Representations of Exponential Solvable Lie Groups Induced from Maximal Subgroups

RONALD L. LIPSMAN

1. Introduction

We are concerned here with the orbital integral formula for the direct integral decomposition of an induced representation of an exponential solvable Lie group. This is an explicit formula for the spectrum and spectral multiplicities and measure in terms of coadjoint data. Such a formula is derived for nilpotent groups in [2] or [8], and for completely solvable groups in [9]. The work in this paper originated at the time the author extended the results of [9] to arbitrary exponential solvable groups [13]. Whereas in [9] the codimension-1 (nonnormal) case was the critical subcase to understand, in exponential solvable groups [13] one must deal with codimension-2 (nonnormal and maximal) subgroups. Indeed, one can carry out the analysis and prove [9, Thm. 2.2, or alternatively Thm. 5.1] in the generality of exponential solvable groups [13]. However, Fujiwara has recently obtained the proof [4] by a different method. In this paper we shall therefore present only the codimension-2 analysis. It has independent interest for the following reasons: (i) exactly as in nilpotent [5] or completely solvable [9] groups, wherein the codimension-1 analysis is used in many different aspects of representation theory and harmonic analysis, one expects the codimension-2 theory to play a similar role in exponential solvable groups; (ii) the codimension-2 theory might be useful in applications to differential equations [7]; and (iii) the most intriguing reason—namely, that although all orbital multiplicity formulas that appear in the theory are known to hold only generically (i.e. almost everywhere), we shall prove that in the case of a *maximal* subgroup (which includes codimension 1 or 2) all the orbital formulas are valid *everywhere.*

Now we explain the point of the paper in more detail. Let G be a simply connected exponential solvable Lie group, and H a closed connected (therefore simply connected) subgroup. Let $\mathfrak{g}$, $\mathfrak{h}$ be the Lie algebras, $\mathfrak{g}^*$, $\mathfrak{h}^*$ their real linear duals, and $p\colon \mathfrak{g}^* \to \mathfrak{h}^*$ the canonical projection. Let ν be an irreducible

---

Received July 25, 1991. Revision received February 5, 1993.
The author was supported by NSF grant #DMS-90-02642.
Michigan Math. J. 40 (1993).

unitary representation of H, and let $\mathcal{O}_\nu$ be the corresponding coadjoint orbit in $\mathfrak{h}^*$. (The reader is referred to [9, p. 128] for a summary of the Kirillov-Bernat parameterization of the dual $\hat{H}$ by coadjoint orbits $\mathfrak{h}^*/H$.) We recall the *orbital spectrum formula* (see [9, Def. 2.1 and formulas (2.1), (2.2)]) for the induced representation:

$$\operatorname{Ind}_H^G \nu = \int_{p^{-1}(\mathcal{O}_\nu)/H}^{\oplus} \pi_\phi \, d\mu_{G,H}^\nu(\phi) = \int_{G\cdot p^{-1}(\mathcal{O}_\nu)/G}^{\oplus} n_\phi^\nu \pi_\phi \, d\tilde{\mu}_{G,H}^\nu(\phi). \tag{1.1}$$

We shall be concerned less with the canonical measures $\mu, \tilde{\mu}$ (see [9] for their definition) than with the multiplicity function

$$n_\phi^\nu = \text{multiplicity of } \pi_\phi \text{ in } \operatorname{Ind}_H^G \nu = \#(G\cdot\phi \cap p^{-1}(\mathcal{O}_\nu))/H. \tag{1.2}$$

Formulas (1.1) and (1.2) are proven in [8] when G is nilpotent, and in [9] when G is completely solvable. The case $\dim G/H = 1$ plays a critical role in both. A generalization to G exponential solvable is carried out in [4]. In this paper we shall consider the situation G exponential solvable, $\dim G/H = 2$ and H is maximal in G. This is worthwhile for the reasons (i) and (ii) indicated above; and also because the comparison and reconciliation of orbital and Mackey parameters that is effected is interesting and because the seven different structures that are manifested are illuminating. But the most intriguing feature of the presentation is the following.

The induced representation $\operatorname{Ind}_H^G \nu$ is typically not of uniform multiplicity; that is, n_ϕ^ν is *not* constant (see e.g. [2; 10]). In the completely solvable case it is known that n_ϕ^ν is either identically infinity or finite and bounded a.e. [12]. It is suspected, but not known, that the same result is true for arbitrary exponential solvable groups. The CG (for Corwin–Greenleaf) condition

$$\dim G\cdot\phi = 2\dim H\cdot\phi - \dim \mathcal{O}_\nu \quad \text{for generic } \phi \in p^{-1}(\mathcal{O}_\nu) \tag{1.3}$$

is necessary and sufficient for finite multiplicity in the completely solvable case [12], but only necessary in the exponential solvable case [2; 8]. However *even in the case of uniform finite multiplicity:* condition (1.3) holds only generically, not everywhere; and the number of H-orbits on $G\cdot\phi \cap p^{-1}(\mathcal{O}_\nu)$ agrees with the (uniform) multiplicity only generically, not everywhere. Illustrative examples of the degenerate pathologies are given in Section 2. In this paper we shall prove the following theorem.

THEOREM 1.1. *Let G be simply connected exponential solvable, and let $H \subset G$ be a maximal closed connected subgroup. Let $\nu \in \hat{H}$ with $\mathcal{O}_\nu$ the corresponding coadjoint orbit. Then:*

(i) *$\operatorname{Ind}_H^G \nu$ is of uniform multiplicity.*

(ii) *The number*

$$\dim G\cdot\phi - 2\dim H\cdot\phi + \dim \mathcal{O}_\nu$$

is a nonnegative constant on $p^{-1}(\mathcal{O}_\nu)$. The multiplicity is finite if and only if that number is zero.

(iii) *The number*

$$\#(G\cdot\phi\cap p^{-1}(\mathcal{O}_\nu))/H$$

is constant on $p^{-1}(\mathcal{O}_\nu)$; *it is the multiplicity of* $\mathrm{Ind}_H^G \nu$.

REMARKS 1.2. (1) Of course in Theorem 1.1 the maximality combined with the exponential solvability forces $\dim G/H = 1$ or 2. It may be that the maximality of H is governing the phenomenon occurring in Theorem 1.1 more so than the exponential solvability. Another famous maximal situation is: G connected semisimple, K a maximal compact subgroup. Theorem 1.1 is true for the quasiregular representation of G on $L^2(G/K)$. It should be interesting to explore other maximal situations outside of exponential solvable Lie groups.

(2) In the course of the proof we shall see that, in codimension 2, the induced representation is actually multiplicity-free except in one (of the seven) situation(s) where it has infinite multiplicity.

(3) There is a companion theorem to Theorem 1.1 for restricted representations $\pi|_H$, $\pi \in \hat{G}$. The same results are true—the variable this time being $\psi \in p(\mathcal{O}_\pi)$. But I shall leave the details of that theorem to another time and place.

Here is the organization of the paper. In Section 2 we prove Theorem 1.1 for codimension-1 subgroups. We also give the promised examples of its failure without maximality—actually in codimension 3. In Section 3 we present some canonical structure results on codimension-2 maximal subgroups. This should be compared with section 3 of [9] where corresponding results are presented for codimension-1 maximal subgroups. In Section 4, we derive the seven possibilities for the structure of a representation induced from a codimension-2 maximal subgroup. We also prove Theorem 1.1 in that generality. As in [9], we have to spend some time reconciling the orbital parameters with Mackey parameters that arise from the canonical structure subgroups determined by H. Finally, in Section 5 we briefly illustrate the seven possibilities with examples.

2. When Generic Means Everywhere— and When It Does Not

In this section we show how codimension-1 inductions are special in that both the CG condition (1.3) and the multiplicity formula (1.2) are valid for *all* points in the orbital spectrum—generic or degenerate. One of the main results of this paper is that the same situation obtains for codimension-2 induction (see Section 4). However, when H is not maximal of codimension 1 or 2, typically both the CG condition and the multiplicity formula fail to hold on sets of lower dimension. We shall illustrate this with two codimension-3 examples later in this section. But first we examine codimension 1.

Let G be simply connected exponential solvable, $H \subset G$ a closed connected (therefore simply connected) subgroup, and suppose $\dim G/H = 1$. As usual, $p: \mathfrak{g}^* \to \mathfrak{h}^*$ denotes the canonical projection. Let $\nu \in \hat{H}$ and let $\mathcal{O}_\nu \subset \mathfrak{h}^*$ be the H-orbit corresponding to ν. One knows [9] that $\mathrm{Ind}_H^G \nu$ is multiplicity-free. Therefore our goal here is to prove that for *every* $\phi \in p^{-1}(\mathcal{O}_\nu)$ we have:

$$\dim G \cdot \phi = 2 \dim H \cdot \phi - \dim \mathcal{O}_\nu \tag{2.1}$$

and

$$G \cdot \phi \cap p^{-1}(\mathcal{O}_\nu) = H \cdot \phi. \tag{2.2}$$

Now we now, from [2], [8], or [9], that *generically* on $p^{-1}(\mathcal{O}_\nu)$ the truth of (2.2) implies that of (2.1). Moreover, we also know [9] that (2.2) is true generically on $p^{-1}(\mathcal{O}_\nu)$. We are now going to show that both are true *everywhere* on $p^{-1}(\mathcal{O}_\nu)$. However, we shall have to handle the two formulas differently. We shall prove (2.1) by demonstrating an equivalent formula. No case-by-case analysis is required, though such an analysis will be necessary to prove formula (2.2).

LEMMA 2.1. *For any $\phi \in p^{-1}(\mathcal{O}_\nu)$, set $\psi = \phi|_{\mathfrak{h}}$. Then formula* (2.1) *is equivalent to*

$$\dim \mathfrak{g}_\phi/\mathfrak{h}_\phi + \dim \mathfrak{h}_\psi/\mathfrak{h}_\phi = \dim \mathfrak{g}/\mathfrak{h}. \tag{2.3}$$

Proof. We have

$$\begin{aligned} \dim G \cdot \phi - 2 \dim H \cdot \phi + \dim \mathcal{O}_\nu &= \dim \mathfrak{g}/\mathfrak{g}_\phi - 2 \dim \mathfrak{h}/\mathfrak{h}_\phi + \dim \mathfrak{h}/\mathfrak{h}_\psi \\ &= \dim \mathfrak{g}/\mathfrak{h} - \dim \mathfrak{g}_\phi + 2 \dim \mathfrak{h}_\phi - \dim \mathfrak{h}_\psi \\ &= \dim \mathfrak{g}/\mathfrak{h} - \dim \mathfrak{g}_\phi/\mathfrak{h}_\phi - \dim \mathfrak{h}_\psi/\mathfrak{h}_\phi. \quad \square \end{aligned}$$

REMARK 2.2. When $\dim \mathfrak{g}/\mathfrak{h} = 1$, both $\dim \mathfrak{g}_\phi/\mathfrak{h}_\phi$ and $\dim \mathfrak{h}_\psi/\mathfrak{h}_\phi$ are 0 or 1 (because of Proposition 2.3(a)). Thus formula (2.3) is true exactly when one of $\mathfrak{g}_\phi/\mathfrak{h}_\phi$, $\mathfrak{h}_\psi/\mathfrak{h}_\phi$ has dimension 0 and one has dimension 1. It could only fail if both have dimension 0 or both have dimension 1. That is not possible, as we now show.

