Conditioned Brownian Motion and Hyperbolic Geodesics in Simply Connected Domains

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0. Introduction

Let D be a simply connected planar domain. Let B_t be Brownian motion in D with lifetime τ_D. If h is a positive harmonic function in D, then the Brownian motion conditioned by h is determined by the transition functions

$$P_t^h(w,z) = \frac{1}{h(w)} P_t^D(w,z)\, h(z),$$

where $P_t^D(w,z)$ are the transition functions for the unconditioned Brownian motion in D. We let P_w^h be the measure on path space induced by the P_t^h and write E_w^h for the corresponding expectation. If $h=1$, the case of killed Brownian motion in D, we simply write P_w and E_w. The following result is due to Cranston and McConnell [9].

THEOREM A. *Let D be any planar domain and denote by $H^+(D)$ the collection of all positive harmonic functions in D. Then*

$$\sup_{\substack{w \in D \\ h \in H^+(D)}} E_w^h(\tau_D) \le C\,\text{area}(D), \tag{0.1}$$

where C is a universal constant.

This result has been extended in several directions. We refer the reader to Bañuelos [6], where a survey of the recent literature on this subject is given. The purpose of this paper is to prove the following theorem.

THEOREM 1. *Let D be a simply connected planar domain. If $z = x+iy \in D$ and if Γ is a geodesic for the hyperbolic metric in D, we let $d_D(z,\Gamma)$ be the hyperbolic distance from z to Γ. There are universal constants c_1 and c_2 such that*

$$c_1 \sup_\Gamma \iint_D e^{-2d_D(z,\Gamma)}\,dx\,dy \le \sup_{\substack{w \in D \\ h \in H^+(D)}} E_w^h(\tau_D) \le c_2 \sup_\Gamma \iint_D e^{-2d_D(z,\Gamma)}\,dx\,dy,$$

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where the suprema outside the integrals are taken over all hyperbolic geodesics in D.

Theorem 1 provides new insight on the behavior of conditioned Brownian motion in simply connected planar domains. The theorem is proved in Section 1. In Section 2, we give some applications. We give new and shorter proofs of some recent results and show that conditioned Brownian motion paths tend to follow hyperbolic geodesics (see Corollaries 1, 2, and 3). We end the paper with some remarks on some estimates of Ancona [1; 2; 3] and their potential usefulness in the study of conditioned Brownian motion. This connection, it seems, has not been noticed before.

We will make frequent use of the conformal invariance of hyperbolic distance and of hyperbolic geodesics as well as the explicit forms of some of these quantities for the infinite strip and the unit disc. The reader unfamiliar with the basics of hyperbolic geometry can find everything that is needed to read this paper in Hayman [14, Chap. 9].

1. Proof of Theorem 1

Let $\Delta = D(0,1)$ be the unit disc and let φ be any conformal mapping of Δ onto D. Define the Littlewood–Paley square function $g_*(\varphi)$ by

$$g_*^2(\varphi) = \frac{1}{\pi}\iint_\Delta \log\left(\frac{1}{|z|}\right)\frac{1-|z|^2}{|1-z|^2}|\varphi'(z)|^2\,dx\,dy.$$

Our Theorem 1 was motivated by the following simple identity, proved in Bañuelos [6]:

$$\sup_{\substack{w\in D\\ h\in H^+(D)}} E_w^h(\tau_D) = \sup_{\varphi} g_*^2(\varphi), \tag{1.1}$$

where the supremum on the right is taken over all conformal mappings φ of Δ onto D. We briefly explain (1.1) for the convenience of the reader. It follows from the form of the transition functions that, with $z = x+iy$,

$$P_w^h\{\tau_D > t\} = \frac{1}{h(w)}\iint_D P_t^D(w,z)\,h(z)\,dx\,dy,$$

and since $P_t^D(w,z)$ is the Dirichlet heat kernel for one half the Laplacian in D we have, after integrating in time, that

$$E_w^h(\tau_D) = \frac{1}{h(w)}\iint_D G_D(w,z)\,h(z)\,dx\,dy,$$

where $G_D(w,z)$ is the Green function for D. Let φ be a conformal mapping of Δ onto D such that $\varphi(0) = w$. Then, by the conformal invariance of $G_D(w,z)$, we obtain that

$$E_w^h(\tau_D) = \frac{1}{\pi h(w)}\iint_\Delta \log\left(\frac{1}{|z|}\right)h(\varphi(z))|\varphi'(z)|^2\,dx\,dy. \tag{1.2}$$

