A Lower Bound for the Gap between the First Two Eigenvalues of Schrödinger Operators on Convex Domains in $\mathbf{S}^n$ or $\mathbf{R}^n$

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Introduction

Let $\mathbf{S}^n$ denote an n-dimensional sphere with canonical metric. Let $\Omega \subset \mathbf{S}^n$ or $\mathbf{R}^n$ be a smooth strictly convex bounded domain, let Δ be the Laplace-Beltrimi operator on $\mathbf{S}^n$ or $\mathbf{R}^n$, and let $V: \Omega \to \mathbf{R}^1$ be a nonnegative convex smooth function. The eigenvalues of the following problem:

$$\begin{aligned} -\Delta u + Vu &= \lambda u \quad \text{in } \Omega, \\ u &= 0 \quad \text{on } \partial\Omega, \end{aligned} \tag{0.1}$$

can be arranged in a nondecreasing order as follows:

$$0 < \lambda_1 < \lambda_2 \leq \lambda_3 \leq \lambda_4 \leq \cdots.$$

It is a significant problem to find a lower bound for $\lambda_2 - \lambda_1$ in terms of the geometry of Ω and the given potential function V. For $\Omega \subset \mathbf{R}^n$, Singer, Wong, Yau, and Yau [9] showed that

$$\lambda_2 - \lambda_1 \geq \frac{\pi^2}{4d^2},$$

where d is the diameter of Ω. This estimate was later improved to

$$\lambda_2 - \lambda_1 \geq \frac{\pi^2}{d^2} \tag{0.2}$$

by Yu and Zhong [10]. Then Lee and Wong [6] proved that (0.2) also holds for $\Omega \subset \mathbf{S}^n$ provided the potential $V \equiv 0$. Our result is in our Theorem 2, which improves (0.2) to

$$\lambda_2 - \lambda_1 \geq \frac{4}{d^2} K(\sigma)^2 \quad \text{for } \Omega \subset \mathbf{S}^n \text{ or } \mathbf{R}^n, \tag{0.3}$$

where the potential V is a nonnegative convex function and where K denotes the complete elliptic integral of the first kind, defined by

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$$K(\sigma)=\int_0^{\pi/2}\frac{dt}{\sqrt{1-\sigma\sin^2 t}} \tag{0.4}$$

(see [1]). It is known that $\sigma>0$ and, what is more, σ can be estimated from below in terms of other quantities occurring in the problem. In particular, since $K(\sigma)>\pi/2$ for $\sigma>0$, (0.3) improves upon (0.2). The approach we employ in this paper is from [10; 11].

1. Main Results

Let f and $\bar{f}$ be the first and second eigenfunctions of problem (0.1), respectively. It is known that $f\neq 0$ in Ω, $\bar{f}$ changes its sign in Ω (see [3]), and $\bar{f}/f$ is smooth up to the boundary $\partial\Omega$ (see [9]). Without loss of generality, we can assume that

$$f>0 \text{ in } \Omega, \quad \inf_\Omega \frac{\bar{f}}{f}=-k, \quad \sup_\Omega \frac{\bar{f}}{f}=1, \quad \text{and} \quad 0<k\leq 1. \tag{1.1}$$

Define

$$v=\left(\frac{\bar{f}}{f}-\frac{1-k}{2}\right)\Big/\frac{1+k}{2}.$$

Then the function v satisfies

$$\begin{gathered}\Delta v=-\lambda(v+a)-2\nabla v\cdot\nabla(\log f) \text{ in } \Omega,\\ \inf_\Omega v=-1, \quad \text{and} \quad \sup_\Omega v=1,\end{gathered} \tag{1.2}$$

where $\lambda=\lambda_2-\lambda_1$ and $a=(1-k)/(1+k)$ so $0\leq a<1$.

Using Li and Yau's method of gradient estimate, one readily derives the following estimate:

$$\frac{|\nabla v|^2}{b^2-v^2}\leq\lambda(1+a), \tag{1.3}$$

where $b>1$ is an arbitrary constant, as was found in [9; 10]. We are going to improve this estimate and then find a new lower bound of $\lambda_2-\lambda_1$.