PROPOSITION 2.3. *In the codimension*-1 *situation, we have:*

(a) $\dim \mathfrak{g}_\phi/\mathfrak{h}_\phi + \dim \mathfrak{h}_\psi/\mathfrak{h}_\phi \leq 1$;
(b) $1 - \dim \mathfrak{g}_\phi + \dim \mathfrak{h}_\psi$ *is even.*

Proof. (a) The following argument—generously supplied by the referee—replaces an earlier (incorrect) version. By the proof of Lemma 2.1, it suffices to prove that $\dim G \cdot \phi - 2 \dim H \cdot \phi + \dim \mathcal{O}_\nu \geq 0$ for all $\phi \in p^{-1}(\mathcal{O}_\nu)$. Even if $G \cdot \phi \cap p^{-1}(\mathcal{O}_\nu)$ is not a manifold, we have $H \cdot \phi \subset p^{-1}(\mathcal{O}_\nu)$. Thus $\mathfrak{h} \cdot \phi \subset T_\phi(p^{-1}(\mathcal{O}_\nu))$, the tangent space at ϕ to the smooth manifold $p^{-1}(\mathcal{O}_\nu)$. Therefore $\mathfrak{h} \cdot \phi \subset \mathfrak{g} \cdot \phi \cap T_\phi(p^{-1}(\mathcal{O}_\nu)) = \mathfrak{g} \cdot \phi \cap p^{-1}(\mathfrak{h} \cdot \psi)$. In [8, p. 450] it is shown that the latter intersection has dimension $\dim \mathfrak{g} \cdot \phi - \dim \mathfrak{h} \cdot \phi + \dim \mathfrak{h} \cdot \psi$. So $\dim \mathfrak{g} \cdot \phi - 2 \dim \mathfrak{h} \cdot \phi + \dim \mathfrak{h} \cdot \psi \geq 0$ and the proof of (a) is done.

(b) The orbits $G\cdot\phi$ and $H\cdot\psi$ are even-dimensional. By the codimension-1 assumption, it must be that

$$\dim \mathfrak{g}/\mathfrak{g}_\phi - \dim \mathfrak{h}/\mathfrak{h}_\psi = 0 \text{ or } 2.$$

That is,

$$1 - \dim \mathfrak{g}_\phi + \dim \mathfrak{h}_\psi = 0 \text{ or } 2.$$

Finally, combining Lemma 2.1, Remark 2.2, and Proposition 2.3(a), we see that (2.1) can only fail if $\mathfrak{g}_\phi = \mathfrak{h}_\phi = \mathfrak{h}_\psi$. But that possibility is excluded by Proposition 2.3(b). □

Unfortunately we cannot derive the universal truth of (2.2) by the same kind of case-free reasoning. So we begin its proof by first considering H to be *normal* in G, an automatic condition of course if G is nilpotent. Then one knows [2; 8] that either:

(i) $\mathrm{Ind}_H^G \nu = \pi$ is irreducible and $\pi = \pi_\Omega$, where $\Omega = \Omega_\pi$ is the unique G-orbit satisfying $\Omega \supset p^{-1}(\mathcal{O}_\nu)$; or

(ii) $\mathrm{Ind}_H^G \nu = \int_{\mathbf{R}}^{\oplus} \pi_t \, dt$, where if $\phi \in p^{-1}(\mathcal{O}_\nu)$ and $X \in \mathfrak{g}$, $X \notin \mathfrak{h}$ then $\pi_t = \pi_{\phi_t}$, $\phi_t = \phi + tX^*$.

In (ii) we understand X^* to mean the functional satisfying $X^*(X) = 1$, $X^*(\mathfrak{h}) = 0$. The orbits of ϕ_t form a 1-parameter disjoint family, and they account for all the G-orbits meeting $p^{-1}(\mathcal{O}_\nu)$. In fact it is proven in [8] that: in case (i) we have $G\cdot\phi \cap p^{-1}(\mathcal{O}_\nu) = H\cdot\phi$; and in case (ii), $G\cdot\phi_t \cap p^{-1}(\mathcal{O}_\nu) = H\cdot\phi_t$ for all $t \in \mathbf{R}$.

Now we drop the normality assumption. Then the structure of the induced representation is elucidated thoroughly in [9]. In that paper it is assumed that G is completely solvable. However, both Proposition 3.2 and Theorem 3.3 of [9] are valid for any codimension-1 pair $H \subset G$ provided G is exponential solvable and H is not normal. Drawing the distinctions somewhat less finely than in [9], we see there are three possibilities:

(i) $\mathrm{Ind}_H^G \nu = \pi$ is irreducible and $\pi = \pi_\Omega$, where $\Omega = \Omega_\pi$ is the unique G-orbit satisfying $\Omega \supset p^{-1}(\mathcal{O}_\nu)$.

(ii) $\mathrm{Ind}_H^G \nu = \int_{\mathbf{R}}^{\oplus} \pi_t \, dt$, where $\phi \in p^{-1}(\mathcal{O}_\nu)$ and there is a nonzero functional $\alpha \in \mathfrak{g}^*$ such that $\pi_t = \pi_{\phi_t}$, $\phi_t = \phi + t\alpha$, $t \in \mathbf{R}$. The orbits $G\cdot\phi_t$, $t \in \mathbf{R}$, form a mutually disjoint family which exhausts the G-orbits that meet $p^{-1}(\mathcal{O}_\nu)$.

(iii) $\mathrm{Ind}_H^G \nu = \pi_1 \oplus \pi_2$, a direct sum of two inequivalent irreducible representations. In this case there are precisely three G-orbits that meet $p^{-1}(\mathcal{O}_\nu)$, only two of which are generic (i.e. of maximal dimension—see [9, p. 138]).

The facts we want are proven completely in cases (i) and (ii) in [9]. There it is shown that, for all $\phi \in p^{-1}(\mathcal{O}_\nu)$,

$$G\cdot\phi \cap p^{-1}(\mathcal{O}_\nu) = H\cdot\phi.$$

The same is proven in [9] for the two generic orbits in case (iii). The only unverified fact that remains to be substantiated is the formula (2.2) for the one degenerate orbit in case (iii). We look more closely at [9, proof of Thm. 3.3]; in particular, we utilize the terminology of Theorem 3.3, case (v) [9, pp. 138–140] (except we substitute H for G_1). Having chosen $\phi \in p^{-1}(\mathcal{O}_\nu)$, there is a nonzero $\alpha \in \mathfrak{h}^\perp$ and a unique $s_0 \in \mathbf{R}$ such that $\phi^0 = \phi + s_0\alpha$ lies in the degenerate orbit. Let $g \cdot \phi^0 \in p^{-1}(\mathcal{O}_\nu)$ for some $g \in G$. Then there is $h \in H$ such that $hg \cdot \phi^0|_{\mathfrak{h}} = \psi$. In particular, $hg \cdot \phi^0 = \phi + s\alpha$ for some $s \in \mathbf{R}$. But the G-orbits $G \cdot (\phi + s\alpha)$ are disjoint from $G \cdot (\phi + s_0\alpha)$ if $s \neq s_0$ [9]. Hence $hg \cdot \phi^0 = \phi^0$; that is, $g \cdot \phi^0 \in H \cdot \phi^0$.

The results of this section so far amount to a proof of Theorem 1.1 in the case $\dim G/H = 1$. We will prove Theorem 1.1 for codimension 2 in Section 4. The remainder of this section is taken up with examples showing that the generic requirement for formulas (2.1) and (2.2) cannot be dropped in codimension 3.

EXAMPLES 2.4. We consider the standard 4-dimensional nilpotent Lie group G. Its Lie algebra $\mathfrak{g}$ is spanned by four generators X, Y, Z, W satisfying nonzero bracket relations

$$[X, Y] = Z, \qquad [X, Z] = W.$$

The elements $\phi \in \mathfrak{g}^*$ are parameterized $\phi_{\xi,\eta,\zeta,\omega} = \xi X^* + \eta Y^* + \zeta Z^* + \omega W^*$, $\xi, \eta, \zeta, \omega \in \mathbf{R}$. The coadjoint action is

$$g \cdot \phi_{\xi,\eta,\zeta,\omega} = \phi_{\xi - y\zeta - (z+xy)\omega,\ \eta + x\zeta + (1/2)x^2\omega,\ \zeta + x\omega,\ \omega}$$

if $g = \exp xX \exp yY \exp zZ \exp wW$. There are three families of coadjoint orbits:

$$\Omega^{\xi,\eta} = G \cdot \phi_{\xi,\eta,0,0}; \quad \xi, \eta \in \mathbf{R} \qquad \text{(points);}$$

$$\Omega_\zeta = G \cdot \phi_{0,0,\zeta,0}; \quad \zeta \neq 0 \qquad \text{(planes);}$$

$$\Omega_{\eta,\omega} = G \cdot \phi_{0,\eta,0,\omega}; \quad \eta \in \mathbf{R},\ \omega \neq 0 \qquad \text{(parabolic sheets).}$$

(i) First let $H = \exp \mathbf{R}Z$. Obviously $\dim G/H = 3$. The irreducible representations of H are characters χ_ζ, $\zeta \in \mathbf{R}$, where $\chi_\zeta(e^{tZ}) = e^{it\zeta}$, $t \in \mathbf{R}$. Take $\zeta = 1$ and write $\nu = \chi_1$. Of course, $\mathcal{O}_\nu = \{\chi_1\}$ and $\dim \mathcal{O}_\nu = 0$. We have

$$p^{-1}(\mathcal{O}_\nu) = \{\phi_{\xi,\eta,1,\omega} : \xi, \eta, \omega \in \mathbf{R}\}.$$

If $\omega \neq 0$ then $G \cdot \phi = \Omega_{\eta - 1/2\omega,\, \omega}$ and $H \cdot \phi = \{\phi_{\xi - z\omega,\eta,1,\omega} : z \in \mathbf{R}\}$, so $\dim G \cdot \phi = 2 \dim H \cdot \phi - \dim \mathcal{O}_\nu = 2$. But if $\omega = 0$ then $G \cdot \phi = \{\phi_{\xi - y,\eta + x,1,0} : x, y \in \mathbf{R}\} = \Omega_1$ while $H \cdot \phi = \phi$; and so formula (2.1) is violated for this degenerate orbit.

(ii) Now set $H = \exp \mathbf{R}Y$. Again $\dim G/H = 3$. This time take $\nu = \chi_0 =$ the identity character of H, $\chi_0(e^{tY}) = 1$, $t \in \mathbf{R}$. Then $\mathcal{O}_\nu = \{0\}$ and $p^{-1}(\mathcal{O}_\nu) = \mathfrak{h}^\perp = \{\phi_{\xi,0,\zeta,\omega} : \xi, \zeta, \omega \in \mathbf{R}\}$. If $\omega\zeta \neq 0$ then $G \cdot \phi = \Omega_{-\zeta^2/2\omega,\, \omega}$, $H \cdot \phi = \{\phi_{\xi,0,\zeta,\omega} : \xi \in \mathbf{R}\}$, and

$$G \cdot \phi \cap p^{-1}(\mathcal{O}_\nu) = \{\phi_{\phi,0,\pm\zeta,\omega} : \xi \in \mathbf{R}\}.$$

Thus $\#(G\cdot\phi\cap p^{-1}(\mathcal{O}_\nu))/H=2$ generically. But if $\omega\zeta=0$ then it is easy to check that

$$G\cdot\phi\cap p^{-1}(\mathcal{O}_\nu)=H\cdot\phi.$$

Thus the number of H-orbits on $G\cdot\phi\cap p^{-1}(\mathcal{O}_\nu)$ differs on the degenerate orbits from the generic value of the multiplicity function.