Since $h(\varphi)$ is positive and harmonic in Δ, we have

$$h(\varphi(z))=\int_T \frac{1-|z|^2}{|z-e^{i\theta}|^2}\,d\mu(\theta), \tag{1.3}$$

where $T=\partial\Delta$ and μ is a positive measure on T with $\mu(T)=h(\varphi(0))=h(w)$. Substituting (1.3) in (1.2) and applying Fubini's theorem, we find that

$$E_w^h(\tau_D)=\frac{1}{h(w)}\int_T g_*^2(\varphi_\theta)\,d\mu(\theta), \tag{1.4}$$

where $\varphi_\theta(z)=\varphi(e^{-i\theta}z)$. Since $\mu(T)=h(w)$, we obtain from (1.4) that

$$\sup_{\substack{w\in D\\ h\in H^+(D)}} E_w^h(\tau_D)\le \sup_\varphi g_*^2(\varphi).$$

The other direction of (1.1) follows from the identity

$$g_*^2(\varphi)=E_w^h(\tau_D),$$

where $w=\varphi(0)$ and $h(z)=K_D(\varphi(1),z)$, the Martin kernel for D at $\varphi(1)$. Here we have used the conformal invariance of both the Green function and the Martin kernel. Thus, the classical Littlewood–Paley square function $g_*^2(\varphi)$ is also, up to a constant, the expected lifetime of Brownian motion starting at $\varphi(0)$ and conditioned to exit D at the Martin boundary point $\varphi(1)$.

Next we show that there are universal constants c_3 and c_4 such that

$$c_3\sup_\varphi g_*^2(\varphi)\le \sup_\varphi \int_\Delta K(z,1)K(z,-1)|\varphi'(z)|^2\,dx\,dy\le c_4\sup_\varphi g_*^2(\varphi), \tag{1.5}$$

where $K(z,e^{i\theta})=(1-|z|^2)/|z-e^{i\theta}|^2$, the Poisson kernel for Δ. To obtain the second inequality in (1.5) we suppose that φ is a conformal mapping of Δ onto D and, for $r<1$, we let M_r be the Möbius transformation of the disc

$$M_r(z)=\frac{z+r}{1+rz}.$$

With $\psi_r=\varphi\circ M_r$, we obtain, for each $r<1$, that

$$\begin{aligned}\sup_\varphi g_*^2(\varphi)&\ge g_*^2(\psi_r)\\ &=\frac{1}{\pi}\iint_\Delta \log\left(\frac{1}{|z|}\right)\frac{1-|z|^2}{|z-1|^2}|\psi_r'(z)|^2\,dx\,dy\\ &=\frac{1}{\pi K(r,1)}\iint_\Delta \log\left|\frac{1-rz}{z-r}\right|K(z,1)|\varphi'(z)|^2\,dx\,dy.\end{aligned}$$

Since $(1/\pi)\log|(1-rz)/(z-r)|$ is the Green function for the disc with pole at r, as $r\to -1$ this quantity divided by $K(r,1)$ goes to $2K(z,-1)$, and the right hand side of (1.5) follows with $c_4=1/2$.

