_k z)^{M+n}}\right|^q d\mu \le C\left(\sum |a_k|^p\right)^{q/p}.$$

In this inequality, replace a_k with $r_k(t)a_k$ (so that the right side does not change) and then integrate with respect to t from 0 to 1. After Fubini's theorem and Khinchine's inequality this yields

$$a_p \int \left(\sum |a_k|^2 \frac{(1-|z_k|^2)^{2M-4/p}}{|1-\bar{z}_k z|^{2M+2n}} \right)^{q/2} d\mu \leq C\left(\sum |a_k|^p\right)^{q/p}.$$

Now we observe that $\chi_{D(z_k)}(z) \leq C(1-|z_k|^2)/|1-\bar{z}_k z|$ for some constant C, and that there is a constant $N = N(\delta_0)$ such that each point z belongs to not more than N of the disks $D(z_k)$ (see Lemma 3). This allows us to estimate

$$\begin{aligned}
&\sum |a_k|^q \frac{\mu(D(z_k))}{(1-|z_k|)^{2q/p+nq}} \\
&\quad = \int \sum \frac{|a_k|^q \chi_{D(z_k)}}{(1-|z_k|)^{2q/p+nq}}\, d\mu \\
&\quad \leq \max\{N^{1-q/2}, 1\} \int \left(\sum |a_k|^2 \frac{\chi_{D(z_k)}}{(1-|z_k|)^{4/p+2n}} \right)^{q/2} d\mu \\
&\quad \leq C \int \left(\sum |a_k|^2 \frac{(1-|z_k|)^{2M-4/p}}{|1-\bar{z}_k z|^{2M+2n}} \right)^{q/2} d\mu \leq C\left(\sum |a_k|^p\right)^{q/p}.
\end{aligned}$$

Now put $b_k = |a_k|^q$ and conclude that the sequence

$$\frac{\mu(D(z_k))}{(1-|z_k|^2)^{2q/p+nq}}$$

belongs to the dual of $l^{p/q}$, whence

$$\sum \left(\frac{\mu(D(z_k))}{(1-|z_k|^2)^{2q/p+nq}} \right)^{p/(p-q)} < \infty.$$

Using $(1-|z_k|^2)^2 \leq CA(D(z_k))$, we can write this as

$$\sum \left(\frac{\mu(D(z_k))}{(1-|z_k|^2)^{2+nq}} \right)^{p/(p-q)} A(D(z_k)) < \infty \tag{2.2}$$

and we see a discrete version of the condition we require. To get the continuous version, observe the following.

(1) Since the argument thus far is independent of the size of δ_0, inequality (2.2) will hold when $\{z_k\}$ has separation constant δ_0 and $D(z_k)$ is replaced with $\tilde{D}(z_k) = D(z_k, 2\delta_0)$.

(2) It is possible (see Lemma 4) to select $\{z_k\}$ with separation constant δ_0 and $\inf_k |z_k| > 0$ such that the collection $\{D(z_k)\}$ covers $\mathbf{D}$.

Now notice that if z is in $D(z_k)$ then the whole disk $D(z)$ lies in $\tilde{D}(z_k)$. Now, whenever $z \in D(z_k)$,

$$\frac{\mu(D(z))}{(1-|z|^2)^{2+nq}} \leq C \frac{\mu(\tilde{D}(z_k))}{(1-|z_k|^2)^{2+nq}},$$

whence

$$\int\left(\frac{\mu(D(z))}{(1-|z|^2)^{2+nq}}\right)^{p/(p-q)} dA(z)$$
$$\leq \sum \int_{D(z_k)}\left(\frac{\mu(D(z))}{(1-|z|^2)^{2+nq}}\right)^{p/(p-q)} dA(z)$$
$$\leq C\sum\left(\frac{\mu(\tilde{D}(z_k))}{(1-|z_k|^2)^{2+nq}}\right)^{p/(p-q)} A(D(z_k)) < +\infty.$$

The first inequality is from observation (2) above and the last inequality is from observation (1) about (2.2). Thus we have shown that (a) implies (b).

To get the other implication, we proceed exactly as in [12]. The following is an outline: First, show that

$$|f^{(n)}(z)|^q \leq \frac{C}{(1-|z|^2)^{2+nq}} \int_{D(z)} |f|^q \, dA.$$

This is a relatively routine normal families argument. Next, integrate this inequality with respect to μ and use Fubini's theorem to obtain

$$\int |f^{(n)}(z)|^q \, d\mu(z) \leq C \int |f|^q \frac{\mu(D(w))}{(1-|w|^2)^{2+nq}} \, dA.$$

Finally, apply Hölder's inequality to get (b) from (a). □

It should be pointed out that this method of using Khinchine's inequality gives the correct necessary condition for inequality (2.1) even when $q > p$, namely that $\mu(D(z))(1-|z|^2)^{-2-nq}$ should be bounded; but that result is obtained much more easily by substituting the function

$$f(z) \stackrel{\text{def}}{=} \left(\frac{1-|a|^2}{(1-\bar{a}z)^2}\right)^{2/p}$$

into (2.1).

Two results used in the proof are the following two essentially well-known lemmas. The arguments used appear in a more general context in [3].

LEMMA 3. *Given $0 < \delta < \epsilon < \infty$, there is a constant $L = L(\delta, \epsilon)$ such that if $\{z_k\}$ satisfies $\inf_{k \neq n} h(z_k, z_n) \geq \delta$ then every $z \in \mathbf{D}$ belongs to at most L disks $D(z_k, \epsilon)$.*

Proof. Because of the conformal invariance of h, we may suppose that $z = 0$ and we need only show that $D(0, \epsilon)$ contains at most L of the points, where L depends only on δ and ϵ. Now the measure $d\nu(z) \stackrel{\text{def}}{=} (1-|z|^2)^{-2} \, dA(z)$ is conformally invariant, so $\nu(D(z, \delta/2))$ is independent of z. The disks $D(z_k, \delta/2)$ are disjoint, and if $z_k \in D(0, \epsilon)$ then $D(z_k, \delta/2) \subset D(0, \epsilon + \delta/2)$. Thus, if L is the number of z_k in $D(0, \epsilon)$ then

$$L \leq \frac{\nu(D(0, \epsilon + \delta/2))}{\nu(D(0, \delta/2))}. \qquad \square$$

LEMMA 4. *If* $\delta_0 > 0$ *then there is a sequence* $\{z_k\} \subset \mathbf{D}$ *such that* $h(z_k, z_n) \geq \delta_0$ *for all* $k \neq n$, $\inf_k |z_k| > 0$, *and*

$$\bigcup_k D(z_k, \delta_0) = \mathbf{D}.$$

Proof. Let z_1 be any point with $h(z_1, 0) = \delta_0/2$, and let z_2 be any point with $h(z_1, z_2) = \delta_0$. Given $z_1, z_2, \ldots, z_{k-1}$, select $z_k \notin \bigcup_{j=1}^{k-1} D(z_j, \delta_0)$ such that $h(z_1, z_k)$ is minimized. This defines the sequence $\{z_k\}$ inductively. Clearly $h(z_k, z_n) \geq \delta_0$ whenever $k \neq n$. If any $z \in \mathbf{D}$ is not in $\bigcup_{k=1}^{\infty} D(z_k, \delta_0)$ then (by the choice of z_k) for all k we have $h(z_1, z_k) \leq h(z_1, z)$. This is impossible because of Lemma 3 with $\delta = \delta_0$ and $\epsilon = h(z_1, z)$. □

As a final note before going on to more complicated embedding theorems, we observe that the Rademacher functions are not the only sequence that could be used in the proof of the theorem: If q happened to be 2 then any bounded orthonormal set could be used, and otherwise any bounded orthonormal sequence satisfying the left half of Khinchine's inequality could be used—for example, the exponentials $\{e^{in_k\theta}\}$ with $\{n_k\}$ any lacunary sequence. However, the inequalities for vector-valued series which are used later may require the stochastic independence that the Rademacher functions possess.

3. Some Technical Results for Mixed Norm Spaces

Having seen the method at work in a case where the result was already known, we turn to some more complicated applications. These will include characterizing measures involved in inequalities like (1.2) but with the functions f in a mixed norm space and its mixed norm on the right-hand side.