3. Basic Structure for a Codimension-2 Maximal Subgroup

We continue to assume G is simply connected exponential solvable and H is a closed connected subgroup. But now we suppose $\dim G/H=2$. Moreover, we assume that H is a *maximal* (connected) subgroup of G. That is, no intermediate subgroup $H\subset G_1\subset G$, $\dim G_1=\frac{1}{2}(\dim G+\dim H)$, exists. In particular, G cannot be completely solvable. We start with the following lemma.

LEMMA 3.1. *H cannot be normal in G.*

Proof. If H is normal in G, then G/H is a 2-dimensional simply connected real solvable Lie group. Hence G/H is either abelian or isomorphic to the $(ax+b)$-group. In either case there is a 1-dimensional normal subgroup of G/H. Its pullback to G would be an intermediate subgroup between H and G. This completes the proof. □

Next comes the basic structure result. This generalizes the codimension-1 structure found in Proposition 3.2 of [9]. In order to preserve some notational similarity with [9], I will use the notation $\mathfrak{g}_1$ for $\mathfrak{h}$.

PROPOSITION 3.2. *Let $\mathfrak{g}$ be exponential solvable, and let $\mathfrak{g}_1\subset\mathfrak{g}$ be a maximal subalgebra of codimension 2. (In particular, by Lemma 3.1, $\mathfrak{g}_1$ is not an ideal.) Then there exist a codimension-1 subalgebra $\mathfrak{g}_0$ of $\mathfrak{g}_1$ which is a codimension-3 ideal in $\mathfrak{g}$, as well as three elements X, Y, Z and a nonzero real number r such that*

$$\mathfrak{g}=\mathfrak{g}_0+\mathbf{R}X+\mathbf{R}Y+\mathbf{R}Z,\quad \mathfrak{g}_1=\mathfrak{g}_0+\mathbf{R}X,$$

$$[X,Y]\equiv Y+rZ \mod \mathfrak{g}_0, \tag{3.1}$$

$$[X,Z]\equiv -rY+Z \mod \mathfrak{g}_0, \tag{3.2}$$

and

$$[Y,Z]\equiv 0 \mod \mathfrak{g}_0. \tag{3.3}$$

Proof. Let $\mathfrak{n}$ be the nilradical of $\mathfrak{g}$. Then $\mathfrak{g}_1+\mathfrak{n}=\mathfrak{g}$ since by hypothesis $\mathfrak{g}_1+\mathfrak{n}$ must be either $\mathfrak{g}_1$ or $\mathfrak{g}$. But if $\mathfrak{g}_1+\mathfrak{n}=\mathfrak{g}_1$, then $\mathfrak{n}\subset\mathfrak{g}_1$ which would force $\mathfrak{g}_1$ to be an ideal. Now $\mathrm{ad}_{\mathfrak{g}/\mathfrak{g}_1}\mathfrak{g}_1$ is irreducible. In fact, by the assumption of exponential solvability, there must exist linearly independent $Y, Z\in\mathfrak{g}$ (both $\notin\mathfrak{g}_1$), a real linear functional $\delta:\mathfrak{g}_1\to\mathbf{R}$, and a real number r such that

$$[W, Y+iZ] \equiv \delta(W)(1-ir)(Y+iZ) \bmod (\mathfrak{g}_1)_c, \; W \in \mathfrak{g}_1. \tag{3.4}$$

By the maximality of $\mathfrak{g}_1$ we must have $\delta \not\equiv 0$ and $r \neq 0$. Moreover, by the equation $\mathfrak{g}_1 + \mathfrak{n} = \mathfrak{g}$, it is no loss of generality to assume $Y, Z \in \mathfrak{n}$.

Set $\mathfrak{g}_0 = \ker \delta$. Since δ is a Lie mapping, we have $\mathfrak{g}_0 \lhd \mathfrak{g}_1$. In fact, $\mathfrak{g}_0$ is an ideal in $\mathfrak{g}$. To show this, since $\mathfrak{g} = \mathfrak{g}_1 + \mathfrak{n}$ it is enough to show that $[\mathfrak{n}, \mathfrak{g}_0] \subset \mathfrak{g}_0$. But clearly

$$\mathfrak{g}_0 = \{X \in \mathfrak{g}_1 : [X, \mathfrak{g}] \subset \mathfrak{g}_1\}.$$

In particular, $[\mathfrak{g}_0, \mathfrak{g}] \subset \mathfrak{g}_1$. Hence, since $\mathfrak{n}$ is an ideal,

$$[\mathfrak{g}_0, \mathfrak{n}] \subset \mathfrak{n} \cap \mathfrak{g}_1.$$

Thus it is enough to show $\mathfrak{n} \cap \mathfrak{g}_1 \subset \mathfrak{g}_0$. Now, since $\mathfrak{n}$ is a nilpotent ideal in $\mathfrak{g}$, this ensures that the only eigenvalue that can arise in $\mathrm{ad}_{\mathfrak{g}} \mathfrak{n}$ is zero. Hence $\delta|_{\mathfrak{g}_1 \cap \mathfrak{n}} \equiv 0 \Rightarrow \mathfrak{g}_1 \cap \mathfrak{n} \subset \mathfrak{g}_0$.

Now return to equation (3.4). There must be a linear functional $\tau: \mathfrak{g}_1 \to (\mathfrak{g}_1)_c$ such that

$$[W, Y+iZ] = \delta(W)(1-ir)(Y+iZ) + \tau(W).$$

But $Y + iZ \in \mathfrak{n}_c$, which is an ideal, implying $\tau(W) \in (\mathfrak{g}_1)_c \cap \mathfrak{n}_c = (\mathfrak{g}_1 \cap \mathfrak{n})_c \subset (\mathfrak{g}_0)_c$. Then, choosing $X \in \mathfrak{g}_1$ to satisfy $\delta(X) = 1$, we see that (3.1) and (3.2) are satisfied. Finally, for $W \in \mathfrak{g}_1$,

$$[W, [Y, Z]] = [[W, Y], Z] + [Y, [W, Z]] \equiv 2\delta(W)[Y, Z] \bmod \mathfrak{g}_1.$$

No such real eigenfunctional can exist (by the maximality of $\mathfrak{g}_1$), and therefore $[Y, Z] \equiv 0 \bmod \mathfrak{g}_1$. Invoking the inclusion $\mathfrak{n} \cap \mathfrak{g}_1 \subset \mathfrak{g}_0$ yet again, we see that (3.3) is also verified. □

We observe that $\mathfrak{g}_0$ and r are uniquely determined by $\mathfrak{g}_1$, but that the elements X, Y, Z are not. Clearly $\mathfrak{g}_1 = \mathfrak{g}_0 + \mathbf{R}X$. We set $\mathfrak{g}_2 = \mathfrak{g}_0 + \mathbf{R}Y$, $\mathfrak{g}_3 = \mathfrak{g}_0 + \mathbf{R}Z$, and $\mathfrak{g}_4 = \mathfrak{g}_0 + \mathbf{R}Y + \mathbf{R}Z$. $\mathfrak{g}_4$ is a codimension-1 ideal in $\mathfrak{g}$ (also uniquely determined by $\mathfrak{g}_1$). We write G, G_1, G_0, G_2, G_3, G_4 for the corresponding simply connected groups. Then we have the diagram shown as Figure 1, where each

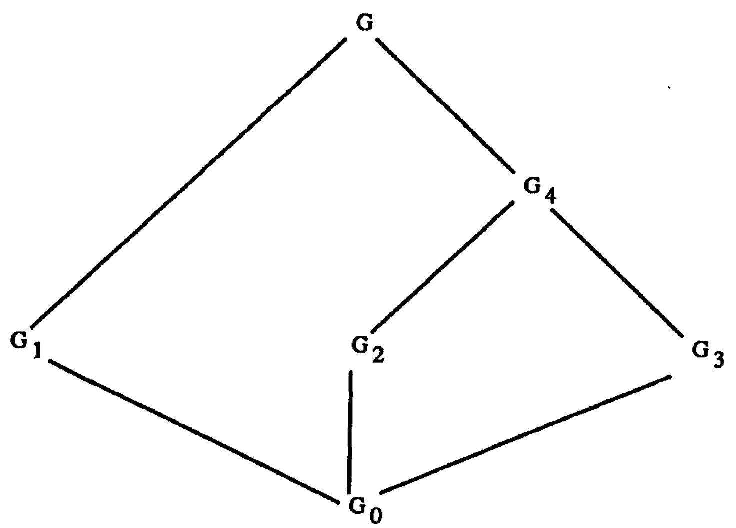

Figure 1

group is codimension 1 in any group lying on a line immediately above it (except codimension $G/G_1=2$). Moreover, both G_0 and G_4 are normal in G. (In fact it is easy to verify that $\mathfrak{g}_4$ is the only codimension-1 subalgebra containing $\mathfrak{g}_0$.) We continue to write $\delta \in \mathfrak{g}_0^\perp(\mathfrak{g}_1)$ and extend it to an element (also denoted δ) of $\mathfrak{g}_4^\perp(\mathfrak{g})$ by setting $\delta(Y)=\delta(Z)=0$. We shall also have occasion to utilize two linear functionals $\alpha, \beta \in \mathfrak{g}_1^\perp(\mathfrak{g})$ determined by $\alpha(Y)=\beta(Z)=1$, $\alpha(Z)=\beta(Y)=0$.

The reader might compare the structure developed here with that of codimension-1 maximal subgroups in [9]. We shall write $\mathfrak{g}_r$ for the Lie algebra $\mathfrak{g}/\mathfrak{g}_0$—that is, the 3-dimensional Lie algebra with generators X, Y, Z satisfying bracket relations $[X,Y]=Y+rZ$ and $[X,Z]=-rY+Z$.

4. Orbital and Mackey Theory in Codimension 2

We continue with the notation and terminology of Section 3. We are ready to enumerate the possibilities for the induced representation

$$\pi = \operatorname{Ind}_{G_1}^G \nu, \quad \nu \in \hat{G}_1.$$

We first fix a functional $\psi \in \mathcal{O}_\nu \subset \mathfrak{h}^*$. We also select a specific $\phi \in p_{\mathfrak{g},\mathfrak{g}_1}^{-1}(\psi)$ by requiring $\phi(Y)=\phi(Z)=0$. We set $\omega=\phi|_{\mathfrak{g}_4}$ and $\theta=\phi|_{\mathfrak{g}_0}$. Finally, we write $\gamma=\gamma_\theta \in \hat{G}_0$ and $\sigma=\sigma_\omega \in \hat{G}_4$ for the corresponding Kirillov–Bernat representations.

In the following theorem the key determining invariant will be the stability group G_γ. A special feature occurs when $G_\gamma=G_4$. Then $G_\gamma/G_0 \cong \mathbf{R}^2$ but there could be a Mackey obstruction to extending γ from G_0 to G_γ. It is well known (see [3]) that the obstruction to extending γ to G_4 is exactly the same as the obstruction to extending χ_θ from $(G_0)_\theta$ to $(G_4)_\theta$, and that is determined by whether the bilinear form $B_\theta(\cdot,\cdot)=\theta[\cdot,\cdot]$ is nondegenerate or not on $(\mathfrak{g}_4)_\theta/(\mathfrak{g}_0)_\theta$. With that in mind, we state our next theorem.