For the left-hand side we follow the argument in Garnett [12, p. 237]. Since $\log(1/|z|)\le C(1-|z|^2)$ for $|z|\ge 1/4$, we obtain that

$$\begin{aligned}\frac{1}{\pi}\iint_{|z|>1/4}&\log\left(\frac{1}{|z|}\right)\frac{1-|z|^2}{|z-1|^2}|\varphi'(z)|^2\,dx\,dy\\ &\le C\iint_{|z|>1/4} K(z,-1)K(z,1)|\varphi'(z)|^2\,dx\,dy\\ &\le C\sup_{\varphi}\iint_{\Delta} K(z,-1)K(z,1)|\varphi'(z)|^2\,dx\,dy.\end{aligned}$$

Furthermore, $|\nabla\varphi(z)|^2$ is harmonic in $|z|\le 1/4$, which yields, with $w=u+iv$, that

$$\begin{aligned}|\nabla\varphi(z)|^2&\le C\iint_{|w-z|<1/4}|\nabla\varphi(w)|^2\,du\,dv\\ &\le C\iint_{|w|<1/2}|\nabla\varphi(w)|^2\left(\frac{1-|w|^2}{|w-1|^2}\right)\left(\frac{1-|w|^2}{|w+1|^2}\right)du\,dv\\ &\le C\sup_{\varphi}\iint_{\Delta}K(z,1)K(z,-1)|\varphi'(z)|^2\,dx\,dy.\end{aligned}$$

Thus,

$$\begin{aligned}\frac{1}{\pi}\iint_{|z|<1/4}&\log\left(\frac{1}{|z|}\right)\frac{1-|z|^2}{|z-1|^2}|\varphi'(z)|^2\,dx\,dy\\ &\le C\left(\sup_{\varphi}\iint_{\Delta}K(z,1)K(z,-1)|\varphi'(z)|^2\,dx\,dy\right)\\ &\quad\times\frac{1}{\pi}\iint_{|z|<1/4}\log\left(\frac{1}{|z|}\right)\frac{1-|z|^2}{|z-1|^2}\,dx\,dy.\end{aligned}$$

The second term on the right-hand side is dominated by

$$\frac{1}{\pi}\iint_{\Delta}\log\left(\frac{1}{|z|}\right)\frac{1-|z|^2}{|z-1|^2}\,dx\,dy=E_0^1(\tau_\Delta)=E_0(\tau_\Delta)=C,$$

and (1.5) follows.

REMARK 1. A sharper result has been independently obtained by P. Griffin, T. McConnell, and G. Verchota (private communication). Namely, they have proved that

$$\sup_{\substack{w\in D\\ h\in H^+(D)}} E_w^h(\tau_D)=2\sup_{\varphi}\iint_{\Delta}K(z,1)K(z,-1)|\varphi'(z)|^2\,dx\,dy.$$

Theorem 1 will now follow from (1.5) and the following lemma.

LEMMA 1. *Let $\Gamma=(-1,1)$ be the hyperbolic geodesic connecting -1 to 1 in Δ. Then for all $z\in\Delta$,*

$$\tfrac{1}{4}K(z,1)K(z,-1)\le e^{-2d_\Delta(z,\Gamma)}\le K(z,1)K(z,-1). \tag{1.6}$$

Proof. We let

$$f(z)=\log\left(\frac{1+z}{1-z}\right),$$

so that $f(z)$ is a conformal mapping of Δ onto the strip $S=\{w: -\pi/2 < \operatorname{Im} w < \pi/2\}$. By the conformal invariance of the hyperbolic metric and the fact that Γ is mapped to the geodesic $\mathbf{R}=(-\infty,\infty)$, we see that

$$\begin{aligned} d_\Delta(z,\Gamma) &= d_S(f(z),\mathbf{R}) = d_S(f(z), \operatorname{Re} f(z)) \\ &= d_S(i \operatorname{Im} f(z), 0). \end{aligned} \tag{1.7}$$