The *mixed norm spaces* are the spaces $A^{p,q}$, defined for any $0 < p, q < \infty$ as the collection of functions f analytic on $\mathbf{D}$ with finite "norm":

$$\|f\|_{p,q} \stackrel{\text{def}}{=} \left(\int_0^1 \left(\int_0^{2\pi} |f(re^{i\theta})|^p \, d\theta\right)^{q/p} dr\right)^{1/q}.$$

As mentioned in the introduction, an embedding theorem for these spaces was proved by Jevtić. To state his result, let us first define a partition of the disk into disjoint sets Q_{ij}, which are of roughly equal size in the hyperbolic metric. For each $j = 1, 2, \ldots$, let us divide the annulus $\{z : 2^{-j} < 1 - |z| \leq 2^{1-j}\}$ into 2^{j+1} equal pieces by means of equally spaced radii. Number these sectors arbitrarily as Q_{ij}, $i = 1, 2, \ldots, 2^{j+1}$. The whole collection is a disjoint cover of $\mathbf{D}$ except for the origin. What I mean by "roughly equal size" is that if z_{ij} are the centers of the Q_{ij} then there exist positive numbers $0 < \delta < \epsilon < \infty$ independent of i and j such that

$$\{z : h(z, z_{ij}) < \delta\} \subset Q_{ij} \subset \{z : h(z, z_{ij}) < \epsilon\}. \tag{3.1}$$

The Q_{ij} will be used to replace the finitely overlapping cover $D(z_k)$ of Section 2.

JEVTIĆ'S THEOREM [7]. *For a measure μ on* $\mathbf{D}$ *and positive numbers $s<p$ and $t<q$, the following are equivalent:*

(a) *There is a constant C independent of f such that*

$$\left(\sum_j\left(\sum_i\int_{Q_{ij}}|f(z)|^s\,d\mu(z)\right)^{t/s}\right)^{1/t}\le C\|f\|_{p,q}$$

for all $f\in A^{p,q}$.

(b) *The measure satisfies*

$$\sum_j\left(\sum_i(\mu(Q_{ij})^{1/s}2^{j(1/p+1/q)})^u\right)^{v/u}<\infty,$$

where $1/u=1/s-1/p$ and $1/v=1/t-1/q$.

Note that Jevtić's theorem reduces to the analogue of (1.1) when $s=t$. Here, I will extend Jevtić's results by putting no restrictions on the exponents involved and by considering the effect of differentiation. Problems like (1.1) were also considered by Gadbois in his thesis [4]. The problem he considered was like Jevtić's in that he too allowed a mixed norm on the left, but of a different type. His solution also placed restrictions on the exponents involved. His result was a characterization of those product measures $\mu\times\nu$ on $[0,2\pi)\times(0,1)$ such that there exists a constant C with

$$\left(\int_{(0,1)}\left(\int_{[0,2\pi)}|f(re^{i\theta})|^s\,d\mu(\theta)\right)^{t/s}d\nu(r)\right)^{1/t}\le C\|f\|_{p,q}$$

for all $f\in A^{p,q}$. His methods required the restriction that $p=s$ and $t\ge q$. Both Jevtić and Gadbois also considered weighted mixed norm spaces. We could also do that, but the statements of already unwieldy inequalities would become more so. Thus, we will leave it to the interested reader to put in the appropriate factors in order to obtain the results for weighted spaces.

Before I can extend these theorems, I will need three results. The first is an extension of the Khinchine-Kahane inequality to quasi-Banach spaces by Kalton [8]; the second is a simple characterization of multipliers between mixed norm sequence spaces due to Kellog [9]; and the third is an analogue of Lemma 1 for mixed norm spaces due to Ricci and Taibleson [15].

LEMMA 5 (Khinchine-Kahane-Kalton). *Let X be a quasi-Banach space with quasi-norm $\|\cdot\|$ and let $0<p<q<\infty$. Then there exist constants $a=a(X,p,q)$ and $B=B(X,p,q)$ such that for any positive integer m and any $x_1,x_2,\dots,x_m\in X$ we have*

$$a\left(\int_0^1\left\|\sum_k r_k(t)x_k\right\|^p dt\right)^{1/p}\le\left(\int_0^1\left\|\sum_k r_k(t)x_k\right\|^q dt\right)^{1/q}$$
$$\le B\left(\int_0^1\left\|\sum_k r_k(t)x_k\right\|^p dt\right)^{1/p}.$$

A quasi-Banach space is a vector space with a quasi-norm. A quasi-norm is a function from X to $[0,\infty)$ which has all the properties of a norm except that the triangle inequality is replaced by:

there exists a constant C such that $\|x+y\| \le C(\|x\|+\|y\|)$ for all $x, y \in X$.

The constants a and B in the lemma actually depend only on p, q, and C. A particular example of a quasi-Banach space is $L^p(d\mu)$ for any $p>0$ and any measure μ. The quasi-norm is the expression

$$\|f\|_{L^p(d\mu)} = \left(\int |f|^p \, d\mu\right)^{1/p}$$

and the constant C in the definition of quasi-norm can be taken to be $\max(1, 2^{1/p-1})$.

Our applications of the Khinchine–Kahane–Kalton inequality will be of the following sort. We will have an inequality of the form

$$\left(\int_0^1 \left(\int_0^{2\pi} \left|\sum_k r_k(t) a_k f_k(r,\theta)\right|^q d\mu(\theta)\right)^{s/q} d\nu(r)\right)^{1/s} \le C\|(a_k)\|, \tag{3.2}$$

where $\|(a_k)\|$ is some norm on a sequence space that can be defined solely in terms of $|a_k|$. If both sides of this inequality are raised to the s power and the resulting inequality integrated in t from 0 to 1, then the integrals with respect to t and r may be interchanged. The inner integral can be estimated from below as follows:

$$\begin{aligned}\int_0^1 \|\sum r_k(t) a_k f_k(r,\cdot)\|_{L^q(d\mu)}^s \, dt &\ge a_{q,s}\left(\int_0^1 \|\sum r_k(t) a_k f_k(r,\cdot)\|_{L^q(d\mu)}^q \, dt\right)^{s/q} \\ &= a_{q,s}\left(\int_0^{2\pi}\int_0^1 |\sum r_k(t) a_k f_k(r,\theta)|^q \, dt \, d\mu(\theta)\right)^{s/q}.\end{aligned}$$

Now the scalar version of Khinchine's inequality can be used. The effect of the Khinchine–Kahane–Kalton inequality is that *the integral with respect to t can be brought to the center of any mixed norm expression* such as (3.2).

For any $0<p<\infty$ and $0<q<\infty$, let $l^{p,q}$ denote the set of doubly indexed sequences of complex numbers (c_{ij}) such that

$$\|(c_{ij})\|_{p,q} \overset{\text{def}}{=} \left(\sum_j \left(\sum_i |c_{ij}|^p\right)^{q/p}\right)^{1/q} < \infty.$$

The index set may be the set of natural numbers, or it may be a finite interval of natural numbers. The spaces $l^{p,\infty}$, $l^{\infty,q}$ and $l^{\infty,\infty}$ are defined similarly, with a $\sup|\cdot|$ replacing $(\sum|\cdot|^p)^{1/p}$ in the appropriate places. We will need the following characterization of duality for the $l^{p,q}$ spaces that includes the cases $0<p,q<1$. One reference that includes all these cases is [14].

LEMMA 6 (Nakamura). *For any* $0<p\leq\infty$, *define*

$$p'=\begin{cases}1 & \text{if } p=\infty,\\ p/(p-1) & \text{if } 1<p<\infty,\\ \infty & \text{if } 0<p\leq 1.\end{cases}$$

Then

$$\sup\left\{\left|\sum_{ij} c_{ij}b_{ij}\right| : \|(c_{ij})\|_{p,q}\leq 1\right\}=\|(b_{ij})\|_{p',q'}. \tag{3.3}$$

Of course, when $p, q<\infty$ or if p or q equals ∞ and the corresponding index set is finite, this implies that the continuous linear functionals on $l^{p,q}$ may be identified with elements of $l^{p',q'}$ with equality of norms. The unmixed norm version of (3.3) is of course the familiar

$$\sup\left\{\left|\sum_{j} c_{j}b_{j}\right| : \|(c_{j})\|_{p}\leq 1\right\}=\|(b_{j})\|_{p'},\quad 0<p\leq\infty. \tag{3.4}$$

We actually need a more general version of Lemma 6. We will say that a sequence (b_{ij}) multiplies $l^{p,q}$ into $l^{s,t}$ if there is a constant C such that

$$\begin{gathered}\left(\sum_j\left(\sum_i|b_{ij}c_{ij}|^s\right)^{t/s}\right)^{1/t}\leq C\left(\sum_j\left(\sum_i|c_{ij}|^p\right)^{q/p}\right)^{1/q}\\ \text{that is,}\quad \|(b_{ij}c_{ij})\|_{s,t}\leq C\|(c_{ij})\|_{p,q}\end{gathered} \tag{3.5}$$

for all $(c_{ij})\in l^{p,q}$ or, equivalently, for all finite sequences (c_{ij}). It is easy to see that if every finite subsequence of (b_{ij}) multiplies $l^{p,q}$ into $l^{s,t}$ *with the same constant* C, then the entire sequence multiplies $l^{p,q}$ into $l^{s,t}$. The collection of multipliers from $l^{p,q}$ to $l^{s,t}$ will be denoted by $M(l^{p,q}, l^{s,t})$. Another obvious fact is that (b_{ij}) belongs to $M(l^{p,q}, l^{s,t})$ if and only if $(|b_{ij}|^s)\in M(l^{p/s,q/s}, l^{1,t/s})$. Finally observe that $M(l^{p,q}, l^{1,1})=l^{p',q'}$ from (3.3) of Lemma 6.