THEOREM 4.1. *One of the following seven mutually exclusive possibilities obtains:*

(i) $G_\gamma=G_0$. *Then* $G\cdot\phi \supset p^{-1}(G_1\cdot\psi)$ *and* $\pi=\pi_\phi$ *is irreducible.*

(ii) $G_0 \subset G_\gamma \subset G_4$, *each of codimension* 1. *Then the linear functionals* $\phi_s=\phi+s\alpha$, $s\in\mathbf{R}$, *lie in distinct G-orbits. These exhaust the G-orbits that meet* $p^{-1}(G_1\cdot\psi)$ *and*

$$\pi = \int_{\mathbf{R}}^{\oplus} \pi_{\phi_s}\, ds.$$

(iii) $\dim G_\gamma/G_0=1$, *but* $G_\gamma \cap G_1 = G_\gamma \cap G_4 = G_0$. *Choose* $\delta \in \mathfrak{g}_4^\perp(\mathfrak{g})$ *as in Section 3. Then the linear functionals* ϕ^u *uniquely determined by* $\phi^u=\phi+u\delta$, $u\in\mathbf{R}$, *lie in distinct G-orbits. These exhaust the G-orbits that meet* $p^{-1}(G_1\cdot\psi)$ *and*

$$\pi = \int_{\mathbf{R}}^{\oplus} \pi_{\phi^u}\, du.$$

(iv) $G_\gamma = G_1$. *Then, as in* (i), $G \cdot \phi \supset p^{-1}(G_1 \cdot \psi)$ *and* $\pi = \pi_\phi$ *is irreducible.*

(v) $G_\gamma = G_4$ *and* B_θ *is degenerate. Then the linear functionals* $\phi_{s,t} = \phi + s\alpha + t\beta$, $s, t \in \mathbf{R}$, *lie in distinct G-orbits. These exhaust the G-orbits that meet* $p^{-1}(G_1 \cdot \psi)$ *and*

$$\pi = \int_{\mathbf{R}^2}^{\oplus} \pi_{\phi_{s,t}}\, ds\, dt.$$

(vi) $G_\gamma = G_4$ *and* B_θ *is nondegenerate. Then* $G \cdot \phi$ *is the only G-orbit meeting* $p^{-1}(G_1 \cdot \psi)$, *but*

$$\pi = \infty \pi_\phi.$$

(vii) $G_\gamma = G$. *Then, as in* (v), *the G-orbits* $G \cdot \phi_{s,t}$ *exhaust the G-orbits that meet* $p^{-1}(G_1 \cdot \psi)$. *But there is a unique* $P_0 = (s_0, t_0)$ *such that* $\dim G \cdot \phi_{s_0, t_0} < \dim G \cdot \phi_{s,t}$, $(s,t) \neq P_0$. *The latter dimensions are constant. Moreover,* $\phi_{s,t}, \phi_{s',t'}$ *lie in the same orbit if and only if* $(s,t), (s',t')$ *lie on the same spiral centered at* P_0*—namely,*

$$P_0 + e^{\lambda(1-ir)}(s + it - P_0), \quad \lambda \in \mathbf{R}.$$

Thus, if $\mathcal{C}$ *denotes the unit circle centered at* P_0, *then*

$$\pi = \int_{\mathcal{C}}^{\oplus} \pi_{\phi_\zeta}\, d\zeta.$$

NOTE. There is a degree of similarity between Theorem 4.1 and [9, Thm. 3.3]. The first four cases correspond quite closely. The unusual case in [9, Thm. 3.3] is the last case (v). In Theorem 4.1 we have three new cases beyond the completely solvable or codimension-1 situation. Case (v) is entirely expected because of codimension 2. It is the last two cases that have no true analog in previously considered situations. Case (vi) yields infinite multiplicity, whereas codimension 1 never departs from multiplicity-free. And case (vii) mimics the prototypical 3-dimensional exponential solvable Lie group—exactly as the $(ax+b)$-group in codimension 1 yields the basic example of case (v) in [9, Thm. 3.3].

Proof. The seven possibilities for the stabilizer (together with the form B_θ) enumerated in the statement of the theorem are manifestly mutually distinct. We handle each case separately. In each case we verify the orbital facts by matching with the Mackey parameters. This differs from the proof of [9, Thm. 3.3], where we actually derived the orbital spectrum formula in each case. We could so that here also, but there is no need since one already knows [4] that the orbital spectrum formula is true. The point here of course is to understand the special features of a codimension-2 induction (from a maximal subgroup) and thereby to prove Theorem 1.1.

Throughout all of what follows we use both normal and codimension-1 theory extensively, that is, [8, Thm. 6.1], [9, Thms. 3.1 and 3.3], and [9, Lemma 2.4]. We shall do so repeatedly in application to the various extensions in Figure 1.

Case (i). $G_\gamma = G_0$. Then $\nu_\psi = \mathrm{Ind}_{G_0}^{G_1} \gamma_\theta$ and so $\pi = \mathrm{Ind}_{G_1}^{G} \nu_\psi = \mathrm{Ind}_{G_0}^{G} \gamma_\theta$ must be irreducible by the Mackey machine. The general orbital spectrum formula tells us that there is one orbit lying over $G_1 \cdot \psi$ generically, but it allows for the possibility of some degenerate orbits also. That nothing of the sort occurs is seen as follows: $\pi = \mathrm{Ind}_{G_0}^{G} \gamma_\theta = \mathrm{Ind}_{G_4}^{G}(\mathrm{Ind}_{G_0}^{G_4} \gamma_\theta)$. We know, from the normal theory applied to $\mathfrak{g}_0 \lhd \mathfrak{g}_4$ (see [9, Lemma 2.4]), that $\mathfrak{g}_\gamma = (\mathfrak{g}_4)_\gamma = \mathfrak{g}_0$, and so

$$(G_0)_\theta \cdot \omega = \omega + \mathfrak{g}_0^\perp(\mathfrak{g}_4).$$

Hence there is a unique G_4-orbit lying over $G_0 \cdot \theta$. Applying the normal theory again to $\mathfrak{g}_4 \lhd \mathfrak{g}$, we also see there is a unique G-orbit lying over $G_4 \cdot \omega$. Hence there is a unique G-orbit lying over $G_0 \cdot \theta$, and so there can be only one G-orbit lying over $G_1 \cdot \psi$. By the orbital spectrum formula we have

$$p^{-1}(G_1 \cdot \psi) = G_1 \cdot \phi.$$

Case (ii). $G_0 \subset G_\gamma \subset G_4$, each of codimension 1. Thus $\mathfrak{g}_\gamma = \mathfrak{g}_0 + y_0 Y + z_0 Z$ for some y_0, z_0 with $y_0^2 + z_0^2 \neq 0$. Replacing Y and Z (resp.) by $y_0 Y + z_0 Z$ and $y_0 Z - z_0 Y$ if necessary, we see it is no loss of generality to assume $\mathfrak{g}_\gamma = \mathfrak{g}_2$. Then $\mathfrak{g}_\theta = (\mathfrak{g}_2)_\theta$, so $(\mathfrak{g}_1)_\theta = (\mathfrak{g}_0)_\theta \Rightarrow \nu_\psi = \mathrm{Ind}_{G_0}^{G_1} \gamma_\theta$. Therefore

$$\pi = \mathrm{Ind}_{G_1}^{G} \nu_\psi = \mathrm{Ind}_{G_0}^{G} \gamma_\theta = \mathrm{Ind}_{G_2}^{G} \mathrm{Ind}_{G_0}^{G_2} \gamma_\theta = \mathrm{Ind}_{G_2}^{G} \int^{\oplus} \sigma_{\omega_s'}\, ds = \int^{\oplus} \mathrm{Ind}_{G_2}^{G} \sigma_{\omega_s'}\, ds,$$

where $\omega_s' \in \mathfrak{g}_2^*$, $\omega_s'|_{\mathfrak{g}_0} = \theta$, and $\omega_s'(Y) = s$. It follows from Mackey theory applied to $G_0 \lhd G$ that the constituents of the above direct integral decomposition are irreducible and pairwise inequivalent. Now the chain $\mathfrak{g}_2 \lhd \mathfrak{g}_4 \lhd \mathfrak{g}$ and the codimension-1 theory applied successively says that if we extend ω_s' to $\mathfrak{g}_4^*$ by sending Z to 0, and then to $\mathfrak{g}^*$ by sending X to $\psi(X)$, we obtain exactly the functionals $\phi_s = \phi + s\alpha$. They must lie in distinct orbits, and we have

$$\pi = \int^{\oplus} \pi_{\phi_s}\, ds.$$

By the orbital spectrum formula we know that, for generic s,

$$G \cdot \phi_s \cap p^{-1}(G_1 \cdot \psi) = G_1 \cdot \phi_s.$$

We prove it is true for all s. First we show no other orbits lie over $G_1 \cdot \psi$. Suppose $\tilde{\phi} \in p^{-1}(G_1 \cdot \psi)$, so $\phi' = g_1 \cdot \tilde{\phi}$ restricts to ψ on $\mathfrak{g}_1$. Let $s_0 = \phi'(Y)$. Then $\phi'|_{\mathfrak{g}_0} = \theta$ and the fact that $(G_0)_\theta \cdot \phi_{s_0} = \phi_{s_0} + \mathfrak{g}_2^\perp$ (from [9, Lemma 2.4]) shows we can select $g_0 \in (G_0)_\theta$ such that $g_0 \cdot \phi_{s_0} = \phi'$. So only the G-orbits $G \cdot \phi_s$ lie over $G_1 \cdot \psi$.

Next we verify that $G \cdot \phi_s \cap p^{-1}(G_1 \cdot \psi) = G_1 \cdot \phi_s$ for *all* s. Of course the right side is contained in the left. Conversely, if $\phi' = g \cdot \phi_s \in p^{-1}(G_1 \cdot \psi)$ then $g_1 \cdot \phi'|_{\mathfrak{g}_1} = \psi$ implies (as above, modifying g_1 by an element of $(G_0)_\theta$ if necessary) that $g_1 \cdot \phi' = \phi_{s_0}$ for some s_0. That is,

$$g_1 \cdot \phi' = g_1 g \cdot \phi_s = \phi_{s_0}.$$

The distinctness of the orbits implies $s = s_0$.

Case (iii). $\dim G_\gamma/G_0=1$ but $G_\gamma\cap G_1=G_\gamma\cap G_4=G_0$. Then $\mathfrak{g}_\gamma=\mathfrak{g}_0+x_0X+y_0Y+z_0Z$ with $x_0(y_0^2+z_0^2)\neq 0$. Again $(\mathfrak{g}_1)_\theta=(\mathfrak{g}_0)_\theta$ implies that $\pi=\operatorname{Ind}_{G_1}^G \nu_\psi=\operatorname{Ind}_{G_0}^G\gamma_\theta$. But this time we have $(\mathfrak{g}_4)_\theta=(\mathfrak{g}_0)_\theta$ also, so $\operatorname{Ind}_{G_0}^{G_4}\gamma_\theta$ is irreducible; write $\sigma_\omega=\operatorname{Ind}_{G_0}^{G_4}\gamma_\theta$, where $\omega\in\mathfrak{g}_4^*$ is defined by $\omega|_{\mathfrak{g}_0}=\theta$, $\omega(Y)=\omega(Z)=0$. Now π is not irreducible (since $G_\gamma\neq G_0$), so by the codimension-1 theory (applied to $\mathfrak{g}_4\lhd\mathfrak{g}$) we must have

$$\pi=\operatorname{Ind}_{G_4}^G\sigma_\omega=\int_{\mathbf{R}}^{\oplus}\pi_{\phi+u\delta}\,du,$$

where $\delta\in\mathfrak{g}_4^\perp(\mathfrak{g})$, $\delta(X)=1$, and ϕ as usual is given by $\phi|_{\mathfrak{g}_1}=\psi$, $\phi(Y)=\phi(Z)=0$. In particular, the functionals $\phi^u=\phi+u\delta$ parameterize the distinct G-orbits that lie over $G_0\cdot\theta$.