We compute $d_S(iy,0)$ for $-\pi/2<y<\pi/2$. Since everything is explicit (see e.g. Hayman [14, Chap. 9]), we have

$$\begin{aligned} d_S(iy,0) &= d_\Delta\left(\frac{1-e^{iy}}{1+e^{iy}}, 0\right) = \log\left(\left(1+\left|\frac{1+e^{iy}}{1+e^{iy}}\right|\right)\Big/\left(1-\left|\frac{1-e^{iy}}{1+e^{iy}}\right|\right)\right) \\ &= \log\left(\frac{|1+e^{iy}|+|1-e^{iy}|}{|1+e^{iy}|-|1-e^{iy}|}\right) \\ &= \log\left(\frac{\sqrt{1+\cos y}+\sqrt{1-\cos y}}{\sqrt{1+\cos y}-\sqrt{1-\cos y}}\right) \\ &= \log\left(\frac{1+\cos y+1-\cos y+2\sqrt{1-\cos^2 y}}{1+\cos y-(1-\cos y)}\right) \\ &= \log\left(\frac{1+\sqrt{1-\cos^2 y}}{\cos y}\right) = \log\left(\frac{1+|\sin y|}{\cos y}\right). \end{aligned}$$

From this it follows that

$$e^{-2d_S(iy,0)} = \frac{\cos^2 y}{(1+|\sin y|)^2}, \tag{1.8}$$

which gives

$$\tfrac{1}{4}\cos^2 y \le e^{-2d_S(iy,0)} \le \cos^2 y. \tag{1.9}$$

In our case,

$$y = \operatorname{Im} f(z) = \frac{1}{2i}\log\left(\frac{1+z}{1-z}\cdot\frac{1-\bar z}{1+\bar z}\right).$$

Substituting this for y, we obtain

$$\begin{aligned} 4\cos^2 y &= 2+e^{2iy}+e^{-2iy} \\ &= 2+\frac{(1+z)(1-\bar z)}{(1-z)(1+\bar z)}+\frac{(1-z)(1+\bar z)}{(1+z)(1-\bar z)} \\ &= \frac{2(1-z^2)(1-\bar z^2)+(1+z)^2(1-\bar z)^2+(1-z)^2(1+\bar z)^2}{|1-z|^2|1+z|^2} \\ &= \frac{2[1+|z|^4-z^2-\bar z^2]+[(1-|z|^2)+(z-\bar z)]^2+[(1-|z|^2)-(z-\bar z)]^2}{|1-z|^2|1+z|^2} \\ &= \frac{2[1+|z|^4-z^2-\bar z^2+(1-|z|^2)^2+(z-\bar z)^2]}{|1-z|^2|1+z|^2} \\ &= 4\frac{1+|z|^4-2|z|^2}{|1-z|^2|1+z|^2} = 4K(z,1)K(z,-1). \end{aligned}$$

That is, we have shown that

$$K(z,1)K(z,-1)=\cos^2(\operatorname{Im} f(z)), \tag{1.10}$$

and this together with (1.7) and (1.9) proves the lemma. □

We now prove the theorem. By (1.1), (1.5), and Lemma 1,

$$\begin{aligned}\sup_{\substack{w\in D\\ h\in H^+(D)}} E_w^h(\tau_D) &\approx \sup_{\varphi}\iint_\Delta e^{-2d_\Delta(z,(1,1))}|\varphi'(z)|^2\,dx\,dy\\ &=\sup_{\varphi}\iint_D e^{-2d_D(z,\varphi(-1,1))}\,dx\,dy\\ &=\sup_{\Gamma}\iint_D e^{-2d_D(z,\Gamma)}\,dx\,dy.\end{aligned}$$

Here again we have used the conformal invariance of hyperbolic geodesics. □

Notice that our proof shows that $c_1=2$.

REMARK 2. It follows from (1.10) that

$$\sup_{\varphi}\iint_D K(z,1)K(z,-1)|\varphi'(z)|^2\,dx\,dy=\sup_{\varphi}\iint_D \cos^2(\operatorname{Im}\varphi(z))\,dx\,dy, \tag{1.11}$$

where the sup is taken over all conformal mappings φ mapping D onto the strip S. Formula (1.11) was independently obtained by P. Griffin, T. McConnell, and G. Verchota (personal communication). In fact, since they have equality in (1.5), with $c_3=c_4=1/2$, this shows that we may take $c_2=8$ in Theorem 1.