LEMMA 7 (Kellog). *Let* $0<p,q,s,t<\infty$. *Then*

$$M(l^{p,q}, l^{1,t})=l^{p',t(q/t)'},$$

and consequently $M(l^{p,q}, l^{s,t})=l^{s(p/s)',t(q/t)'}$.

Proof. This is essentially Kellog's proof but cast in a considerably more compact form. Suppose (b_{ij}) is a finite sequence satisfying inequality (3.5) with $s=1$ for all finite sequences (c_{ij}). That is,

$$\left(\sum_j\left(\sum_i|b_{ij}c_{ij}|\right)^{t}\right)^{1/t}\leq C\left(\sum_j\left(\sum_i|c_{ij}|^p\right)^{q/p}\right)^{1/q}. \tag{3.6}$$

Now let (a_j) be an arbitrary nonnegative sequence, and if $p>1$ put

$$c_{ij}=\begin{cases} b_{ij}^{p'-1}(\sum_i b_{ij}^{p'})^{-1/p}a_j^{1/t} & \text{if } b_{ij}\neq 0,\\ 0 & \text{otherwise;}\end{cases}$$

if $p \le 1$, put

$$c_{ij} = \begin{cases} a_j^{1/t} & \text{if } i \text{ is the first integer with } b_{ij} = \max_k b_{kj}, \\ 0 & \text{otherwise.} \end{cases}$$

Then from (3.6) we obtain

$$\sum_j \|(b_{ij})_i\|_{p'}^t a_j \le C^t \Big(\sum_j a_j^{q/t}\Big)^{t/q}.$$

But from (3.4), sup-ing over (a_j) of $l^{q/t}$ norm 1, we see that

$$\|(\|(b_{ij})_i\|_{p'}^t)_j\|_{(q/t)'} \le C^t.$$

That is, $\|(b_{ij})\|_{p',t(q/t)'} \le C$. As remarked, once we get this for finite sequences, it follows for all sequences with the same constant.

For the converse, if $(b_{ij}) \in l^{p',t(q/t)'}$ then (3.6) follows from two successive applications of (3.4), the first in the i index with exponents p and p' and the second in the j index with exponents q/t and $(q/t)'$. □

It should be pointed out that the expression $u = s(p/s)'$ satisfies $u = \infty$ if $s \ge p$, and that $1/u = 1/s - 1/p$ if $s < p$, with a similar remark for $v = t(q/t)'$. Thus Jevtić's theorem, part (b), states that a certain sequence multiplies $l^{p,q}$ into $l^{s,t}$. Our results will extend Jevtić's by obtaining the same conclusion for all positive exponents.

The following theorem to be found in [15, Thm. 1.5], stated in somewhat greater generality. It is the analogue of Lemma 1 for mixed norm spaces. For a sequence $\{z_k\}$ in the unit disk, re-index it in the following way: Let $\{z_{ij} : i = 1, 2, \ldots\}$ denote the points of $\{z_k\}$ that belong to the annulus $\{z : 2^{-j} \le 1 - |z| < 2^{-j+1}\}$.

LEMMA 8. *Define the operator S on double sequences by*

$$S((a_{ij})) = \sum_{ij} a_{ij} \frac{(1 - |z_{ij}|^2)^{M - 1/p - 1/q}}{(1 - \bar{z}_{ij} z)^M},$$

where $M > \max\{1/q, 1\} + 1/p$. Then S is bounded from $l^{p,q}$ to $A^{p,q}$ whenever $\{z_{ij}\}$ is separated.

4. Embedding Theorems for the Mixed Norm Spaces

We precede the results on mixed norm spaces with some more preparatory material. A continually recurring phenomenon (which we saw in Section 2) is that the sufficient condition is easiest to obtain in continuous form while the necessary condition is most readily obtained in a discrete form. Rather than convert from discrete to continuous or the reverse, we first state a result that allows us to obtain both necessary and sufficient conditions readily in discrete form. The continuous version will still be given (if it is reasonably accessible), but the proof that it is equivalent to the discrete version will be omitted, being similar to the last part of the necessity proof in Theorem 1.

LEMMA 9. *For $f \in A^{p,q}$ let $|f|_{ij} = \sup\{|f(z)|: z \in Q_{ij}\}$. Then there exist constants c_1 and C_2 such that*

$$c_1\|f\|_{p,q} \le \|(|f|_{ij}2^{-(j/p+j/q)})\|_{l^{p,q}} \le C_2\|f\|_{p,q}. \tag{4.1}$$

For a proof, see [15, Lemma 6.3]. The first inequality is straightforward, using the inequality $|f| \le \sum_{ij}|f|_{ij}\chi_{Q_{ij}}$. The other inequality can be obtained by first estimating $|f|_{ij}^p$ by a multiple of $2^{2j}\int_{D_{ij}}|f|^p\,dA$, where $D_{ij} = \{z: h(z, z_{ij}) < 2\epsilon\}$ with z_{ij} and ϵ as in (3.1). Then the inner sum in the definition of the $l^{p,q}$ norm of $(|f|_{ij}2^{-j(1/p+1/q)})$ is easily estimated by an integral over an annulus. Then it is easy to get $\|f\|_{p,q}$ as an upper estimate because $\int|f(re^{i\theta})|^p\,d\theta$ increases with r.

Another result that we will need is a simple estimate of the derivatives of a function in $A^{p,q}$:

$$\|(|f^{(n)}|_{ij}2^{-j(n+1/p+1/q)})\|_{l^{p,q}} \le C\|(|f|_{ij}2^{-j(1/p+1/q)})|\,\|_{l^{p,q}}. \tag{4.2}$$

This comes from estimating $|f^{(n)}|_{ij}$ by 2^{nj} times the sum of $|f|_{kl}$ over adjacent Q_{kl}, using the Cauchy integral formula.

We are now ready for one of the main theorems of this section.

THEOREM 2. *Let $0 < p, q, s < \infty$ and let μ be a finite positive measure on $\mathbf{D}$. Then the following are equivalent.*

(a) *There is a constant C such that*

$$\left(\int_{\mathbf{D}}|f^{(n)}|^s\,d\mu\right)^{1/s} \le C\|f\|_{p,q} \tag{4.3}$$

for all $f \in A^{p,q}$.

(b) *The sequence $(\mu(Q_{ij})2^{sj(n+1/p+1/q)})$ belongs to $l^{(p/s)',(q/s)'}$.*

(c) *The function $k(z)$ belongs to $L^{(p/s)',(q/s)'}$, where*

$$k(z) = \begin{cases} \mu(D(z))(1-|z|^2)^{-2-sn} & \text{if } s<p,\ s<q,\\ \mu(D(z))(1-|z|^2)^{-1-s/p-sn} & \text{if } p\le s<q,\\ \mu(D(z))(1-|z|^2)^{-1-s/q-sn} & \text{if } q\le s<p,\\ \mu(D(z))(1-|z|^2)^{-s/p-s/q-sn} & \text{if } p\le s,\ q\le s.\end{cases}$$

Before proving Theorem 2, let us indicate why (c) is the appropriate continuous version of (b). This is not so much of a difficulty with the fourth formula for $k(z)$: condition (b) says that $\mu(Q_{ij})(1-|z_{ij}|^2)^{-s/p-s/q-sn}$ is bounded while (c) says the same thing of $\mu(D(z))(1-|z|^2)^{-s/p-s/q-sn}$. Since any $D(z)$ (respectively Q_{ij}) can be covered by a fixed finite number of Q_{ij} (respectively $D(z)$), it is easy to see the equivalence of (b) and (c) in this case. Consider, then, the first case. By the same reasoning we get

$$k(z) \sim \sum \mu(Q_{ij})2^{j(2+ns)}\chi_{Q_{ij}}(z)$$

in the sense, at least, that their norms are equivalent. But if we calculate the $L^{(p/s)',(q/s)'}$ norm of this sum (keeping in mind that the lengths of the sides of the Q_{ij} are proportional to 2^{-j}) we get the same expression as the $l^{(p/s)',(q/s)'}$ norm of the sequence in (b).