Next we show that the orbits $G\cdot\phi^u$, $\phi^u=\phi+u\delta$, also parameterize the G-orbits that lie over $G_1\cdot\psi$. Set $\phi_{s,t}=\phi+s\alpha+t\beta$, $s,t\in\mathbf{R}$. Clearly $p^{-1}(\psi)=\{\phi_{s,t}: s,t\in\mathbf{R}\}$. Thus we must prove that, for all s,t, there exists a u such that $G\cdot\phi_{s,t}=G\cdot\phi^u$. We must solve the equation

$$g\cdot(\phi+s\alpha+t\beta)=\phi+u\delta. \tag{4.0}$$

In order to do so, I recall from the $\mathfrak{g}_0\lhd\mathfrak{g}$ theory that one must have (by [9, Lemma 2.4])

$$(G_0)_\theta\cdot\phi_{s,t}=\phi_{s,t}+\mathfrak{g}_\gamma^\perp. \tag{4.1}$$

Claim: $G_\theta\cdot\phi_{s,t}=(G_0)_\theta\cdot\phi_{s,t}$. In fact, it is enough to show by (4.1) that $\dim\mathfrak{g}_\theta/\mathfrak{g}_{\phi_{s,t}}=2$. The reasoning does not depend on s,t, so we verify that $\dim\mathfrak{g}_\theta/\mathfrak{g}_\phi=2$. We know (see [9, Thm. 3.1]) that

$$\dim\mathfrak{g}\cdot\phi=\dim\mathfrak{g}_4\cdot\omega=\dim\mathfrak{g}_0\cdot\theta+4\Rightarrow\dim(\mathfrak{g}_0)_\theta=\dim\mathfrak{g}_4+1.$$

But we also know that $\dim\mathfrak{g}_\theta/(\mathfrak{g}_0)_\theta=1$. Combining these we obtain the claim. Now, upon restriction to $\mathfrak{g}_0$, equation (4.0) requires that $g\in G_\theta$. Therefore

$$\phi_{s,t}-\phi-u\delta\in\mathfrak{g}_\gamma^\perp,$$

which forces

$$ux_0=sy_0+tz_0. \tag{4.2}$$

In fact, reading backwards, we see that if given any s,t we choose u to satisfy (4.2), then because $G_\theta\cdot\phi_{s,t}=\phi_{s,t}+\mathfrak{g}_\gamma^\perp$ we have

$$G_\theta\cdot\phi_{s,t}=G_\theta\cdot\phi^u=(G_0)_\theta\cdot\phi^u. \tag{4.3}$$

Finally we demonstrate that for all u we have

$$G\cdot\phi^u\cap p^{-1}(G_1\cdot\psi)=G_1\cdot\phi^u.$$

The inclusion $\supset$ follows from (4.3). The reverse inclusion comes about as follows. Let $\phi'=g\cdot\phi^u\in p^{-1}(G_1\cdot\psi)$. Then $g_1g\cdot\phi^u|_{\mathfrak{g}_1}=\psi$. Hence $g_1g\cdot\phi^u=\phi+s\alpha+t\beta$ for some s,t. But the above reasoning shows that then we must have $ux_0=sy_0+tz_0$. This in turn forces the existence of $g_0\in(G_0)_\theta$ such that $g_1g\cdot\phi^u=g_0\cdot\phi^u\Rightarrow\phi'\in G_1\cdot\phi^u$.

Case (iv). $G_\gamma = G_1$. Then $\mathfrak{g}_\theta = (\mathfrak{g}_1)_\theta \supsetneqq (\mathfrak{g}_0)_\theta$. Therefore ν_ψ is not induced, but rather $\nu_\psi|_{G_0} = \gamma_\theta$. Therefore

$$\pi|_{G_4} = (\mathrm{Ind}_{G_1}^{G} \nu_\psi)|_{G_4} = \mathrm{Ind}_{G_0}^{G_4}(\nu_\psi|_{G_0}) = \mathrm{Ind}_{G_0}^{G_4} \gamma_\theta = \sigma_\omega$$

is irreducible because $(\mathfrak{g}_4)_\theta = (\mathfrak{g}_0)_\theta$. Thus π itself must be irreducible. Applying the codimension-1 theory to $\mathfrak{g}_4 \lhd \mathfrak{g}$, we see that

$$\pi|_{G_4} = \sigma_\omega \Rightarrow \pi = \pi_{\phi + u\delta} \text{ for some } u \in \mathbf{R}.$$

In fact $u = 0$. To see this, we can generalize the argument of [9, Thm. 3.1, case (iv)]. Let $\mathfrak{b}$ be a real polarization for ψ satisfying the Pukanszky condition. We have $\psi[\mathfrak{b}, \mathfrak{b}] = 0$ and

$$(\mathfrak{g}_1)_\psi \subset \mathfrak{b} \subset \mathfrak{g}_1, \qquad \dim \mathfrak{g}_1/\mathfrak{b} = \dim \mathfrak{b}/(\mathfrak{g}_1)_\psi.$$

We claim that $\mathfrak{b}$ is also a real polarization for ϕ satisfying Pukanszky. Again using the normal theory, we have

$$\dim \mathfrak{g}\cdot\phi = \dim \mathfrak{g}_4\cdot\omega = \dim(\mathfrak{g}_0)_\theta + 4 = \dim \mathfrak{g}_1\cdot\psi + 4.$$

Also,

$$\mathfrak{g}_\phi \subset \mathfrak{g}_\theta = (\mathfrak{g}_1)_\theta \Rightarrow \mathfrak{g}_\phi \subset (\mathfrak{g}_1)_\psi.$$

Therefore,

$$\dim \mathfrak{g} - \dim \mathfrak{g}_\phi = \dim \mathfrak{g}_1 - \dim(\mathfrak{g}_1)_\psi + 4$$

and

$$\dim \mathfrak{g} = \dim \mathfrak{g}_1 + 2$$

yield

$$\dim(\mathfrak{g}_1)_\psi/\mathfrak{g}_\phi = 2.$$

Then $\mathfrak{b}$ is a real polarization for ϕ. (Note: it is not a real polarization for ϕ^u if $u \neq 0$.) Next we show that $\mathfrak{b}$ satisfies Pukanszky. We know $B\cdot\phi \subset \phi + \mathfrak{b}^\perp(\mathfrak{g})$. Let $\zeta \in \mathfrak{b}^\perp(\mathfrak{g})$. Then $\zeta_1 = \zeta|_{\mathfrak{g}_1} \in \mathfrak{b}^\perp(\mathfrak{g}_1)$. Since $\mathfrak{b}$ satisfies Pukanszky for ψ, there must exist $b \in B$ such that $b\cdot\psi = \psi + \zeta_1$. Suppose $b\cdot\phi = \phi + \zeta'$. Then

$$\zeta - \zeta' = s\alpha + t\beta \quad \text{for some } s, t \in \mathbf{R}.$$

Set $\phi_1 = b\cdot\phi$, $\psi_1 = \phi_1|_{\mathfrak{g}_1}$, and $\theta_1 = \phi_1|_{\mathfrak{g}_0}$. Then

$$(G_0)_{\theta_1}\cdot\phi_1 = \phi_1 + \mathfrak{g}_1^\perp.$$

We also know ($\mathfrak{g}_0 \lhd \mathfrak{g}_1$) that $(\mathfrak{g}_1)_{\psi_1} = (\mathfrak{g}_1)_{\theta_1} \Rightarrow (G_0)_{\theta_1} \subset (G_1)_{\psi_1} \subset B$ (since $\mathfrak{b}$ is also a real polarization for ψ_1 [1, p. 69]). Hence there exists $g_1 \in (G_1)_{\psi_1} \subset B$ such that $g_1\cdot\phi_1 = \phi_1 + s\alpha + t\beta$. Thus

$$\begin{aligned} g_1 b\cdot\phi &= g_1\cdot\phi_1 = \phi_1 + s\alpha + t\beta = b\cdot\phi + s\alpha + t\beta \\ &= \phi + \zeta' + s\alpha + t\beta \\ &= \phi + \zeta. \end{aligned}$$

This verifies the Pukanszky condition, and therefore

$$\pi = \mathrm{Ind}_{G_1}^{G} \nu_\psi = \mathrm{Ind}_{G_1}^{G} \mathrm{Ind}_{B}^{G_1} \chi_\psi = \mathrm{Ind}_{B}^{G} \chi_\phi = \pi_\phi.$$

Now we show that $G\cdot\phi$ is the only orbit over ψ, and in that case

$$G\cdot\phi\cap p^{-1}(G_1\cdot\psi)=G_1\cdot\phi$$

follows automatically from the truth of the orbital spectrum formula (although as in (i), it is very easy to demonstrate this directly). Suppose $\phi'\in p^{-1}(G_1\cdot\psi)$. Then $g_1\cdot\phi'|_{\mathfrak{g}_1}=\psi \Rightarrow g_1\cdot\phi'=\phi+s\alpha+t\beta$. But $(G_0)_\theta\cdot\phi=\phi+\mathfrak{g}_1^\perp \Rightarrow g_1\cdot\phi'=g_0\cdot\phi$ for some $g_0\in G_0\subset G_1$.

Case (v). $G_\gamma=G_4$ and B_θ is degenerate. Then $(\mathfrak{g}_1)_\theta=(\mathfrak{g}_0)_\theta$ implies, as in cases (i)–(iii), that

$$\pi=\operatorname{Ind}_{G_1}^G\nu_\psi=\operatorname{Ind}_{G_0}^G\gamma_\theta.$$

But since we are assuming that B_θ is degenerate, the representation γ_θ has an extension to G_4 as an ordinary representation. Thus, if we set $\omega\in p_{\mathfrak{g}_4,\mathfrak{g}_0}^{-1}(\theta)$, $\omega(Y)=\omega(Z)=0$, and $\omega_{s,t}=\omega+s\alpha+t\beta$, then clearly

$$\operatorname{Ind}_{G_0}^{G_4}\gamma_\theta=\int_{\mathbf{R}^2}^{\oplus}\sigma_{\omega_{s,t}}\,ds\,dt;$$

in fact,

$$\sigma_{\omega_{s,t}}=\sigma_\omega\chi_{s,t},$$

where $\chi_{s,t}$ is the unique character of G_4 trivial on G_0 and satisfying

$$\chi_{s,t}(e^{yY}e^{zZ})=e^{i(sy+tz)}.$$

Moreover, by Mackey theory we must have that $\operatorname{Ind}_{G_4}^G\sigma_{\omega_{s,t}}$ is irreducible. Thus $\phi_{s,t}=\phi+s\alpha+t\beta$ parameterizes the distinct G-orbits, and

$$\pi=\int_{\mathbf{R}^2}^{\oplus}\pi_{\phi_{s,t}}\,ds\,dt.$$

We have clearly accounted for all the G-orbits over ψ. We conclude this case by verifying that $G\cdot\phi_{s,t}\cap p^{-1}(G_1\cdot\psi)=G_1\cdot\phi_{s,t}$ for all s,t. One inclusion is obvious. As for the reverse, if $\phi'=g\cdot\phi_{s,t}\in p^{-1}(G_1\cdot\psi)$ then

$$\begin{aligned}g_1\cdot\phi'|_{\mathfrak{g}_1}=\psi &\Rightarrow g_1\cdot\phi'=\phi_{s_0,t_0} \text{ for some } s_0,t_0\in\mathbf{R}\\ &\Rightarrow \phi_{s_0,t_0}=g_1g\cdot\phi_{s,t}\Rightarrow s=s_0 \text{ and } t=t_0\end{aligned}$$

(since the orbits are all distinct).