2. Some Applications of Theorem 1

We start with the following theorem due to Xu [16], which shows that for some domains there is a converse to the result of Cranston and McConnell.

THEOREM 2 (Xu). *Suppose that θ_1 and θ_2 are Lipschitz functions on* **R** *with $\theta_1(x)\le\theta_2(x)$, and that*

$$D=\{x+iy:\theta_1(x)<y<\theta_2(x)\}$$

is a simply connected domain. Then

$$\sup_{\substack{w\in D\\ h\in H^+(D)}} E_w^h(\tau_D)\ge c\operatorname{area}(D),$$

where c is constant depending only on the Lipschitz character of θ_1 and θ_2.

Proof. We produce a geodesic Γ in D for which

$$\iint_D e^{-2d_D(z,\Gamma)}\,dx\,dy\ge c\operatorname{area}(D). \tag{2.1}$$

Theorem 2 will then follow from Theorem 1. Let ψ be the conformal mapping of D onto the strip $S=\{w: -\pi/2<\operatorname{Im} w<\pi/2\}$ which takes the graphs of $\theta_1(x)$ and $\theta_2(x)$ onto the lines $\operatorname{Im} w=-\pi/2$ and $\operatorname{Im} w=\pi/2$, respectively. Let Γ be the geodesic in D which is mapped by ψ onto $\mathbf{R}$. Let $D_1 \subset D$ be defined by

$$D_1=\{x+iy: \tfrac{3}{4}\theta_1(x)+\tfrac{1}{4}\theta_2(x)<y<\tfrac{1}{4}\theta_1(x)+\tfrac{3}{4}\theta_2(x)\}.$$

Let $\omega_R(\cdot, A)$ denote the harmonic measure of a subset A of the boundary of a domain R with respect to R. Suppose that $z_0=x_0+iy_0 \in D_1$ and denote by E that part of the boundary of D on the graph of $\theta_1(x)$. Let m be the larger of the Lipschitz characters of θ_1 and θ_2. Then

$$\omega_D(z_0, E) \geq \omega_{D^*}(z_0, E^*),$$

where

$$\Omega^*=\{x+iy: \theta_1(x_0)+m|x-x_0|<y<\theta_2(x_0)+m|x-x_0|\}$$

and

$$E^*=\{x+iy: y=\theta_2(x_0)+m|x-x_0|\}.$$

Furthermore, from the conformal invariance of harmonic measure we obtain, for $z_0 \in D_1$, that

$$\omega_{D^*}(z_0, E^*)=\omega_{\Omega_0}(it, E_0)$$

for some t with $\frac{1}{4}<t<\frac{3}{4}$ where $\Omega_0=\{x+iy: m|x|<y<1+m|x|\}$ and $E_0=\{x+iy: y=1+m|x|\}$. Consequently, for $z_0 \in D_1$,

$$\omega_D(z_0, E) \geq \omega_{\Omega_0}(it, E_0) \geq \omega_{\Omega_0}(\tfrac{1}{4}, E_0)=c_0,$$

where c_0 depends only on m. Then, by conformal invariance again,

$$\begin{aligned} c_0 \leq \omega_D(z_0, E) &= \omega_S(\psi(z_0), \{\operatorname{Im} w=\pi/2\}) \\ &= \frac{1}{\pi} \operatorname{Im} \psi(z_0)+\frac{1}{2} \end{aligned}$$

and we have $\operatorname{Im} \psi(z_0) \geq -\pi/2+c_0\pi$. Similarly we obtain

$$\operatorname{Im} \psi(z_0) \leq \pi/2-c_0\pi$$

for the same constant c_0, so that

$$|\operatorname{Im} \psi(z_0)| \leq \pi/2-c_0\pi \tag{2.2}$$

for $z_0 \in D_1$.