Define $F:[\arcsin(-1/b), \arcsin(1/b)]\to\mathbf{R}^1$ by

$$F(t_0)=\max_{\substack{x\in\bar{\Omega}\\ t(x)=t_0}}\frac{|\nabla v|^2}{b^2-v^2},$$

where $t(x)=\arcsin(v(x)/b)$. Then $F[t(x)]$ is continuous on $\bar{\Omega}$. The estimate (1.3) becomes

$$F(t)\leq\lambda(1+a) \quad \forall t\in[\arcsin(-1/b), \arcsin(1/b)]. \tag{1.3'}$$

Throughout we let (unless otherwise stated)

$$I=(\arcsin(-1/b), \arcsin(1/b)),$$

$$c=a/b,$$

$$d = \text{the diameter of } \Omega,$$

$$t(x) = \arcsin \frac{v(x)}{b},$$

$$\alpha = \inf_{\substack{x \in \bar{\Omega} \\ \tau \in T_x \bar{\Omega}, |\tau| = 1}} \{[\nabla^2(-\log f)(x)](\tau, \tau)\},$$

$$\delta = \delta_M = \alpha / M, \quad \text{where M is an upper bound of } \lambda.$$

REMARK 1. $\alpha > 0$, by [2; 4; 5; 9] and Lemma 5 of this paper, and $\delta_\lambda = \alpha/\lambda \geq 1/4$ for $\Omega = B_r \subset \mathbf{S}^n$ or $\mathbf{R}^n$ and $V \equiv 0$, by [7].

REMARK 2. For some of the upper bounds one can refer to [7; 8; 9].

Now define a function G by

$$F(t) = \lambda(1+a)G(t) \quad \text{for } t \in \bar{I},$$

and, if $a \neq 0$, define a function H by

$$F(t) = \lambda(1 + cH(t)) \quad \text{for } t \in \bar{I}.$$

Then (1.3′) becomes

$$G(t) \leq 1 \quad \text{and} \quad H(t) \leq b \quad \text{for every } t \in \bar{I}. \tag{1.3″}$$

We now prove the following theorem.

THEOREM 1. *For $t \in \bar{I}$,*

$$G(t) \leq A + 2B \sin t - B \cos^2 t =: W(t) \tag{1.4}$$

and

$$H(t) \leq \frac{(4/\pi)(t + \cos t \sin t) - 2 \sin t}{\cos^2 t} - E \cos^2 t =: U(t), \tag{1.5}$$

where

$$A = \frac{1+c}{1+a} - 2Bc, \qquad B = \min\left\{\frac{4\delta}{15}, \frac{1}{4}\right\},$$

$$E = \min\{\delta/a, m\},$$

$$m = \inf_{(-\pi/2, 0)} \frac{1 + \sin^2 t - (4/\pi)(\cos t + t \sin t)}{\cos^4 t \cdot \sin t}.$$

REMARK 3. $m \geq \pi/9\sqrt{3}$ by Lemma 4 of this paper.

Proof of Theorem 1. There exist some $P \in \mathbf{R}^1$ and $t_0 \in I$ such that

$$P = \max_{t \in \bar{I}} (G(t) - W(t)) = G(t_0) - W(t_0).$$

Therefore,

$$G(t) \leq W(t) + P$$

for every $t \in I$, and

$$G(t_0) = W(t_0) + P.$$

If $P > 0$ then, according to Lemma 1 and Lemma 2 (which we will state and prove in the next section), we obtain

$$\begin{aligned}
W(t_0)+P &= G(t_0) \\
&\le \left\{ \frac{1}{1+a} + \frac{c}{1+a}\sin t - \frac{2\delta}{1+a}\cos^2 t \right. \\
&\quad - \frac{1}{2(1+a)}(W+P)'\cos t[(2+a)\sin t + c] \\
&\quad \left. + \frac{1}{2}(W+P)''\cos^2 t \right\}\Bigg|_{t=t_0} \qquad \text{(by Lemma 1)} \\
&= \frac{1}{1+a} + \frac{c}{1+a}\sin t - \frac{2\delta}{1+a}\cos^2 t \\
&\quad - \frac{1}{2(1+a)} W'\cos t[(2+a)\sin t + c] + \frac{1}{2}W''\cos^2 t \Bigg|_{t=t_0} \\
&\le W(t_0) \qquad \text{(by Lemma 2).}
\end{aligned}$$