Case (vi). $G_\gamma=G_4$ but B_θ is nondegenerate. The Mackey machine says, because the projective irreducible representation of $\mathbf{R}^2$ is infinite-dimensional, that

$$\operatorname{Ind}_{G_0}^{G_4}\gamma_\theta=\infty\sigma_\omega.$$

Therefore

$$\pi=\operatorname{Ind}_{G_1}^G\nu_\psi=\operatorname{Ind}_{G_0}^G\gamma_\theta=\operatorname{Ind}_{G_4}^G\infty\sigma_\omega=\infty\operatorname{Ind}_{G_4}^G\sigma_\omega,$$

an infinite multiple of an irreducible representation. Since $\phi\in p_{\mathfrak{g}_4,\mathfrak{g}}^{-1}(\omega)$, it must be that $\operatorname{Ind}_{G_4}^G\sigma_\omega=\pi_\phi$ and so $\pi=\infty\pi_\phi$. We must verify that $G\cdot\phi$ is the only orbit over $G_1\cdot\phi$. This follows from known facts—namely: by the projective representation theory of $\mathbf{R}^2$, $G_4\cdot\omega$ is the only orbit over $G_0\cdot\theta$; then, by codimension-1 theory, $G\cdot\phi$ is the only orbit over $G_4\cdot\omega$; and, as in case (i), this implies that $G\cdot\phi$ is the only orbit over $G_1\cdot\psi$. Since the orbital spectrum formula is known *a priori*, the desired multiplicity equation must *a fortiori* be true. However, unlike cases (i) and (iv), here I will actually give an independent proof (which in fact I will use later). Namely, I prove that

$$\dim(G\cdot\phi\cap p^{-1}(G_1\cdot\psi))>\dim G_1\cdot\phi.$$

By [8] (generic is no problem since there is only one orbit), this is equivalent to showing that

$$\dim \mathfrak{g}\cdot\phi+\dim \mathfrak{g}_1\cdot\psi>2\dim \mathfrak{g}_1\cdot\phi.$$

We know

$$\dim \mathfrak{g}_1\cdot\psi=\dim \mathfrak{g}_0\cdot\theta+2=\dim \mathfrak{g}_4\cdot\omega=\dim \mathfrak{g}\cdot\phi-2.$$

In particular,

$$\begin{aligned}\dim \mathfrak{g}_0-\dim(\mathfrak{g}_0)_\theta&=\dim \mathfrak{g}-\dim \mathfrak{g}_\phi-4\\&=\dim \mathfrak{g}_0-\dim \mathfrak{g}_\phi-1\\&\Rightarrow \dim(\mathfrak{g}_0)_\theta=\dim \mathfrak{g}_\phi+1.\end{aligned}$$

But

$$(G_0)_\theta\cdot\phi=\phi+\mathfrak{g}_4^\perp\Rightarrow \dim(\mathfrak{g}_0)_\theta-\dim(\mathfrak{g}_0)_\phi=1.$$

Hence $\mathfrak{g}_\phi=(\mathfrak{g}_0)_\phi$. But

$$\begin{aligned}\mathfrak{g}_\phi=(\mathfrak{g}_0)_\phi\Rightarrow(\mathfrak{g}_1)_\phi=\mathfrak{g}_\phi\Rightarrow \dim \mathfrak{g}_1\cdot\phi&=\dim \mathfrak{g}_1-\dim(\mathfrak{g}_1)_\theta\\&=\dim \mathfrak{g}-2-\dim \mathfrak{g}_\phi\\&=\dim \mathfrak{g}\cdot\phi-2.\end{aligned}$$

Thus

$$\begin{aligned}\dim \mathfrak{g}\cdot\phi+\dim \mathfrak{g}_1\cdot\psi&=2\dim \mathfrak{g}\cdot\phi-2\\&>2\dim \mathfrak{g}\cdot\phi-4\\&=2\dim \mathfrak{g}_1\cdot\phi.\end{aligned}$$

Case (vii). $G_\gamma=G$. In this case, the quotients $\mathfrak{g}_\theta/(\mathfrak{g}_0)_\theta$, $(\mathfrak{g}_1)_\theta/(\mathfrak{g}_0)_\theta$, and $(\mathfrak{g}_4)_\theta/(\mathfrak{g}_0)_\theta$ have dimensions 3, 1, and 2, respectively. In particular,

$$\pi|_{G_4}=\operatorname{Ind}_{G_1}^{G}\nu_\psi|_{G_4}=\operatorname{Ind}_{G_0}^{G_4}\gamma_\theta, \tag{4.4}$$

but unlike case (iv) the latter is not irreducible. We structure the ensuing argument somewhat like the case (v) of [9]. To begin, though, we need to rule out a splitting of the case (as occurs when $\mathfrak{g}_\gamma=\mathfrak{g}_4$).

THEOREM 4.2. *G/G_0 supports no co-cycles. In particular, there is no obstruction to extending γ_θ to G.*

Proof. In fact, there does not exist a nonsplit central 1-dimensional extension of $\mathfrak{g}_r$:

$$0\to\mathbf{R}\to\tilde{\mathfrak{g}}\to\mathfrak{g}_r\to 0.$$

If so, the Lie algebra $\tilde{\mathfrak{g}}$ would have generators T, X, Y, Z satisfying bracket relations: T is central and

$$[X,Y]=Y+rZ+t_1T,$$
$$[X,Z]=-rY+Z+t_2T,$$
$$[Y,Z]=t_3T.$$

If $t_3\neq 0$ then the Jacobi identity applied to X, $[Y,Z]$ forces $[X,T]=2T$. This contradicts the centrality of T, so $t_3=0$. Then, replacing Y and Z by $Y+\tau_1T$ and $Z+\tau_2T$, where $\tau_1=(t_1-rt_2)/(r^2+1)$ and $\tau_2=(rt_1+t_2)/(r^2+1)$, we can split the extension. □

Continuing with the argument for the proof of Theorem 4.1, we know that γ_θ must extend to an ordinary representation of G; hence it extends to an ordinary representation on G_4, whence from (4.4) we have

$$\pi|_{G_4}=\int_{\mathbf{R}^2}^{\oplus}\sigma_{\omega_{s,t}}\,ds\,dt, \tag{4.5}$$

with $\omega_{s,t}$ as in case (v). Now we examine which of the representations $\operatorname{Ind}_{G_4}^{G}\sigma_{\omega_{s,t}}$ are irreducible and the equivalences among them. First we claim that there is a unique point $P_0=(s_0,t_0)$ such that $\operatorname{Ind}_{G_4}^{G}\sigma_{\omega_{s,t}}$ is irreducible if and only if $(s,t)\neq P_0$. In fact, we need to see when $\mathfrak{g}_{\omega_{s,t}}=(\mathfrak{g}_4)_{\omega_{s,t}}$. First we have $(\mathfrak{g}_4)_\theta=(\mathfrak{g}_4)_{\omega_{s,t}}$. This is because of the equations

$$\dim\mathfrak{g}_4\cdot\omega_{s,t}=\dim\mathfrak{g}_0\cdot\theta \quad\text{and}\quad \mathfrak{g}_4=(\mathfrak{g}_4)_\theta+\mathfrak{g}_0,$$

which imply that

$$\dim\mathfrak{g}_4/(\mathfrak{g}_4)_\theta=\dim\mathfrak{g}_0/(\mathfrak{g}_0)_\theta=\dim\mathfrak{g}_4/(\mathfrak{g}_4)_{\omega_{s,t}}.$$

Hence $(\mathfrak{g}_4)_{\omega_{s,t}}=(\mathfrak{g}_4)_\theta$. Therefore

$$\mathfrak{g}_{\omega_{s,t}}\subset\mathfrak{g}_\theta=(\mathfrak{g}_1)_\theta+(\mathfrak{g}_4)_\theta=(\mathfrak{g}_1)_\theta+(\mathfrak{g}_4)_{\omega_{s,t}}.$$

We examine $\mathfrak{g}_{\omega_{s,t}}\cap(\mathfrak{g}_1)_\theta$. There exist $a\neq 0$ and $W\in\mathfrak{g}_0$ such that $aX+W\in(\mathfrak{g}_1)_\theta$. Then

$$[aX+W,Y]=a(Y+rZ)+U_1 \quad \text{for some } U_1\in\mathfrak{g}_0;$$
$$[aX+W,Z]=a(-rY+Z)+U_2 \quad \text{for some } U_2\in\mathfrak{g}_0.$$

Set $w_j=\theta(U_j)$. Then

$$\begin{aligned}\omega_{s,t}[aX+W,\mathfrak{g}_4]\supset\omega_{s,t}&[aX+W,\operatorname{sp}\{Y,Z\}]\\ &=0\Leftrightarrow a(s+rt)+w_1=a(-rs+t)+w_2=0\\ &\Leftrightarrow(s,t)=\frac{-a^{-1}}{1+r^2}(w_1-rw_2,rw_1+w_2)\\ &\Leftrightarrow s+it=\frac{-a^{-1}w}{1+ir},\ w=w_1+iw_2.\end{aligned}$$

Define $P_0=(-a^{-1}w)/(1-ir)$. Then, using $\mathfrak{g}=\mathbf{R}(aX+W)+\mathfrak{g}_4$, $\mathfrak{g}_4=\mathfrak{g}_0+\mathbf{R}Y+\mathbf{R}Z$, and $aX+W\in(\mathfrak{g}_1)_\theta$ we see:

$$\text{if } (s,t)=P_0 \text{ then } \omega_{s,t}[aX+W,\mathfrak{g}_4]=0 \Rightarrow \mathfrak{g}_{\omega_{s,t}} \supsetneq (\mathfrak{g}_4)_{\omega_{s,t}};$$
$$\text{if } (s,t)\neq P_0 \text{ then } \mathfrak{g}_{\omega_{s,t}}=(\mathfrak{g}_4)_{\omega_{s,t}}.$$

Therefore

$$\operatorname{Ind}_{G_4}^G \sigma_{\omega_{s,t}} \text{ is irreducible} \Leftrightarrow (s,t)\neq P_0.$$

The resulting representations are clearly $\pi_{\phi_{s,t}}$, $\phi_{s,t}|_{\mathfrak{g}_4}=\omega_{s,t}$, and $\phi|_{\mathfrak{g}_1}=\psi$. Two of these, say $\phi_{s,t}$ and $\phi_{s',t'}$, will be equivalent precisely when there exist $g\in G \ni g\cdot\phi_{s,t}=\phi_{s',t'}$. Alternatively, we can characterize when the equation $g\cdot\omega_{s,t}=\omega_{s',t'}$ is satisfied. And for that, since

$$G=\exp \mathbf{R}(aX+W)G_4,$$

it is enough to examine the action of $\exp\lambda(aX+W)$, $\lambda\in\mathbf{R}$, on $\omega_{s,t}$. For this we continue the use of complex notation. We know that

$$\exp\lambda(aX+W)\cdot\omega_{s,t}|_{\mathfrak{g}_0}=\omega_{s,t}|_{\mathfrak{g}_0}=\theta.$$