By the conformal invariance of the hyperbolic geodesic Γ and by (1.9) and (2.2), we have, for $z_0 \in D_1$, that

$$\begin{aligned} e^{-2d_D(z_0, \Gamma)} &= e^{-2d_S(\psi(z_0), \mathbf{R})} \\ &\geq \tfrac{1}{4}\cos^2(\operatorname{Im} \psi(z_0)) \\ &\geq c, \end{aligned} \tag{2.3}$$

where c is a constant depending only on c_0. The theorem now follows from (2.3), since $\operatorname{area}(D_1)=\frac{1}{2}\operatorname{area}(D)$. □

REMARK 2.1. It should be mentioned that Xu's proof, which is more complicated than ours, also contains similar estimates on harmonic measure.

Theorem 2 shows that for such strip domains the lifetime estimate holds if and only if the domain has finite area. However, Xu [16] also gave the first example of a domain of infinite area which has finite lifetime. In [11] Davis gave a similar example of a domain of infinite area which is intrinsic ultracontractive, a property which implies the lifetime estimate. In Bañuelos [5], a class of domains which is called "uniformly Hölder" was introduced and intrinsic ultracontractivity was proved for these domains. Bass and Burdzy [7] observed that some of these domains also have infinite volume (we refer the reader to Bañuelos [6] for a more careful account of the literature on this subject). Theorem 1 can be used to show relatively easily that uniformly Hölder domains have finite lifetime. We shall not give the details here, however, in order to keep this note short and elementary. Instead we present a further, elementary example of a domain of infinite area with finite lifetime. More precisely, we have this next theorem.

THEOREM 3. *Let D be the simply connected domain given by*

$$D = O_1 \cup O_2 \cup O_3,$$

where

$$O_1 = \left\{ z = x+iy : x > \frac{3}{2}, 0 < y < \frac{1}{x^2} \right\};$$

$$O_2 = \bigcup_{n \geq 2} \left\{ z = x+iy : |x-n| < \frac{1}{n^2}, 0 < y \leq 2 \right\};$$

$$O_3 = \bigcup_{n \geq 2} \left\{ z = x+iy : |x-n| < \frac{1}{2}, 2 < y < 3 \right\}.$$

Then

$$\sup_{\substack{w \in D \\ h \in H^+(D)}} E_w^h(\tau_D) < \infty.$$

Before we prove the theorem we introduce the quasi-hyperbolic distance and make some observations. Let $W(D) = \{Q_j\}$ be a Whitney decomposition of a domain D. This is a decomposition of D into cubes with the following properties:

(i) the cubes have disjoint interiors;
(ii) if two cubes are not disjoint (i.e., if they have touching edges) then they have comparable length with constants independent of the cubes ($\frac{1}{4}$ and 4 will do); and
(iii) the length of each cube is proportional to its distance from the boundary of D.

Let Q_0 and Q_1 be two Whitney cubes. We say that $Q_0 \to Q(1) \to \cdots \to Q(m) = Q_1$ is a Whitney chain connecting Q_0 to Q_1 of length m if $Q(i) \in W(D)$ and if

$Q(i)$ and $Q(i+1)$ have touching edges for each i. We define the Whitney distance $d_W(Q_0, Q_1)$ to be the length of the shortest Whitney chain connecting Q_0 to Q_1. If $z_1, z_2 \in D$, we define the quasi-hyperbolic distance between these points to be $\rho_D(z_1, z_2) = d_W(Q_1, Q_2)$, where $z_1 \in Q_1$ and $z_2 \in Q_2$. It follows easily from the Koebe distortion theorem and the Schwarz lemma (see Vuorinen [15, Chap. 3]) that $\rho_D(z_1, z_2) \approx d_D(z_1, z_2)$.