Thus $P \le 0$. This contradicts the assumption that $P > 0$. Therefore we know that $P \le 0$. (1.4) is proved. (1.5) is proved by an analogous argument, using Lemmas 3 and 4 instead of Lemmas 1 and 2. □

THEOREM 2. *Assume that $\tilde{\delta}$ is a constant such that $\tilde{\delta} \le \delta$; then*

$$\lambda \ge \frac{4}{d^2}\left(\int_0^{\pi/2} \frac{dt}{\sqrt{1-\sigma \sin^2 t}}\right)^2 =: \frac{4}{d^2} K(\sigma)^2,$$

where $\sigma > 0$ depends only on $\tilde{\delta}$.

REMARK 4. $\sigma \ge (2/15\pi)^2(\alpha/M)^2$. This was proved in [7].

COROLLARY. $\lambda > \pi^2/d^2$.

This is the main result in [10].

Proof of Theorem 2. By (1.4) we get

$$\frac{|\nabla v|^2}{b^2 - v^2} \le \lambda(1+a)G(t) \le \lambda(1+a)W(t).$$

Let q_1 and q_2 be two points in $\bar{\Omega}$ such that $v(q_1) = -1$ and $v(q_2) = 1$, and let η be a minimal geodesic joining them. Then

$$d \ge \int_{q_1}^{q_2} d\eta \ge \int_{-1}^{1} \frac{dv}{|\nabla v|} \ge \int_{\arcsin(-1/b)}^{\arcsin(1/b)} \frac{dt}{\sqrt{\lambda(1+a)W(t)}}.$$

Therefore,

$$\sqrt{\lambda}\geq\frac{1}{d}\int_{\arcsin(-1/b)}^{\arcsin(1/b)}\frac{dt}{\sqrt{(1+a)W(t)}}$$
$$=\frac{1}{d}\int_{0}^{\arcsin(1/b)}\left(\frac{1}{\sqrt{(1+a)W(t)}}+\frac{1}{\sqrt{(1+a)W(-t)}}\right)dt$$
$$=\frac{1}{d}\int_{0}^{\arcsin(1/b)}\frac{1}{\sqrt{(1+a)(A-B\cos^2 t)}}$$
$$\left(\frac{1}{\sqrt{1+(2Bc\sin t/(A-B\cos^2 t))}}+\frac{1}{\sqrt{1-(2Bc\sin t/(A-B\cos^2 t))}}\right)dt$$
$$\geq\frac{2}{d}\int_{0}^{\arcsin(1/b)}\frac{dt}{\sqrt{(1+a)(A-B\cos^2 t)}}$$
$$=\frac{2}{d}\int_{0}^{\arcsin(1/b)}\frac{dt}{\sqrt{1+c-2Bc(1+a)-(1+a)B\cos^2 t}}$$
$$\geq\frac{2}{d}\int_{0}^{\arcsin(1/b)}\frac{dt}{\sqrt{1+c-2Bc(1+a)-B\cos^2 t}}.$$

Let $b\to 1$; then

$$\lambda\geq\frac{4}{d^2}\left(\int_0^{\pi/2}\frac{dt}{\sqrt{1+a-2Ba(1+a)-B\cos^2 t}}\right)^2. \tag{1.6}$$

If $a\neq 0$, then (1.5) and an argument analogous to that above shows that

$$\lambda\geq\frac{4}{d^2}\left(\int_0^{\pi/2}\frac{dt}{\sqrt{1-aE\cos^2 t}}\right)^2. \tag{1.7}$$

Choose $\epsilon=\epsilon(B)$ so that

$$\int_0^{\pi/2}\frac{dt}{\sqrt{1+s-2Bs(1+s)-B\cos^2 t}}\geq\int_0^{\pi/2}\frac{dt}{\sqrt{1-(B/2)\cos^2 t}} \tag{1.8}$$

holds for $0\leq s\leq\epsilon$, and let

$$\sigma=\min\{B/2,\min(\delta,\epsilon m)\}=\min\{2\delta/15,1/8,\epsilon m\}.$$