Also, we know

$$[aX+W,Y+iZ]=a(1-ir)(Y+iZ)+U, \quad U=U_1+iU_2.$$

Therefore

$$\omega_{s,t}[aX+W,Y+iZ]=a(1-ir)(s+it)+w.$$

Exponentiating and setting $g_\lambda=\exp\{-\lambda(aX+W)\}$, we obtain

$$g_\lambda\cdot\omega_{s,t}|_{\mathfrak{g}_0}=\theta \quad \text{and}$$
$$g_\lambda\cdot\omega_{s,t}(Y+iZ)=e^{\lambda a(1-ir)}(s+it)+\frac{e^{\lambda a(1-ir)}-1}{a(1-ir)}w.$$

Thus

$$g_\lambda\cdot\omega_{s,t}=\omega_{s,t}\Leftrightarrow s+it=\frac{-w}{a(1-ir)}=P_0,$$

and obviously the orbits are spirals around the point P_0: if we set $\sigma+i\tau=s+it-P_0$ then its orbit is

$$P_0+e^{\lambda a(1-ir)}(\sigma+i\tau), \quad \lambda\in\mathbf{R}.$$

It follows from (4.5) and [6] that

$$\pi=\int_{\mathfrak{C}}^{\oplus}\pi_{\phi_\zeta}\,d\zeta,$$

where $\mathfrak{C}$ is the unit circle in $\mathbf{C}$, $\phi_\zeta|_{\mathfrak{g}_1}=\psi$, and $\phi_\zeta(Y+iZ)=P_0+\zeta$ (or alternatively $\pi_{\phi_\zeta}=\operatorname{Ind}_{G_4}^G\omega_{P_0+\zeta}$).

Now we turn to orbital parameters and the multiplicity formula. Since the representations π_{ϕ_ζ}, $\zeta\in\mathfrak{C}$, are inequivalent (by the above Mackey theory), they lie in distinct orbits. It is clear that the only orbits lying above ψ are $G\cdot\phi_{P_0}$ and $G\cdot\phi_\zeta$, $\zeta\in\mathfrak{C}$. The latter are generic, which is further corroborated by their dimensions:

$$\dim \mathfrak{g}_0 \cdot \theta = \dim \mathfrak{g}_4 \cdot \omega_{s,t} \quad \text{for all } s, t;$$

$$= \begin{cases} \dim \mathfrak{g} \cdot \phi_\zeta - 2, & \zeta \in \mathcal{C}; \\ \dim \mathfrak{g} \cdot \phi_{P_0}. \end{cases}$$

Furthermore, we have

$$G \cdot \phi_\zeta \cap p^{-1}(G_1 \cdot \psi) = G_1 \cdot \phi_\zeta, \quad \zeta \in \mathcal{C}.$$

Indeed, if $\phi' = g \cdot \phi_\zeta \in p^{-1}(G_1 \cdot \psi)$, then $g_1 \cdot \phi'|_{\mathfrak{g}_1} = \psi$. Thus $g_1 \cdot \phi' = \phi + \zeta' \varpi$, $\varpi = \frac{1}{2}(\alpha - i\beta)$, for some $\zeta' \in \mathbf{C}$. Also,

$$\phi_\zeta = \phi + (P_0 + \zeta)\varpi.$$

But this says that ϕ_ζ and $\phi + \zeta'\varpi = g_1 \cdot \phi' = g_1 g \cdot \phi_\zeta$ are in the same orbit. Previously we saw that the orbits are swept out by elements of $(G_1)_\theta$. Hence there exists $g_1' \in (G_1)_\theta$ such that

$$g_1 \cdot \phi' = \phi + \zeta'\varpi = g_1' \cdot \phi_\zeta \Rightarrow \phi' \in G_1 \cdot \phi_\zeta.$$

Finally, we verify that the same formula obtains for the degenerate orbit—namely,

$$G \cdot \phi_{P_0} \cap p^{-1}(G_1 \cdot \psi) = G_1 \cdot \phi_{P_0}.$$

Let $g \cdot \phi_{P_0} \in p^{-1}(G_1 \cdot \psi)$ for some $g \in G$. Then there exists $g_1 \in G_1 \ni g_1 g \cdot \phi_{P_0}|_{\mathfrak{g}_1} = \psi$. In particular, $g_1 g \cdot \phi_{P_0} = \phi + \zeta\varpi$ for some $\zeta \in \mathbf{C}$. But any orbit $G \cdot \phi_\zeta = G \cdot (\phi_{P_0} + \zeta\varpi)$ is disjoint from $G \cdot \phi_{P_0}$ for $\zeta \neq 0$. Hence $g_1 g \cdot \phi_{P_0} = \phi_{P_0}$. That is, $g \cdot \phi_{P_0} \in G_1 \cdot \phi_{P_0}$. □

This completes the proof of Theorem 4.1. In fact we have done more. A review of each of the seven cases reveals that we have actually proven completely items (i) and (iii) in Theorem 1.1 for codimension-2 maximal subgroups. Having observed that, we now attend to the remaining item (ii).

Proof of Theorem 1.1. We already observed in Remark 1.2(1) that the maximality of H in an exponential solvable group G forces $\dim G/H$ to be 1 or 2. The theorem is proven for codimension 1 in Section 2 (relying of course heavily on [9]). As just observed, for codimension 2, parts (i) and (iii) of the theorem are demonstrated within the proof of Theorem 4.1. It remains to prove part (ii); that is, that the CG number

$$\dim G \cdot \phi - 2 \dim H \cdot \phi + \dim \mathcal{O}_\nu$$

is constant on $p^{-1}(\mathcal{O}_\nu)$, and that its vanishing characterizes finite multiplicity. It turns out to be more difficult than the corresponding proof in codimension 1 (Section 2). Originally, we gave a non-case-by-case argument in analogy with Proposition 2.3. But as the referee pointed out, there was a gap in the argument. I shall convey it here as far as it is accurate, and then complete the proof with case-by-case reasoning. We begin my mimicking the arguments in Section 2 as far as possible. (To utilize the notation of Theorem 4.1, we again replace $\mathfrak{h}$ by $\mathfrak{g}_1$.)

Fix $\psi \in \mathfrak{g}_1^*$ and let $\phi \in p^{-1}(G_1 \cdot \psi)$ be any linear functional in the pullback of the orbit. Lemma 2.1 says, since $\dim \mathfrak{g}/\mathfrak{g}_1 = 2$, that

$$\dim G \cdot \phi - 2 \dim G_1 \cdot \phi + \dim G_1 \cdot \psi = 0$$
$$\Leftrightarrow 2 - (\dim \mathfrak{g}_\phi/(\mathfrak{g}_1)_\phi + \dim(\mathfrak{g}_1)_\psi/(\mathfrak{g}_1)_\phi) = 0.$$

Of course both expressions to the left of the equality signs can only be nonnegative. The possibilities for the dimensions of $\mathfrak{g}_\phi/(\mathfrak{g}_1)_\phi$ and $(\mathfrak{g}_1)_\psi/(\mathfrak{g}_1)_\phi$ are obviously 0, 1, or 2. Furthermore, the analog of Proposition 2.3(b) clearly yields that

$$2 - \dim \mathfrak{g}_\phi + \dim(\mathfrak{g}_1)_\psi \text{ is even;}$$

a consequence of the fact that $\dim \mathfrak{g}/\mathfrak{g}_\phi - \dim \mathfrak{g}_1/(\mathfrak{g}_1)_\psi$ can only be 0, 2, or 4. The corollary of all these conclusions is that the following table of values encompasses the only possibilities.

	$\dim \mathfrak{g}_\phi/(\mathfrak{g}_1)_\phi$	$\dim(\mathfrak{g}_1)_\psi/(\mathfrak{g}_1)_\phi$	$2-(\dim \mathfrak{g}_\phi/(\mathfrak{g}_1)_\phi + \dim(\mathfrak{g}_1)_\psi/(\mathfrak{g}_1)_\phi)$
(1)	2	0	0
(2)	1	1	0
(3)	0	2	0
(4)	0	0	2

The gap in the original argument ignored the theoretical possibility that generically we could be in row (3), for example, and have a degenerate functional whose $\mathfrak{g}$ and $\mathfrak{g}_1$ stabilizers each increase in dimension by 2—thereby bouncing us to row (4). (Or generically we are in row (2) and have a degenerate functional whose $\mathfrak{g}$ stabilizer has unchanged dimension, but whose $\mathfrak{g}_1$ stabilizer increased in dimension by 1.) We proceed to a case-by-case analysis which shows those eventualities do not occur.

Once again, we must show that for $\psi \in \mathfrak{g}_1^*$, the numbers

$$n(\psi, \phi) = \dim G \cdot \phi - 2 \dim G_1 \cdot \phi + \dim G_1 \cdot \psi$$

are constant on $p^{-1}(G_1 \cdot \psi)$, equal 2 in case (vi) of Theorem 4.1, and equal 0 in every other case. In case (vi) we know that only one orbit $G \cdot \phi$ lies over $G_1 \cdot \psi$, and from the additional computation in part (vi) of the proof of Theorem 4.1, we know that $n(\psi, \phi) = 2$. For the remaining six cases we employ the reasoning of [8, Prop. 3.4]. In every case, we know that for any $\phi \in p^{-1}(G_1 \cdot \psi)$ we have

$$G \cdot \phi \cap p^{-1}(G_1 \cdot \psi) = G_1 \cdot \phi.$$

Therefore

$$\begin{aligned} \dim \mathfrak{g}_1 \cdot \phi &= \dim G_1 \cdot \phi = \dim G \cdot \phi \cap p^{-1}(G_1 \cdot \psi) \\ &\leq \dim \mathfrak{g} \cdot \phi \cap p^{-1}(\mathfrak{g}_1 \cdot \psi) \\ &= \dim \mathfrak{g} \cdot \phi - \dim \mathfrak{g}_1 \cdot \phi + \dim \mathfrak{g}_1 \cdot \psi. \end{aligned}$$

But generically, the inequality is an equality. Therefore, for generic $\phi \in p^{-1}(G_1 \cdot \psi)$, the numbers $n(\psi, \phi) = 0$. We examine the six cases to see that $n(\psi, \phi)$ vanishes for *all* $\phi \in p^{-1}(G_1 \cdot \psi)$.

There is nothing to do in cases (i) and (iv) since there is only one orbit lying over $G_1 \cdot \psi$ in those cases. We complete cases (ii), (iii), and (v) by proving that in those instances the numbers $\dim \mathfrak{g}_\phi$ and $\dim(\mathfrak{g}_1)_\phi$ are constant on $p^{-1}(G_1 \cdot \psi)$. We rely heavily on the notation and proof of Theorem 4.1. Choose the specific functional $\phi \in p^{-1}(G_1 \cdot \psi)$ that satisfies $\phi|_{\mathfrak{g}_1} = \psi$, $\phi(Y) = \phi(Z) = 0$.