The domain D of Theorem 3 consists of an infinitely long corridor O_1 from which infinitely many narrow corridors lead to identical rooms. The total area of all the corridors is finite. We shall say that a point $z = x + iy \in D$ is inside the nth room if $|x-n| < \frac{1}{2}$ and $y > \frac{3}{2}$. We claim that there is an integer K such that any geodesic Γ can contain points inside at most K rooms. This follows from the equivalence of hyperbolic distance and quasi-hyperbolic distance. In fact, suppose that Γ is a geodesic containing a point in the nth room and in two other rooms. Then there are points $z_i = x_i + iy_i$ on Γ with $|x_i - n| < 1/n^2$ for $i = 1, 2, 3, 4$ and with $y_1 = y_4 = 1$ and $y_2 = y_3 = \frac{3}{2}$. Since Γ is a geodesic,

$$\begin{aligned} d_D(z_1, z_4) &= d_D(z_1, z_2) + d_D(z_2, z_3) + d_D(z_3, z_4) \\ &\geq d_D(z_1, z_2) + d_D(z_3, z_4). \end{aligned} \tag{2.4}$$

Now $d_D(z_1, z_4) \leq C_1 \rho_D(z_1, z_4) \leq k$, where k is the length of the Whitney chain joining z_1 to z_4 along the straight line segment $[z_1, z_4]$. Furthermore,

$$\begin{aligned} d_D(z_1, z_2) + d_D(z_3, z_4) &\geq c_2 \rho_D(z_1, z_2) + c_2 \rho_D(z_3, z_4) \\ &\geq c_2(k + c_3 n^2). \end{aligned}$$

This contradicts (2.4) if n is sufficiently large. Thus there is an integer K such that any geodesic can enter at most K rooms.

Next suppose that $z_1 = x_1 + iy_1$ is such that $|x_1 - n| < 1/2$ and $2 < y_1 < 3$ for some $n \geq 2$. If z_2 is not inside the nth room, then it follows once more from the equivalence of the hyperbolic and quasi-hyperbolic metrics (by counting Whitney cubes inside the rectangle) that $d_D(z_1, z_2) \geq cn^2$. Alternatively, we can argue as follows. If γ is a curve in D of shortest hyperbolic length joining z_1 to z_2, then γ contains a subarc γ_1 which joins the line segment $[n - 1/n^2 + 2i, n + 1/n^2 + 2i]$ to the line segment $[n - 1/n^2 + 3/2i, n + 1/n^2 + 3/2i]$. Since γ is part of a geodesic,

$$d_D(z_1, z_2) = L(\gamma) \geq L(\gamma_1) \geq (\pi/2)(n^2 - 1),$$

where L denotes the hyperbolic length of a curve and the final inequality follows from Lemma 6 in Hayman [13]. Thus, in fact, $d_D(z_1, z_2) \geq \pi n^2/4$.

Proof of Theorem 3. Let Γ be any hyperbolic geodesic in D. From the trivial estimate $e^{-2d_D(z,\Gamma)} \leq 1$ we have that

$$\iint_{D \cap \{\operatorname{Im} z \leq 2\}} e^{-2d_D(z,\Gamma)}\, dx\, dy \leq \operatorname{area}(D \cap \{\operatorname{Im} z \leq 2\}) = A < \infty.$$

Suppose that Γ does not enter the nth room. If z_1 is in this room and $\operatorname{Im} z_1 > 2$ then for any z_2 on γ we have $d_D(z_1, z_2) \geq \pi n^2/4$ as we have shown. Therefore $d(z_1, \Gamma) \geq \pi n^2/4$ in this case, and the contribution from this room to the expected lifetime does not exceed $e^{-\pi n^2/2}$ since the area of the room is 1. This estimate holds for all but possibly K rooms of total area K. In summary, then,

$$\iint_D e^{-2d_D(z,\Gamma)}\,dx\,dy \leq A + K + \sum_{n\geq 2} e^{-\pi n^2/2} = C < \infty,$$

and Theorem 3 is proved. □

Next we make the connection of Theorem 1 to Whitney cubes and the quasi-hyperbolic distance for D. The first use of these quantities in the study of the lifetime of conditioned Brownian motion seems to have been in Bañuelos [4]. They have subsequently been used very effectively by several people, and we again refer to Bañuelos [6] for an account on the literature. Note that if Γ is any hyperbolic geodesic and if Q is any Whitney cube with center z_Q then

$$d_D(z, \Gamma) \approx d_D(Q, \Gamma) \approx d_D(z_Q, \Gamma) \tag{2.5}$$

for all $z \in Q$. This follows, for instance, from the equivalence of the hyperbolic and quasi-hyperbolic distances. From (2.5) and Lemma 1 we immediately obtain the following.