Proof. Assume (a). Using Lemma 8 from the previous section, the function

$$f(z)=\sum_{ij} a_{ij}\bar{z}_{ij}^{-n}\frac{(1-|z_{ij}|^2)^{M-1/p-1/q}}{(1-\bar{z}_{ij}z)^M} \tag{4.4}$$

satisfies $\|f\|_{p,q}\le C\|(a_{ij})\|_{p,q}$. The constant C, of course, depends on p, q, M, and n. We suppose that $a_{ij}=0$ except for a finite number of indices. Differentiating this expression n times and inserting the result into inequality (4.3), we have

$$\int\left|\sum_{ij} a_{ij}\frac{(1-|z_{ij}|^2)^{M-1/p-1/q}}{(1-\bar{z}_{ij}z)^{M+n}}\right|^s d\mu(z)\le C^s\|(a_{ij})\|_{p,q}^s.$$

Replacing a_{ij} by $r_{ij}(t)a_{ij}$, where r_{ij} is a re-indexing of the Rademacher sequence, and integrating with respect to t gives

$$\int\left(\sum_{ij}\left|a_{ij}\frac{(1-|z_{ij}|^2)^{M-1/p-1/q}}{(1-\bar{z}_{ij}z)^{M+n}}\right|^2\right)^{s/2} d\mu(z)\le C\|(a_{ij})\|_{p,q}^s.$$

Again we use $(1-|z_{ij}|^2)/|1-\bar{z}_{ij}z|\ge c\chi_{Q_{ij}}(z)$ and $1-|z_{ij}|^2\ge c2^{-j}$ to obtain

$$\int\sum_{ij}|a_{ij}|^s 2^{js(n+1/p+1/q}\chi_{Q_{ij}}(z)\,d\mu(z)\le C\|(a_{ij})\|_{p,q}^s.$$

Now put $b_{ij}=|a_{ij}|^s$ and integrate to get

$$\sum_{ij} b_{ij}2^{js(n+1/p+1/q)}\mu(Q_{ij})\le C\|(b_{ij})\|_{p/s,q/s}$$

for all positive finite sequences (b_{ij}). By Lemma 6,

$$\|\mu(Q_{ij})2^{js(n+1/p+1/q)}\|_{(p/s)',(q/s)'}\le C.$$

Thus (b) follows from (a).

Now assume (b) and estimate as follows:

$$\begin{aligned}
&\int|f^{(n)}|^s\,d\mu\\
&\quad=\sum\int_{Q_{ij}}|f^{(n)}|^s\,d\mu\\
&\quad\le\sum|f^{(n)}|_{ij}^s\mu(Q_{ij})\\
&\quad=\sum|f^{(n)}|_{ij}^s 2^{-sj(n+1/p+1/q)}\mu(Q_{ij})2^{sj(n+1/p+1/q)}\\
&\quad\le\|(|f^{(n)}|_{ij}^s2^{-sj(n+1/p+1/q)})_{ij}\|_{p/s,q/s}\|(\mu(Q_{ij})2^{sj(n+1/p+1/q)})_{ij}\|_{(p/s)',(q/s)'}
\end{aligned}$$

by Lemma 6. The second factor in the last line is finite, and the first factor is the same as

$$\|(|f^{(n)}|_{ij}2^{-j(n+1/p+1/q)})_{ij}\|_{p,q}^s$$

which (as remarked at the beginning of this section) is dominated by

$$C\|(|f|_{ij}2^{-j(1/p+1/q)})_{ij}\|_{p,q}^{s}\le C\|f\|_{p,q}^{s}.$$

Thus (a) follows from (b) with a constant C which is a multiple of the $l^{(p/s)',(q/s)'}$ norm of $(\mu(Q_{ij})2^{js(n+1/p+1/q)})_{ij}$ raised to the $1/s$ power. □

We turn now to the mixed-measure generalizations of Theorem 2 (the extensions of Gadbois's and Jevtić's results). Since the proofs are very similar to that of Theorem 2, substituting only Lemma 7 for Lemma 6 and the Khinchine-Kahane-Kalton inequality for the scalar Khinchine inequality, the proofs will be a little less detailed. Let us first observe that

$$\{(r,\theta): re^{i\theta}\in Q_{ij}\}=R_j\times I_{ij},$$

where $R_j=(1-2^{1-j},1-2^{-j}]$ and, for fixed j, I_{ij} are 2^{j+1} disjoint intervals of length $2^{-j}\pi$.

THEOREM 3. *Let $0<p,q,s,t<\infty$, let μ be a positive measure on the interval $[0,2\pi)$, and let ν be a positive measure on the interval $(0,1)$. Then the following are equivalent.*

(a) *There is a constant C such that*

$$\left(\int_{(0,1)}\left(\int_{[0,2\pi)}|f^{(n)}(re^{i\theta})|^{s}\,d\mu(\theta)\right)^{t/s}d\nu(r)\right)^{1/t}\le C\|f\|_{p,q} \tag{4.5}$$

for all $f\in A^{p,q}$.

(b) *The sequence $(\mu(I_{ij})^{1/s}\nu(R_j)^{1/t}2^{j(n+1/p+1/q)})_{ij}$ multiplies $l^{p,q}$ into $l^{s,t}$. That is,*

$$(\mu(I_{ij})^{1/s}\nu(R_j)^{1/t}2^{j(n+1/p+1/q)})_{ij}\in l^{s(p/s)',t(q/t)'}.$$

Note that we omit the continuous version. Unless a more efficient form can be found, the continuous version requires the listing of nine different cases compared to the four cases found in Theorem 2.

Proof. Assume (a). Let f be as in equation (4.4). Substitute its nth derivative into (4.5) to obtain

$$\left(\int_{(0,1)}\left(\int_{[0,2\pi)}\left|\sum_{ij}a_{ij}\frac{(1-|z_{ij}|^2)^{M-1/p-1/q}}{(1-\bar{z}_{ij}z)^{M+n}}\right|^{s}d\mu(\theta)\right)^{t/s}d\nu(r)\right)^{1/t}\le C\|(a_{ij})\|_{p,q}.$$

Do the familiar application of Khinchine's inequality, but using the Khinchine-Kahane-Kalton version as outlined in Section 3. This yields

$$\int_{(0,1)}\left(\int_{[0,2\pi)}\sum_{ij}|a_{ij}|^{s}2^{js(n+1/p+1/q)}\chi_{R_j\times I_{ij}}(r,\theta)\,d\mu(\theta)\right)^{t/s}d\nu(r)\le C\|(a_{ij})\|_{p,q}^{t}.$$

After performing the indicated integrations, this becomes

$$\sum_{j}\left(\sum_{i}|a_{ij}|^{s}2^{js(n+1/p+1/q)}\mu(I_{ij})\right)^{t/s}\nu(R_j)\le C\|(a_{ij})\|_{p,q}^{t}. \tag{4.6}$$

A moment's thought reveals that this is precisely the inequality needed for the membership of $(\mu(I_{ij})^{1/s}\nu(R_j)^{1/t}2^{j(n+1/p+1/q)})_{ij}$ in $M(l^{p,q},l^{s,t})$.

The converse implication is obtained by estimating the integral in (4.5) by the left-hand side of (4.6) with $a_{ij} = |f^{(n)}|_{ij} 2^{-j(n+1/p+1/q)}$. Since

$$(\mu(I_{ij})^{1/s} \nu(R_j)^{1/t} 2^{j(n+1/p+1/q)})_{ij} \in M(l^{p,q}, l^{s,t})$$

is also sufficient for (4.6) and since this choice of a_{ij} makes the right-hand side of (4.6) less than $C\|f\|^t_{p,q}$, we see that (b) implies (a). □

The extension of Jevtić's theorem is obtained in an almost identical manner, and I will limit myself to a statement of the result.

THEOREM. *Let $0 < s, t, p, q < \infty$ and let μ be a measure on* **D**. *Then the following are equivalent.*

(a) *There is a constant C independent of f such that*

$$\left(\sum_j \left(\sum_i \int_{Q_{ij}} |f^{(n)}(z)|^s \, d\mu(z)\right)^{t/s}\right)^{1/t} \leq C\|f\|_{p,q}$$

for all $f \in A^{p,q}$.