Case (ii). The orbits over $G_1 \cdot \psi$ are parameterized by $\phi_s = \phi + s\alpha$. Using [8, Thm. 0.1], we see that

$$\dim G_1 \cdot \psi = \dim G_0 \cdot \theta + 2 = \dim G_4 \cdot \omega_s = \dim G \cdot \phi_s - 2 \quad \text{for all } s \in \mathbf{R}.$$

Hence $\dim(\mathfrak{g})_{\phi_s}$ is constant. Next we observe that, since $\phi_s|_{\mathfrak{g}_0} = \theta$, for all s we have $\mathfrak{g}_{\phi_s} \subset \mathfrak{g}_\theta \subset \mathfrak{g}_\gamma \subset \mathfrak{g}_4$. Therefore, $(\mathfrak{g}_1)_{\phi_s} \subset \mathfrak{g}_1 \cap \mathfrak{g}_4 = \mathfrak{g}_0$. But since $\alpha(\mathfrak{g}_0) = 0$, we have $\phi_s[\mathfrak{g}, \mathfrak{g}] = \phi[\mathfrak{g}, \mathfrak{g}]$. That is, $(\mathfrak{g}_1)_{\phi_s}$ is independent of s.

Case (iii). The orbits over $p^{-1}(G_1 \cdot \psi)$ are parameterized by $\phi^u = \phi + u\delta$, $u \in \mathbf{R}$. But $\delta(\mathfrak{g}_4) = 0$ and $[\mathfrak{g}, \mathfrak{g}] \subset \mathfrak{g}_4$. It follows that $\phi^u[\mathfrak{g}, \mathfrak{g}] = \phi[\mathfrak{g}, \mathfrak{g}]$. In particular, neither $\mathfrak{g}_{\phi^u}$ nor $(\mathfrak{g}_1)_{\phi^u}$ depends on u at all.

Case (v). This case is very similar to (ii). Namely, the orbits over $G_1 \cdot \psi$ are parameterized by $\phi_{s,t} = \phi + s\alpha + t\beta$. Also, for any s, t we have

$$\dim G_1 \cdot \psi = \dim G_0 \cdot \theta + 2 = \dim G_4 \cdot \omega_{s,t} + 2 = \dim G \cdot \phi_{s,t}.$$

Thus $\dim \mathfrak{g}_{\phi_{s,t}}$ does not depend on s, t. Also as in (ii), $\mathfrak{g}_{\phi_{s,t}} \subset \mathfrak{g}_4 \Rightarrow (\mathfrak{g}_1)_{\phi_{s,t}} \subset \mathfrak{g}_0$, and since $\alpha(\mathfrak{g}_0) = \beta(\mathfrak{g}_0) = 0$, we obtain that $(\mathfrak{g}_1)_{\phi_{s,t}}$ is independent of s, t.

It remains to consider case (vii). Unlike the other cases, the dimensions of the stabilizers do experience jumps at a degenerate point. In fact, the orbits over $G_1 \cdot \psi$ are parameterized by $\phi_{s,t}$ again. Moreover, we saw in the proof of Theorem 4.1 that

$$\dim \mathfrak{g} \cdot \phi_{s,t} = \begin{cases} \dim \mathfrak{g}_0 \cdot \theta + 2, & s, t \neq P_0, \\ \dim \mathfrak{g}_0 \cdot \theta, & s, t = P_0. \end{cases}$$

For $s, t \neq P_0$, the number $\dim(\mathfrak{g}_1)_{\phi_{s,t}}$ remains constant. For if it jumped anywhere, the corresponding $n(\psi, \phi)$ would be rendered negative—an impossibility. When $s, t = P_0$, $\dim \mathfrak{g}_{\phi_{P_0}}$ jumps by 2. We must show that $\dim(\mathfrak{g}_1)_{\phi_{P_0}}$ jumps only by 1, not 2. But note that since $\phi_{s,t}|_{\mathfrak{g}_0} = \theta$, for any s, t and $\mathfrak{g}_\gamma = \mathfrak{g}$ we have (by [9, Lemma 2.4] yet one more time) that

$$(G_0)_\theta \cdot \phi_{s,t} = \phi_{s,t} \quad \text{for any } s, t.$$

In particular, $(\mathfrak{g}_0)_{\phi_{s,t}} = (\mathfrak{g}_0)_\theta$ for any s, t. But then we have

$$\begin{aligned} \dim(\mathfrak{g}_0)_\theta = \dim(\mathfrak{g}_0)_{\phi_{s,t}} \le \dim(\mathfrak{g}_1)_{\phi_{s,t}} &\le \dim(\mathfrak{g}_0)_{\phi_{s,t}} + 1 \\ &= \dim(\mathfrak{g}_0)_\theta + 1, \quad s, t \neq P_0. \end{aligned}$$

This ensures that the jump in $\dim(\mathfrak{g}_1)_{\phi_{P_0}}$ is at most 1 (in fact it is exactly 1). This completes the proof of Theorem 1.1. □

REMARK 4.2. As mentioned in the introduction, it is possible to prove analogs of Theorems 1.1 and 4.1 for restricted representations. One can also develop an analog of [11, Sect. 5]; that is, we can use restrictions to distinguish the like cases of Theorem 4.1. For example, one can distinguish between cases (i) and (iv)—in both of which the induced representation is irreducible—by examining the difference in the resulting restricted representations. We leave the details to another time.

5. Examples

We give examples of each of the seven structures of a codimension-2 induced representation from a maximal subgroup. We also give an example to demonstrate that maximality, even in codimension 2, is critical for the truth of Theorem 1.1.

In the following, "sp" denotes the real linear span. Note that $[X, W_1]=(1-i)W_1$ is shorthand for $[X, Y_1]=Y_1+Z_1$, $[X, Z_1]=-Y_1+Z_1$.

(1) $\mathfrak{g}=\mathrm{sp}\{X, W_1, W_2, W_3\}$, $W_j=Y_j+iZ_j$ complex:
$[X, W_1]=(1-i)W_1$, $[X, W_3]=(1-i)W_3$, $[W_1, W_2]=W_3$;
$\mathfrak{g}_1=\mathrm{sp}\{X, W_2, W_3\}$, $\mathfrak{g}_0=\mathrm{sp}\{W_2, W_3\}$, $\mathfrak{g}_4=\mathrm{sp}\{W_1, W_2, W_3\}$,
$\psi=W_3^*$.

(2) $\mathfrak{g}=\mathrm{sp}\{X, Y, Z, W, U, V\}$:
$[X, Y+iZ]=(1-i)(Y+iZ)$, $[Y, Z]=W$, $[X, W]=2W$,
$[Y, U]=V$, $[X, V]=V$;
$\mathfrak{g}_1=\mathrm{sp}\{X, W, U, V\}$, $\mathfrak{g}_0=\mathrm{sp}\{W, U, V\}$, $\mathfrak{g}_2=\mathrm{sp}\{Z, W, U, V\}$,
$\mathfrak{g}_3=\mathrm{sp}\{Y, W, U, V\}$, $\mathfrak{g}_4=\{Y, Z, W, U, V\}$, $\psi=V^*+W^*$.

(3) $\mathfrak{g}=\mathrm{sp}\{X, W_1, W_2, W_3\}$, $W_j=Y_j+iZ_j$ complex:
$[X, W_1]=(1-i)W_1$, $[X, W_2]=-(1-i)W_2$, $[W_1, W_2]=W_3$;
$\mathfrak{g}_1=\mathrm{sp}\{X, W_2, W_3\}$, $\mathfrak{g}_0=\mathrm{sp}\{W_2, W_3\}$, $\mathfrak{g}_4=\mathrm{sp}\{W_1, W_2, W_3\}$,
$\psi=Z_2^*+Z_3^*$.

(4) Same as (3) except $\psi=Z_3^*$.

(5) $\mathfrak{g}=\mathrm{sp}\{X, Y, Z, W\}$:
$[X, Y+iZ]=(1-i)(Y+iZ)$, $[X, W]=W$;
$\mathfrak{g}_1=\mathrm{sp}\{X, W\}$, $\mathfrak{g}_0=\mathrm{sp}\{W\}$, $\mathfrak{g}_4=\mathrm{sp}\{Y, Z, W\}$, $\psi=W^*$.

(6) $\mathfrak{g}=\mathrm{sp}\{X, Y, Z, W\}$:
$[X, Y+iZ]=(1-i)(Y+iZ)$, $[Y, Z]=W$, $[X, W]=2W$;
$\mathfrak{g}_1=\mathrm{sp}\{X, W\}$, $\mathfrak{g}_0=\mathrm{sp}\{W\}$, $\mathfrak{g}_4=\mathrm{sp}\{Y, Z, W\}$, $\psi=W^*$.

(7) $\mathfrak{g}=\mathrm{sp}\{X, Y, Z\}$:
$[X, Y+iZ]=(1-i)(Y+iZ)$;
$\mathfrak{g}_1=\mathrm{sp}\{X\}$, $\mathfrak{g}_0=\{0\}$, $\mathfrak{g}_4=\mathrm{sp}\{Y, Z\}$, $\psi=0$.

Finally, here is the example to illustrate the dependence of Theorem 1.1 on maximality—even in codimension 2. Actually, it is enough to look at the $(ax+b)$-group G and take G_1 to be trivial. Certainly $\dim G/G_1=2$. The induced representation $\mathrm{Ind}_{G_1}^{G} 1$ is the regular representation of G which is an *infinite multiple* of the direct sum of two inequivalent infinite-dimensional representations. Thus the generic functionals $\phi \in \mathfrak{g}^* = \mathfrak{g}_1^{\perp} = p^{-1}(0)$ satisfy $\dim G\cdot\phi \cap p^{-1}(0) > \dim G_1\cdot\phi$. But, for the linear functionals $\phi \in \mathfrak{g}^*$ which are trivial on the nilradical of $\mathfrak{g}$, their orbits are points that satisfy

$$G\cdot\phi \cap p^{-1}(0) = G_1\cdot\phi.$$

References

[1] P. Bernat et al., *Répresentations des groupes de Lie résolubles,* Dunod, Paris, 1972.
[2] L. Corwin, F. Greenleaf, and G. Grelaud, *Direct integral decompositions and multiplicities for induced representations of nilpotent Lie groups,* Trans. Amer. Math. Soc. 304 (1987), 549–583.
[3] M. Duflo, *Sur les extensions des représentations irréductibles des groupes de Lie nilpotents,* Ann. Sci. École Norm. Sup. (4) 5 (1972), 71–120.
[4] H. Fujiwara, *Représentations monomiales des groupes de Lie résolubles exponentiels,* The orbit method in representation theory, pp. 61–84, Birkhäuser, Basel, 1990.
[5] A. A. Kirillov, *Unitary representations of nilpotent Lie groups,* Russian Math. Surveys 17 (1962), 53–104.
[6] A. Kleppner and R. Lipsman, *The Plancherel formula for group extensions,* Ann. Sci. École Norm. Sup. (4) 5 (1972), 459–516.
[7] R. Lipsman, *Solvability of invariant differential operators with variable coefficients,* Comm. Partial Differential Equations 10 (1985), 1261–1316.
[8] ———, *Orbital parameters for induced and restricted representations,* Trans. Amer. Math. Soc. 313 (1989), 433–473.
[9] ———, *Induced representations of completely solvable Lie groups,* Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4) 17 (1990), 127–164.
[10] ———, *The Corwin-Greenleaf parity result fails for completely solvable Lie groups,* Proc. Amer. Math. Soc. 108 (1990), 1131–1132.
[11] ———, *Restricting representations of completely solvable Lie groups,* Canad. J. Math. 42 (1990), 790–824.
[12] ———, *The multiplicity function on exponential and completely solvable homogeneous spaces,* Geom. Dedicata 39 (1991), 155–161.
[13] ———, *Induced representations of exponential solvable Lie groups,* unpublished manuscript.

Department of Mathematics
University of Maryland
College Park, MD 20742