Then

$$\lambda\geq\frac{4}{d^2}\left(\int_0^{\pi/2}\frac{dt}{\sqrt{1-\sigma\cos^2 t}}\right)^2=\frac{4}{d^2}\left(\int_0^{\pi/2}\frac{dt}{\sqrt{1-\sigma\sin^2 t}}\right)^2,$$

using (1.6) and (1.8) if $a\leq\epsilon$ and using (1.7) if $a>\epsilon$. □

2. Several Lemmas

LEMMA 1. *If a C^∞ function $y: I\to\mathbf{R}^1$ satisfies the following conditions:*

(i) *$y(t)\geq G(t)$ for every $t\in\bar{I}$;*

(ii) *there exists some $x_0\in\bar{\Omega}$ such that $t(x_0)=t_0$ and $y(t_0)=G(t_0)$;*

(iii) $y(t)>0$ *for every* $t\in\bar{I}$; *and*
(iv) $y'(t_0)(\sin t_0+c)\ge 0$,
then the following inequality is valid:

$$G(t_0)\le\frac{1}{1+a}+\frac{c}{1+a}\sin t_0-\frac{2\delta}{1+a}\cos^2 t_0$$
$$-\frac{1}{2(1+a)}y'(t_0)\cos t_0[(2+a)\sin t_0+c]+\frac{1}{2}y''(t_0)\cos^2 t_0. \tag{2.1}$$

Proof. Define $J:\bar{\Omega}\to\mathbf{R}^1$ by

$$J(x)=\left\{\frac{|\nabla v|^2}{b^2-v^2}-\lambda(1+a)\,y(t)\right\}\cos^2 t, \tag{2.2}$$

where $t=\arcsin(v(x)/b)$. Obviously,

$$J(x)\le 0 \text{ for every } x\in\bar{\Omega}, \quad\text{and}\quad J(x_0)=0.$$

Therefore J achieves its maximum on $\bar{\Omega}$ at x_0.

If $\nabla v(x_0)=0$, then

$$0=J(x_0)=-\lambda(1+a)\,y(t_0).$$

This contradicts (iii). Therefore

$$\nabla v(x_0)\ne 0. \tag{2.3}$$

(2.3) and Lemma 1 in [10] imply $x_0\in\Omega$. By the maximum principle, we have

$$\nabla J(x_0)=0 \tag{2.4}$$

and

$$\Delta J(x_0)\le 0 \tag{2.5}$$

Rewrite (2.2) as

$$J(x)=\frac{1}{b^2}|\nabla v|^2-\lambda(1+a)\,y(t)\cos^2 t. \tag{2.2$'$}$$

(2.4) and (2.5) are equivalent to

$$\frac{2}{b^2}\sum_i v_i v_{ij}\bigg|_{x_0}=\lambda(1+a)\cos t[y'\cos t-2y\sin t]t_j\bigg|_{x_0} \tag{2.4$'$}$$

and

$$0\ge\left\{\frac{2}{b^2}\sum_{i,j}v_{ij}^2+\frac{2}{b^2}\sum_{i,j}v_i v_{ijj}-\lambda(1+a)[(y'\Delta t+y''|\nabla t|^2)\cos^2 t\right.$$
$$\left.\left.-4y'\cos t\cdot\sin t|\nabla t|^2+y\Delta\cos^2 t]\right\}\right|_{x_0}. \tag{2.5$'$}$$

Choose a normal coordinate around x_0 such that $v_1(x_0)\ne 0$ and $v_i(x_0)=0$ for $i\ge 2$. Then (2.4′) gives

$$\begin{cases} v_{11}|_{x_0}=\dfrac{b\lambda(1+a)}{2}[y'\cos t-2y\sin t]\bigg|_{x_0}, \\ v_{1i}|_{x_0}=0,\quad i\ge 2. \end{cases} \tag{2.4$''$}$$