(b) *The sequence $(\mu(Q_{ij})^{1/s} 2^{j(n+1/p+1/q)})_{ij}$ multiplies $l^{p,q}$ into $l^{s,t}$; that is, it belongs to $l^{s(p/s)', t(q/t)'}$.*

A more general version of inequality (4.5) can be conceived:

$$\left(\int_{(0,1)} \left(\int_{[0,2\pi)} |f^{(n)}(re^{i\theta})|^s \, d\mu_r(\theta)\right)^{t/s} d\nu(r)\right)^{1/t} \leq C\|f\|_{p,q},$$

where the measures μ_r vary measurably with r. I do not yet have a good characterization of such sets of measures, but I can show, in much the same way, that there is a single necessary and sufficient condition that is entirely divorced from analytic functions. It is the following:

$$\sum_j \int_{R_j} \left(\sum_i |a_{ij}|^s 2^{js(n+1/p+1/q)} \mu_r(I_{ij})\right)^{t/s} d\nu(r) \leq C\|(a_{ij})\|^t_{p,q}$$

for all sequences (a_{ij}). The sufficiency of the condition arises from the usual upper estimate $|f| \leq \sum |f|_{ij} \chi_{Q_{ij}}$, while the necessity follows from the same Khinchine inequality argument as in Theorem 3.

5. Tent Spaces of Analytic Functions

The tent spaces T^p_q were introduced in [2]. We will see that they contain no analytic functions, but they just fail to: If a weighted version is introduced with a weight which is a positive power of the distance to the boundary, then they contain a rich set of analytic functions. This is a generalization of the observation that the derivatives of H^p functions belong to a weighted T^p_2 space. In this section we develop a sufficiently rich theory of analytic tent spaces for us to formulate and solve the appropriate versions of problems (1.1) and (1.2) in the next section. For reasons that should become obvious, we will shift our domain of analyticity from the disk to the upper half-plane $\mathbf{U} = \{z = x + iy : y > 0\}$.

The tent spaces T_q^p are defined in [2] as spaces of measurable functions on $\mathbf{R}_+^{n+1} = \{(x, y) : x \in \mathbf{R}^n,\ y > 0\}$. They are defined for $0 < p < \infty$ and $0 < q < \infty$ as follows:

First define for t in $\mathbf{R}^n$, $\Gamma(t) = \{(x, y) \in \mathbf{R}_+^{n+1} : |x - t| < y\}$, a cone in $\mathbf{R}_+^{n+1}$ with vertex at $(t, 0)$. Then let

$$\|f\|_{T_q^p} \stackrel{\text{def}}{=} \left(\int_{\mathbf{R}^n}\left(\int_{\Gamma(t)} |f|^q \frac{dx\,dy}{y^{n+1}}\right)^{p/q} dt\right)^{1/p}. \tag{5.1}$$

Finally, T_q^p is the class of measurable functions f with $\|f\|_{T_q^p}$ finite. More generally, let $\Gamma_\beta(t) = \{(x, y) : |x - t| < \beta y\}$. Then an equivalent norm is obtained if in (5.1) we replace $\Gamma(t)$ with $\Gamma_\beta(t)$ for any $\beta > 0$.

There are technical problems with the definition when either p or q is infinite and, when $q < 1$, different choices for the definition can be reasonably justified. Nevertheless we will need the same definition for all positive finite values of p and q in the generalized version considered below.

There is a duality theorem whereby the dual of T_q^p is $T_{q'}^{p'}$ when $1 < p, q < \infty$, the pairing being

$$\langle f, q\rangle = \int_{\mathbf{R}_+^{n+1}} f\bar{g}\frac{dx\,dy}{y} = c_n \int_{\mathbf{R}^n}\int_{\Gamma(t)} f\bar{g}\frac{dx\,dy}{y^{n+1}}\,dt.$$

A theory of tent spaces in which measures other than volume measure are permitted has been begun independently, in [13] and [5]. Unfortunately, two slightly different formulations were chosen, one to simplify the definition of the norm and the other apparently to simplify the duality pairing and other formulas. In order to promote standardization of notation, I will abandon my approach and follow that of [5]. Thus, let $T_q^p(\mu)$ denote the space of Borel measurable functions on $\mathbf{R}_+^{n+1}$ such that the following norm is finite:

$$\|f\|_{T_q^p(\mu)} \stackrel{\text{def}}{=} \left(\int_{\mathbf{R}^n}\left(\int_{\Gamma(t)} |f|^q \frac{d\mu}{y^n}\right)^{p/q} dt\right)^{1/p}. \tag{5.2}$$

Thus $T_q^p = T_q^p(y^{-1}\,dx\,dy)$. We will explicitly allow all positive real values of q and p in this definition, even though an argument can be made that the definition ought to be changed when $q \le 1$. There is a duality result which says that for $1 < p, q < \infty$, the dual of $T_q^p(\mu)$ is $T_{q'}^{p'}(\mu)$ with the pairing $\langle f, g\rangle = \int_{\mathbf{R}_+^{n+1}} f\bar{g}\,d\mu$. It is also true that the norm on $T_q^p(\mu)$ is equivalent to the one obtained on replacing $\Gamma(t)$ with $\Gamma_\beta(t)$.

In the context of the unit disk $\mathbf{D}$, the definitions become

$$\|f\|_{T_q^p(\mu)} \stackrel{\text{def}}{=} \left(\int_0^{2\pi}\left(\int_{\Gamma(\theta)} |f(z)|^q \frac{d\mu(z)}{1 - |z|^2}\right)^{p/q} d\theta\right)^{1/p}.$$

and $\Gamma(\theta)$ is the Stoltz angle at θ, which is defined for real θ as the convex hull of the set $\{e^{i\theta}\} \cup \{z : |z| < \sqrt{1/2}\}$. (The radius $\sqrt{1/2}$ is chosen so that the angle made by the boundary of $\Gamma(\theta)$ at $e^{i\theta}$ is $\pi/2$. Other angles less than π, corresponding to other choices of radius less than 1, would produce the same space with equivalent norm.) The symbols T_q^p, without a measure explicitly given, denote $T_q^p((1 - |z|^2)^{-1}\,dA)$.

There are no nonzero analytic functions in T_q^p on the disk or $\mathbf{R}_+^2$. (Nor are there any Riesz systems in T_q^p on $\mathbf{R}_+^{n+1}$.) Let us recall that the pseudohyperbolic metric on $\mathbf{U}=\mathbf{R}_+^2$ is given by $\rho(z,w)=|z-w|/|1-\bar{w}|$ and the hyperbolic metric h by $h=\log((1+\rho)/(1-\rho))$. To see that T_q^p contains no nonzero analytic functions, observe that the disk $D(t+iy)=\{w: h(w,t+iy)<\delta_0\}$ belongs to $\Gamma(t)$ when δ_0 is sufficiently small. Thus, if $f\in T_q^p$, then

$$|f(t+iy)|^q \le \frac{C}{A(D(t+iy))}\int_{D(t+iy)}|f|^q\,dA \le C\int_{\Gamma(t)}|f|^q\frac{dA}{y^2}, \qquad (5.3)$$

whence we conclude that $f\in H^p$. Now let $y_n=2^{-n}$ and choose δ_0 such that the disks $D(t+iy_n)$ are disjoint. Then inequality (5.3) can be strengthened to

$$\sum_n |f(t+iy_n)|^q \le C\int_{\Gamma(t)}|f|^q\frac{dA}{y^2}<\infty, \quad \text{a.e. } (t),$$

and we see that $f(t+iy_n)\to 0$ for almost all t. Thus $f=0$. However, for some measures μ very close to dA/y, $T_q^p(\mu)$ contains nonzero analytic functions. For instance, the derivatives of functions in H^p belong to $T_2^p(y\,dA)$. This is just the usual square function (or Lusin area function) characterization of H^p. In fact, an easy calculation shows that any function of the form $g(z)(z-\bar{z}_0)^{-M}$ with g bounded and M sufficiently large belongs to

$$AT_q^p(\alpha) \overset{\text{def}}{=} \{f: f \text{ is analytic on } \mathbf{U} \text{ and } f\in T_q^p(y^{\alpha-1}\,dA)\}$$

if $\alpha>0$. Moreover, these functions span $AT_q^p(\alpha)$. The proof of this is a routine calculation in which the lemmas to follow play a role.

Since we are going to solve the analogues of (1.1) and (1.2) for the spaces $AT_q^p(\alpha)$ the first step is, as might be expected, to obtain the analogue of Lemma 1. The second step is to obtain the analogue of Lemma 6, the duality result. This is mostly already done in [13] and [5], but neither article considered the duality of $T_q^p(\mu)$ when $p<1$. We will only need the duality for certain discrete measures.