COROLLARY 1. *Let Γ be a hyperbolic geodesic in D joining the two points ξ_1 and ξ_2 on the Martin boundary of D. Let T_Q be the total time that Brownian motion, starting at ξ_1 and conditioned to go to ξ_2, spends in Q. There are constants c_1, c_2, independent of Q and Γ such that*

$$\tfrac{1}{4} e^{-c_1 d_D(z_Q,\Gamma)}|Q| \leq E_{\xi_1}^{\xi_2}(T_Q) \leq 4e^{-c_2 d_D(z_Q,\Gamma)}|Q|, \tag{2.6}$$

where $|Q|$ denotes the area of Q.

Our proof can be easily modified to give the following.

COROLLARY 2. *Let z_1 and z_2 be two points in D with $d_D(z_1, z_2) \geq 1$. Let $\gamma = \gamma(z_1, z_2)$ be the curve of smallest hyperbolic length joining z_1 to z_2, and let Q be any Whitney cube. Let T_Q be the total time that Brownian motion, starting at z_1 and conditioned to go to z_2, spends in Q. There are constants c_1, c_2, c_3, c_4, independent of Q and γ, such that*

$$c_1 e^{-c_2 d_D(z_Q,\gamma)}|Q| \leq E_{z_1}^{z_2}(T_Q) \leq c_3 e^{-c_4 d_D(z_Q,\gamma)}|Q|. \tag{2.7}$$

From Corollary 2 and from Theorem 1.1 in Davis [10] we deduce the next corollary.

COROLLARY 3. *Let z_1, z_2, γ, and Q be as in Corollary 2. Let P_Q be the probability that the conditioned Brownian motion from z_1 to z_2 ever hits Q. Then*

$$c_1 e^{-c_2 d_D(z_Q,\gamma)} \leq P_Q \leq c_3 e^{-c_4 d_D(z_Q,\gamma)}. \tag{2.8}$$

These corollaries clarify our statement in the introduction that conditioned Brownian motion paths "follow" hyperbolic geodesics. For if the hyperbolic geodesic $\gamma(z_1, z_2)$ enters the Whitney cube Q then the conditioned Brownian motion will hit the cube with at least probability c_1, and the expected time that it will spend in Q is proportional to the area of the cube.

In [1] and [3], Ancona presents several estimates for Green functions on manifolds of negative curvature in terms of the intrinsic metric. Many of these estimates, when interpreted in terms of the quasi-hyperbolic metric, continue to hold for domains in $\mathbf{R}^n$ which satisfy the Hardy inequality. This latter inequality itself follows from a uniform boundary capacity condition [1]. It seems not to have been noticed before that some of these estimates can be very useful in the study of the lifetime of conditioned Brownian motion. For example, the argument used to derive the inequality after Remark 2.1 in [2], together with a version of Proposition 2.5 in [3], can be adapted to prove our Corollary 2. Also, it follows from the proof of Proposition 4.1 in [3] (or Lemma 8 in [2]) that if D is a domain in $\mathbf{R}^2$ satisfying the Hardy inequality then, for any $Q \in W(D)$ and any $x \in \partial Q^*$, where $Q^* = \frac{3}{2}Q$,

$$G(x, y) \leq C \tag{2.9}$$

for all $y \in Q$. Here Q^* is the cube concentric with Q and $l(Q^*) \equiv \frac{3}{2} l(Q)$. The inequality (2.9) can be used to prove that if $D \in UH(\alpha)$ for some $0 < \alpha < 2$ (see Bañuelos [5] for a definition); then the Cranston–McConnell estimate holds for D. The inequality (2.9) is also the key estimate for the conditional gauge theorem proved by Cranston [8] for planar domains. In $\mathbf{R}^2$, it is proved in Ancona [1] that the uniform capacity condition is equivalent to the Hardy inequality, which in turn is equivalent to (2.9).

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