Under the given local coordinate at x_0,

$$\sum_{i,j} v_i v_{ijj} = \nabla v \cdot \nabla(\Delta v) + R(\nabla v, \nabla v) \geq \nabla v \nabla(\Delta v), \tag{2.6}$$

where $R(\cdot, \cdot)$ is the Ricci curvature of $\mathbf{S}^n$ or $\mathbf{R}^n$. We have:

$$\nabla v \cdot \nabla(\Delta v) = -\lambda v_1^2 - 2v_1^2(\log f)_{11} - 2v_1 v_{11}(\log f)_1, \tag{2.7}$$

$$\frac{\nabla v}{b} = \Delta \sin t = \cos t \Delta t - \sin t |\nabla t|^2, \tag{2.8}$$

$$\Delta t = \frac{1}{\cos t}\left[\frac{\Delta v}{b} + \sin t |\nabla t|^2\right], \tag{2.9}$$

$$\Delta \cos^2 t = \Delta\left(1 - \frac{v^2}{b^2}\right) = -\frac{2}{b^2} v \nabla v - 2\cos^2 t |\nabla t|^2, \tag{2.10}$$

$$|\nabla t|^2 = \frac{|\nabla v|^2}{b^2 - v^2} = \lambda(1+a)y. \tag{2.11}$$

Putting (2.6)–(2.11) into (2.5′), we obtain, at x_0,

$$\begin{aligned} 0 \geq{} & \frac{2}{b^2} \sum_{i,j} v_{ij}^2 - 2\lambda^2(1+a)y\cos^2 t + \lambda^2(1+a)y\cos t(\sin t + c) \\ & -\lambda^2(1+a)yy'\cos t \sin t - \lambda^2(1+a)^2 yy''\cos^2 t \\ & +4\lambda^2(1+a)^2 yy'\cos t \sin t - 2\lambda^2(1+a)y\sin t(\sin t + c) \\ & +2\lambda^2(1+a)^2 y^2 \cos^2 t. \end{aligned} \tag{2.5″}$$

Putting (2.4″) into (2.5″), we obtain, at x_0,

$$\begin{aligned} 0 \geq{} & \tfrac{1}{2}\lambda^2(1+a)^2(y')^2\cos^2 t + 2\lambda^2(1+a)^2 y^2 \\ & -2\lambda^2(1+a)y\cos^2 t + \lambda^2(1+a)y'\cos t[\sin t + c + (1+a)y\sin t] \\ & -\lambda^2(1+a)^2 yy''\cos^2 t - 2\lambda^2(1+a)y\sin t(\sin t + c) \\ & +4\delta\lambda^2(1+a)y\cos^2 t, \end{aligned} \tag{2.5‴}$$

where we have used

$$-\frac{4}{b^2} v_1 v_{11}(\log f)_1 + \frac{2\lambda(1+a)}{b}(\log f)_1[y'\cos t - 2y\sin t]\Big|_{x_0} = 0$$

and

$$\frac{(-\log f)_{11}}{\lambda} \geq \frac{(-\log f)_{11}}{M} \geq \delta.$$

Dividing both sides of (2.5‴) by $\lambda^2(1+a)y$, we obtain

$$\begin{aligned} G(t_0) &= y(t_0) \\ &\leq \frac{1}{1+a} + \frac{c}{1+a}\sin t_0 - \frac{2\delta}{1+a}\cos^2 t_0 \end{aligned}$$

$$-\frac{1}{2(1+a)}y'(t_0)\cos t_0\left[\frac{\sin t_0+c}{y(t_0)}+(1+a)\sin t_0\right]$$
$$+\frac{1}{2}y''(t_0)\cos^2 t_0$$
$$\leq \frac{1}{1+a}+\frac{c}{1+a}\sin t_0-\frac{2\delta}{1+a}\cos^2 t_0$$
$$-\frac{1}{2(1+a)}y'(t_0)\cos t_0[(2+a)\sin t_0+c]+\frac{1}{2}y''(t_0)\cos^2 t_0.$$

In the last inequality we have used the conditions

$$y'(t_0)(\sin t_0+c)\geq 0 \quad \text{and} \quad y(t_0)=G(t_0)\leq 1. \qquad \square$$

LEMMA 2. *Let A, B be as in Theorem 1. Then the function*

$$W(t)=A+2Bc\sin t-B\cos^2 t$$

satisfies the following system: For every $t\in \bar{I}$,

$$\begin{cases}(1+a)W''\cos^2 t-W'\cos t[(2+a)\sin t+c] \\ \qquad -2(1+a)W+2+2c\sin t-4\delta\cos^2 t\leq 0, \\ W'(t)(\sin t+c)\geq 0, \quad \textit{and} \\ W(t)>0.\end{cases} \tag{2.12}$$