The analogues of the sets Q_{ij} of the previous sections are the following, also called Q_{ij}:

$$Q_{ij}=\{z=x+iy: 2^{-j}\le y<2^{-j+1},\ i2^{-j}\le x<(i+1)2^{-j}\}$$

for all integers $-\infty<i,j<\infty$. Let z_{ij} denote the center of the square Q_{ij}. Occasionally we will refer to these points as z_k when it is unnecessary to consider the location of individual points in the sequence. We always let x_k and y_k denote the real and imaginary part of z_k; similarly, x_{ij} and y_{ij} are the real and imaginary parts of z_{ij}.

Let τ denote the measure $\sum y_k\delta_k$, where δ_k is the measure with unit mass concentrated at the point z_k. Thus $T_q^p(\tau)$ is a space of sequences $(a_k)_k$ which satisfy

$$\int\Bigl(\sum_{z_k\in\Gamma(t)}|a_k|^q\Bigr)^{p/q}dt\le\infty.$$

The duality we will need is for these spaces, and the analogue of Lemma 1 will have coefficients in these spaces.

PROPOSITION 1. *Define an operator S on sequences by*

$$S((a_k))=\sum a_k \frac{y_k^{M-\alpha/q}}{(z-\bar{z}_k)^M}.$$

If $M>\max\{1,q/p,1/q,1/p\}+\alpha/q$ then S is a bounded operator from $T_q^p(\tau)$ to $AT_q^p(\alpha)$.

The proof will show that the only requirement on $\{z_k\}$ is that it be separated. A result very similar to this was shown in [13]. In that paper, derivatives of the Poisson kernel replace the powers (or derivatives) of the Cauchy kernel that appear here. Also, the value of q was restricted to be 2 in [13]. However, the method to be used here and the one used there are the same. We first state a lemma, the proof of which is straightforward. The corresponding result for the unit ball in $\mathbf{C}^n$ may be found in [17, pp. 17–18]; or see [12, Lemma 4] for the unit disk.

LEMMA 10. *Let $-1<a<M-2$. Then there is a constant depending only on a and M such that*

$$\int_{\mathrm{U}} \frac{y^a}{|z-\bar{w}|^M}\,dA(z)\le \frac{C}{(\operatorname{Im} w)^{M-a-2}},$$

and if $\{z_k\}$ is any separated sequence (i.e., $\inf_{k\neq l} h(z_k,z_l)>0$) then there exists a constant C depending only on a, M, and the separation constant such that

$$\sum_k \frac{y_k^{a+2}}{|z_k-\bar{w}|^M}\le \frac{C}{(\operatorname{Im} w)^{M-a-2}}.$$

We also observe that the function $|z-\bar{b}|$ satisfies a Harnak inequality. That is, $|z-\bar{b}|/|w-\bar{b}|<(1+\rho(z,w))/(1-\rho(z,w))$. Note that this upper bound is independent of b. In particular, if $z=x+iy$ belongs to $\Gamma(t)$ then $|t+iy-\bar{z}_k|< C|z-\bar{z}_k|$, with the constant independent of k. We use this observation in the following lemma.

LEMMA 11. *Let $-1<a<M-2$. Then there is a constant C, depending only on a and M, such that*

$$\int_{\Gamma(t)} \frac{y^{a-1}}{|z-\bar{b}|^{M-1}}\,dA(z)\le \frac{C}{(|t-\operatorname{Re} b|+\operatorname{Im} b)^{M-a-2}}.$$

Proof. Using the Harnak inequality mentioned above, we can replace the $z=x+iy$ in the integrand by $t+iy$ at the expense of multiplying the integral by a constant. The integrand is then independent of x, and integration gives

$$\int_{\Gamma(t)} \frac{y^{a-1}}{|z-\bar{b}|^{M-1}}\,dA(z)\le C\int_0^\infty \frac{y^a}{(|t-\operatorname{Re} b|^2+(y+\operatorname{Im} b)^2)^{(M-1)/2}}\,dy.$$

If we now estimate the denominator using

$$(|t-\operatorname{Re} b|^2+(y+\operatorname{Im} b)^2)^{1/2}$$
$$\geq \begin{cases} c(|t-\operatorname{Re} b|+\operatorname{Im} b) & \text{when } y<|t-\operatorname{Re} b|+\operatorname{Im} b, \\ cy & \text{when } y>|t-\operatorname{Re} b|+\operatorname{Im} b, \end{cases}$$

then easy integrations yield the required estimate. □

Proof of Proposition 1. First assume that $q>1$. Then estimate the absolute value of $S((a_k))(z)$ using Hölders inequality:

$$\left|\sum a_k \frac{y_k^{M-\alpha/q}}{(z-\bar{z}_k)^M}\right|^q \leq \sum |a_k|^q \frac{y_k^{M-\alpha/q}}{|z-\bar{z}_k|^M}\left(\sum \frac{y_k^{M-\alpha/q}}{|z-\bar{z}_k|^M}\right)^{q/q'}. \tag{5.4}$$

Now we estimate the second sum on the right using Lemma 10:

$$\sum \frac{y_k^{M-\alpha/q}}{|z-\bar{z}_k|^M} \leq \frac{C}{y^{\alpha/q}}.$$

Putting this in (5.4) and then integrating with respect to $y^{\alpha-2}\,dA(z)$ on the cone $\Gamma(t)$ gives

$$\int_{\Gamma(t)} |S((a_k))|^q y^{\alpha-2}\,dA \leq C \sum |a_k|^q y_k^{M-\alpha/q} \int_{\Gamma(t)} \frac{y^{\alpha-\alpha/q'-2}\,dA(z)}{|z-\bar{z}_k|^M}.$$

At this point we use Lemma 11 to obtain

$$\int_{\Gamma(t)} |S((a_k))|^q y^{\alpha-2}\,dA \leq C \sum |a_k|^q \frac{y_k^{M-\alpha/q}}{(|t-x_k|+y_k)^{M-\alpha/q}}. \tag{5.5}$$

The final step is to apply Proposition 1 of [13] with $n=1$ and $s=p/q$. The requirement for applying that result is that the exponent (in this case $M-\alpha/q$) divided by n (here $n=1$) be larger than $\max\{1, 1/s\}$ (here $\max\{1, q/p\}$). That is, $M>\max\{1, q/p\}+\alpha/q$. In that proposition we let $\nu=\sum |a_k|^q \delta_k$ and its conclusion is the inequality below:

$$\int_{-\infty}^{\infty}\left(\sum |a_k|^q \frac{y_k^{M-\alpha/q}}{(|t-x_k|+y_k)^{M-\alpha/q}}\right)^{p/q} dt$$
$$=\int_{-\infty}^{\infty}\left(\int_U \frac{y^{M-\alpha/q}}{(|t-x|+y)^{M+\alpha/q}}\,d\nu(x,y)\right)^{p/q} dt$$
$$\leq C\int_{-\infty}^{\infty} \nu(\Gamma(t))^{p/q}\,dt$$
$$=C\int_{-\infty}^{\infty}\left(\sum_{z_k\in\Gamma(t)} |a_k|^q\right)^{p/q} dt=\|(a_k)\|^p_{T_q^p(\tau)}.$$

Combining this with inequality (5.5) finishes the proof for $q>1$. In case $q\leq 1$, replace the first two steps (use of Hölder's inequality followed by use of Lemma 10) with the single estimate

$$\left(\sum |a_k| \frac{y_k^{M-\alpha/q}}{|z-\bar{z}_k|^M}\right)^q \leq \sum |a_k|^q \frac{y_k^{Mq-\alpha}}{|z-\bar{z}_k|^{Mq}}$$

and continue as before. This time the exponent of $y_k/(|t-x_k|+y_k)$ on the right-hand side of (5.5) is $Mq-\alpha$, and the requirement to apply Proposition 1 of [L2] is that $Mq-\alpha>\max\{1, q/p\}$; that is, $M>\max\{1/p, 1/q\}+\alpha/q$. □

The duality result we need is a combination of the following lemma and proposition. The first is to be found in [13]. The second will be proved here.

LEMMA 12. *Let* $1<p<\infty$ *and* $1<q<\infty$. *Then*

$$\sup\{|\sum a_k b_k y_k| : \|(b_k)\|_{T_q^p(\tau)}=1\} \tag{5.6}$$

defines a norm on sequences (a_k) *equivalent to the* $T_{q'}^{p'}(\tau)$ *norm. If p is as above but* $0<q\leq 1$, *then the supremum in* (5.6) *defines a norm equivalent to the following:*

$$\left(\int_{-\infty}^{\infty}\left(\sup_{\Gamma(t)}|a_k|\right)^{p'} dt\right)^{1/p'}$$

(This last expression might perhaps be taken as the definition of $T_\infty^{p'}(\tau)$. Unfortunately that definition, if extended to all measures, would conflict with the definition of T_∞^p given in [2]. In [13], the space defined by this expression was called $\tilde{T}_\infty^{p'}\{z_n\}$. However, in our present notation we will call it $\tilde{T}_\infty^{p'}(\tau)$.)

What is needed now is the extension of Lemma 12 to the case $0<p\leq 1$.

PROPOSITION 2. *If* $p\leq 1$ *then the norm defined by* (5.6) *is equivalent to the norm defined by* $\sup|a_k|y_k^{1-1/p}$.

Proof. One inequality is straightforward: Apply (5.6) with $b_k=0$ when $k\neq k_0$ and $b_{k_0}=y_{k_0}^{-1/p}$. Then $\|(b_k)\|_{T_q^p(\tau)}$ is bounded and bounded away from zero independent of k_0. Thus the supremum in (5.6) is greater than a constant times $|a_{k_0}b_{k_0}|y_{k_0}$, which is equal to $|a_{k_0}|y_{k_0}^{1-1/p}$. The other inequality is considerably more involved and begins with an obvious geometric lemma. If z is a point in $\mathbf{U}$, let I_z denote the set of points t such that $z\in\Gamma(t)$. For an interval I on $\partial\mathbf{U}$, let $\hat{I}$ denote the set of z in $\mathbf{U}$ such that $I_z\subset I$. (In other terms, $\hat{I}$ is the interior of the isosceles triangle whose base is I and whose base angles are 45 degrees—the so-called "tent" over I.) □

LEMMA 13. *If I is an interval on* $\partial\mathbf{U}$ *and* $t\in I$, *then there is an endpoint a of I such that* $\Gamma(t)\setminus\hat{I}\subset\Gamma_3(a)$.

Proof. The figure in $\mathbf{R}_+^2$ is easy to draw and the lemma verified by simple geometry. The endpoint a of I is simply that endpoint nearest t. □

Returning to the proof of Proposition 1, let (a_k) satisfy $|a_k|\leq y_k^{1/p-1}$ and let (b_k) be an arbitrary sequence in $T_q^p(\tau)$. We need to show that

$$\sum a_k b_k y_k \leq C\|(b_k)\|_{T_q^p(\tau)}. \tag{5.7}$$

(Without loss of generality, a_k and b_k are positive.) For each integer m let

$$E_m=\left\{t:\left(\sum_{z_k\in\Gamma_3(t)}|b_k|^q\right)^{1/q}>2^m\right\},$$

and let $I_{1m}, I_{2m}, \ldots$ denote the decomposition of E_m into disjoint open intervals. Let G_m denote $\bigcup_l \hat{I}_{lm}$. Because the $T_q^p(\tau)$ norm is equivalent to one with $\Gamma(t)$ replaced with $\Gamma_3(t)$, the norm of (b_k) is equivalent to the expression

$$\left(\sum_m 2^{mp}|E_m|\right)^{1/p},$$

where $|E|$ denotes the linear measure of E. Now we begin to estimate the sum on the left side of inequality (5.7). Because $\bigcup_m G_m\backslash G_{m+1}=\mathbf{U}$, we have

$$\begin{aligned}\sum a_k b_k y_k&=\sum_m \sum_{z_k\in G_m\backslash G_{m+1}} a_k b_k y_k\\&=\frac{1}{2}\sum_m\sum_l \sum_{z_k\in \hat{I}_{lm}\backslash G_{m+1}} a_k b_k\int_{-\infty}^{\infty}\chi_{I_{z_k}}\,dt\\&=\frac{1}{2}\sum_m\sum_l\int_{I_{lm}}\sum_{z_k\in\Gamma(t)\cap\hat{I}_{lm}\backslash G_{m+1}} a_k b_k\,dt.\end{aligned}\tag{5.8}$$

The second equality comes about because $y_k=|I_{z_k}|/2$, and the last equality is because $t\in I_{z_k}$ if and only if $z_k\in\Gamma(t)$ and because $\Gamma(t)\cap\hat{I}_{lm}=\emptyset$ unless $t\in I_{lm}$. Suppose that $q>1$. Then the last sum in the above chain can be estimated by

$$\sum_{z_k\in\Gamma(t)\cap\hat{I}_{lm}\backslash G_{m+1}} a_k b_k\le\left(\sum_{z_k\in\Gamma(t)\backslash G_{m+1}} b_k^q\right)^{1/q}\left(\sum_{z_k\in\Gamma(t)\cap\hat{I}_{lm}} a_k^{q'}\right)^{1/q'}.\tag{5.9}$$

Now the set $\Gamma(t)\backslash G_{m+1}$ is contained $\Gamma_3(t)$, and if t is not in E_{m+1} then the first factor on the right is less than 2^{m+1}. On the other hand, if t does belong to E_{m+1} then it belongs to some $I_{i,m+1}$. In that case $\Gamma(t)\backslash G_{m+1}$ is contained in $\Gamma(t)\backslash\hat{I}_{i,m+1}$ which, by Lemma 13, is contained in $\Gamma_3(a)$ for an endpoint a of $I_{i,m+1}$. And again the factor involving b_k is less than 2^{m+1} because $a\notin E_{m+1}$. Let us now investigate the second factor in (5.9): By our choice of notation, each z_k is a z_{ij} and $2^{-j}<y_{ij}<2^{-j+1}$. Now in $\Gamma(t)$ there are at most three z_{ij} with any given value of y_{ij}, and in $\hat{I}_{lm}$ the largest y_{ij} can be is $|I_{lm}|/2$. Thus

$$\begin{aligned}\left(\sum_{z_k\in\Gamma(t)\cap\hat{I}_{lm}} a_k^{q'}\right)^{1/q'}&\le\left(\sum_{z_k\in\Gamma(t)\cap\hat{I}_{lm}} y_k^{(1/p-1)q'}\right)^{1/q'}\\&\le C\left(\sum_{2^{-j}<|I_{lm}|}2^{-j(1/p-1)q'}\right)^{1/q'}\\&\le C|I_{lm}|^{1/p-1}.\end{aligned}$$

Putting these two estimates into (5.9) gives

$$\sum_{z_k\in\Gamma(t)\cap\hat{I}_{lm}\backslash G_{m+1}} a_k b_k\le C2^m|I_{lm}|^{1/p-1}.$$

Putting this in (5.8) gives

$$\begin{aligned}\sum a_k b_k y_k &\leq C \sum_m \sum_l \int_{I_{lm}} 2^m |I_{lm}|^{1/p-1}\,dt \\ &= C\sum_m \sum_l 2^m |I_{lm}|^{1/p} \\ &\leq C\left(\sum_m \sum_l 2^{mp}|I_{lm}|\right)^{1/p} \quad \text{(because } p\leq 1) \\ &= C\left(\sum_m 2^{mp}|E_m|\right)^{1/p} \\ &\leq C\|(b_k)\|_{T_q^p(\tau)}.\end{aligned}$$

Finally, the case where $q \leq 1$ is obtained similarly, replacing (5.9) with

$$\sum_{z_k\in\Gamma(t)\cap \hat{I}_{lm}\setminus G_{m+1}} a_k b_k \leq \left(\sum_{z_k\in\Gamma(t)\setminus G_{m+1}} b_k^q\right)^{1/q} \sup_{z_k\in \hat{I}_{lm}} a_k. \qquad \square$$

6. The Embedding Theorem for Tent Spaces of Analytic Functions

We need only one more observation before we solve our analogue of (1.2) for $AT_q^p(\alpha)$. Let $|f|_{ij}$ be defined as in the previous section:

$$|f|_{ij} = \sup\{|f(z)| : z\in Q_{ij}\}.$$

Then the $T_q^p(\alpha)$ norm of an analytic function f is equivalent to the $T_q^p(\tau)$ norm of the sequence $(|f|_{ij} y_{ij}^{\alpha/q})_{ij}$. To see that the sequence norm dominates the function's norm, simply observe that the integral $\int_{\Gamma(t)} |f|^q y^{\alpha-2}\,dA$ is less than $\sum |Q_{ij}| \max_{z\in Q_{ij}}[|f(z)|y^{\alpha-2}]$, where the sum is over all i, j with Q_{ij} meeting $\Gamma(t)$. This is easily bounded above by the corresponding sum over all i, j with z_{ij} in $\Gamma_2(t)$, and the rest is routine. The other inequality is equivalent to the easy half of Theorem 2 of [13]. (Although it was done there only for $q = 2$, the same estimates work for all q.) Also, the $T_q^p(\tau)$ norm of the sequence $(|f^{(n)}|_{ij} y_{ij}^{n+\alpha/q})_{ij}$ is dominated by that of $(|f|_{ij} y_{ij}^{\alpha/q})_{ij}$. The argument is the same as for the corresponding statement in Section 4, inequality (4.2), based on Cauchy's integral formula.

We are now ready for the main theorem of this section. Again, let δ_0 be any convenient positive number and let $D(z) = \{w : h(z, w) < \delta_0\}$.

THEOREM 3. *Let $0 < p, q, s < \infty$, let $\alpha > 0$, and let μ be a positive measure on* $\mathbf{U}$. *Then the following are equivalent.*

(a) *There is a constant C such that*

$$\left(\int_{\mathbf{U}} |f^{(n)}|^s\,d\mu\right)^{1/s} \leq C\|f\|_{T_q^p(\alpha)}$$

for all $f \in AT_q^p(\alpha)$.

(b) *The sequence*

$$\mu_{ij} \stackrel{\text{def}}{=} \mu(Q_{ij})\, y_{ij}^{-s\alpha/q - sn - 1},$$

or equivalently $\mu(Q_{ij})2^{j(s\alpha/q+sn+1)}$, *satisfies one of the following:*

(i) *If* $s<p,q$ *then* $(\mu_{ij}) \in T_{(q/s)'}^{(p/s)'}(\tau)$. *Put another way,* $(\mu_{ij}^{1/s}) \in T_v^u(\tau)$ *where* $1/u = 1/s - 1/p$ *and* $1/v = 1/s - 1/q$. *That is,*

$$\int \Bigg(\sum_{z_{ij}\in\Gamma(t)} \mu(Q_{ij})^{u/s} 2^{j(\alpha/q+n+1/s)u} \Bigg)^{v/u} dt < \infty.$$

(ii) *If* $q \le s < p$ *then* (μ_{ij}) *belongs to* $\tilde{T}_\infty^{(p/s)'}(\tau)$. *That is, if* $1/u = 1/s - 1/p$ *then*

$$\int \Bigg(\sup_{z_{ij}\in\Gamma(t)} \mu(Q_{ij}) 2^{j(s\alpha/q+sn+1)} \Bigg)^{u/s} dt < \infty.$$

(iii) *If* $p \le s$ *(q arbitrary) then* $(\mu_{ij} y_{ij}^{1-s/p})$ *is a bounded sequence. That is,*

$$\mu(Q_{ij}) \le C 2^{-js(\alpha/q+n+1/p)}$$

for all i, j.

(c) *For each of these same three cases, the function*

$$k(z) = \mu(D(z))\, y^{-s\alpha/q - sn - 1}$$

satisfies

(i) $k(z)$ *belongs to* $T_{(q/s)'}^{(p/s)'}$,

(ii) $k(z)$ *belongs to* $\tilde{T}_\infty^{(p/s)'}$,

(iii) $k(z)y^{1-1/p}$ *is bounded.*

Proof. We first prove the implication (b) ⇒ (a). If $f \in AT_q^p(\alpha)$ write $|f^{(n)}| \le \sum |f^{(n)}|_{ij}\chi_{Q_{ij}}$ so that

$$\begin{aligned}\int |f^{(n)}|^s\, d\mu &\le \sum |f^{(n)}|_{ij}^s \mu(Q_{ij}) \\ &= \sum |f^{(n)}|_{ij}^s\, y_{ij}^{s\alpha/q+n} \mu(Q_{ij})\, y_{ij}^{-s\alpha/q-n-1} y_{ij} \\ &= \sum a_{ij}\mu_{ij} y_{ij},\end{aligned} \tag{6.1}$$

where $(a_{ij}) = (|f^{(n)}|_{ij}^s y_{ij}^{s\alpha/q+n}) \in T_{q/s}^{p/s}(\tau)$. The conditions on μ in (b)(i)–(b)(iii) are, according to Lemma 12 and Proposition 2, sufficient to obtain

$$\begin{aligned}\int |f^{(n)}|^s\, d\mu &\le \sum a_{ij}\mu_{ij} y_{ij} \\ &\le C\|(a_{ij})\|_{T_{q/s}^{p/s}(\tau)} \le C\|f\|_{T_q^p(\alpha)}^s.\end{aligned}$$

To obtain (a) ⇒ (b), apply the inequality in (a) to the function

$$f(z) = \sum a_{ij}\bar{z}_{ij}^{-n} \frac{y_{ij}^{M-\alpha/q}}{(z-\bar{z}_{ij})^M},$$

using Proposition 1 to get

$$\int \left| \sum a_{ij} \frac{y_{ij}^{M-\alpha/q}}{(z-\bar{z}_{ij})^{M+n}} \right|^s d\mu(z) \leq C\|(a_{ij})\|^s_{T^p_q(\tau)}.$$

Applying the Khinchine inequality technique gives us

$$\int \left(\sum |a_{ij}|^2 \frac{y_{ij}^{2M-2\alpha/q}}{|z-\bar{z}_{ij}|^{2M+2n}} \right)^{s/2} d\mu(z) \leq C\|(a_{ij})\|^s_{T^p_q(\tau)}.$$

Now, using $y_{ij} > c2^{-j}$ and $|z-\bar{z}_{ij}|^{-1} > c2^j \chi_{Q_{ij}}(z)$, we obtain

$$\sum |a_{ij}|^s 2^{js(\alpha/q+n)} \mu(Q_{ij}) \leq C\|(a_{ij})\|^s_{T^p_q(\tau)}.$$

If we write $b_{ij} = |a_{ij}|^s$ this becomes

$$\sum b_{ij} \mu_{ij} y_{ij} \leq C\|(b_{ij})\|_{T^{p/s}_{q/s}(\tau)}$$

for all nonnegative sequences (b_{ij}) in $T^{p/s}_{q/s}(\tau)$. From Lemma 12 and Proposition 2 we conclude that conditions (b)(i)–(b)(iii) must hold. The equivalence of (b) and (c) is left to the reader. □

We note that since the H^p norm of an analytic function is equivalent to the $T^p_2(2)$ norm of its derivative, Theorem 4 includes the main result of [13] as well as some of the results of [10] and [18].

7. Further Remarks

It is possible to use these same techniques to investigate a nearly unending array of embedding theorems. One might ask, for example, when $AT^p_q(\alpha) \subset T^r_s(\mu)$, that is when there is a constant C such that

$$\left(\int_{-\infty}^{\infty} \left(\int_{\Gamma(t)} |f|^s \, d\mu \right)^{r/s} dt \right)^{1/r} \leq C\|f\|_{T^p_q(\alpha)}.$$

The conclusion would be: whenever

$$\left(\int \left(\sum_{z_{ij} \in \Gamma(t)} a_{ij} \mu(Q_{ij}) 2^{j\alpha s/q} \right)^{r/s} dt \right)^{s/r} \leq C\|(a_{ij})\|_{T^{p/s}_{q/s}(\tau)},$$

that is, whenever $\mu(Q_{ij})2^{j\alpha s/q}$ multiplies $T^{p/s}_{q/s}(\tau)$ into $T^{r/s}_1(\tau)$. All that remains is a characterization of the space of multipliers. Another question might be: When is $AT^p_q(\alpha) \subset A^{s,t}$ (in the disk $\mathbf{D}$)? That is, when is

$$\left(\int_0^1 \left(\int_0^{2\pi} |f(re^{i\theta})|^s \, d\theta \right)^{t/s} dr \right)^{1/t} \leq C\|f\|_{AT^p_q(\alpha)}?$$

And the answer is: whenever $2^{j(1/s+\alpha/q+1/t)}$ multiplies $T^p_q(\tau)$ into $l^{s,t}$. (Both sequence spaces are indexed on the collection of pairs (i,j) such that Q_{ij} exists.) It is not claimed that the solutions of these sequence space problems are easy, or even any easier than the original embedding problems. We merely wish to point out that the methods we have used here always seem to produce the same necessary condition (using Khinchine's inequality) as the sufficient condition (using usually straightforward estimates). And the condition is purely a problem of multipliers in the associated sequence spaces.

We would like to close with a mention of the original embedding theorem due to Carleson: the embedding of H^p into $L^p(\mu)$. The methods used here work also in that case, but they are perhaps not as reasonable to apply as the other methods available. The Khinchine inequality part of the argument is completely unnecessary, and so the corresponding decomposition theorem (the analogue of Lemmas 1 and 8 and Proposition 1) becomes unnecessary. The other part of the argument becomes little more than a disguised and slightly complicated version of one of the standard proofs based on the non-tangential maximal function characterization of H^p. However, the embedding of H^p into $L^q(\mu)$ when $q < p$ does benefit from these methods (see [13, Sec. 7D]).

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Department of Mathematics
University of Arkansas
Fayetteville, AK 72701