Proof.

$$W'(t)=2B\cos t(\sin t+c);$$
$$W'(t)(\sin t+c)=2B\cos t(\sin t+c)^2\geq 0;$$
$$W''(t)=2B(1-c\sin t-2\sin^2 t).$$

Therefore,

$$(1+a)W''\cos^2 t-W'\cos t[(2+a)\sin t+c]-2(1+a)W+2+2c\sin t-4\delta\cos^2 t$$
$$=\cos^2 t\{2B[2+2a-c^2-(4+2a)c\sin t-(4+3a)\sin^2 t]-4\delta\}$$
$$+2\{1-(1+a)A+[1-2(1+a)B]c\sin t\}$$
$$\leq \cos^2 t\left\{\frac{2(1+a)(8+6a+ac^2)B}{4+3a}-4\delta\right\}$$
$$+2\{-c+2Bc(1+a)+[1-2(1+a)B]c\}$$
$$\leq 4\cos^2 t\{(15/4)B-\delta\}\leq 0,$$

and

$$y|_{\min}=y|_{\sin t=-c}=A-B(1+c^2)$$
$$=(1+c)^2\left(\frac{1}{(1+a)(1+c)}-B\right)\geq (1+c)^2\left(\frac{1}{(1+a)(1+c)}-\frac{1}{4}\right)>0. \qquad \square$$

LEMMA 3. *Assume that $a\neq 0$. Let the C^∞ function $y: \bar{I}\to \mathbf{R}^1$ satisfy the conditions:*

(i) $y(t) \geq H(t)$ *for every* $t \in \bar{I}$;
(ii) *there exists some* $x_0 \in \bar{\Omega}$ *such that* $t(x_0) = t_0$ *and* $y(t_0) = H(t_0)$;
(iii) $y(t) > -1/c$ *for every* $t \in \bar{I}$; *and*
(iv) $y'(t_0) \geq 0$.

Then the following inequality is valid:

$$H(t_0) \leq \sin t_0 - \frac{2\delta}{c}\cos^2 t_0 - y'(t_0)\cos t_0 \sin t_0 + \frac{1}{2}y''(t_0)\cos^2 t_0. \tag{2.13}$$

Proof. An argument analogous to that used in the proof of Lemma 1 gives

$$\begin{aligned} H(t_0) &= y(t_0) \\ &\leq \sin t_0 - \frac{2\delta}{c}\cos^2 t_0 - \frac{1}{2}y'(t_0)\cos t_0\left[\frac{\sin t_0 + c}{1+cy(t_0)} + \sin t_0\right] \\ &\quad + \frac{1}{2}y''(t_0)\cos^2 t_0. \end{aligned}$$

By (iii), (iv), the above inequality, and

$$\frac{\sin t_0 + c}{1+cy(t_0)} \geq \sin t_0,$$

we obtain (2.13). □

Lemma 4. *Let E be as in Theorem 1. Then the function*

$$U(t) =: U_1(t) + U_2(t) =: \frac{(4/\pi)(t+\cos t\cdot\sin t) - 2\sin t}{\cos^2 t} - E\cos^2 t$$

satisfies, for $t \in [-\pi/2, \pi/2]$,

$$\begin{cases} U(-\pi/2) = -1, \quad U(\pi/2) = 1, \quad |U(t)| \leq 1; \\ U'(t) \geq 0; \quad \textit{and} \\ \sin t - (2\delta/c)\cos^2 t - U'\cos t \sin t + \frac{1}{2}U''\cos^2 t \leq U(t). \end{cases} \tag{2.14}$$

Proof. One checks readily that the function U_1 satisfies, for $t \in [-\pi/2, \pi/2]$,

$$\begin{cases} U_1(-\pi/2) = -1, \quad U_1(\pi/2) = 1; \\ U_1'(t) \geq 0; \quad \text{and} \\ \sin t - U_1'\cos t \sin t + \frac{1}{2}U_1''\cos^2 t = U_1; \end{cases}$$

as well as

$$\begin{aligned} &-\frac{2\delta}{c}\cos^2 t - U_2 - U_2'\cos t \sin t + \frac{1}{2}U_2''\cos^2 t \\ &= \left\{-\frac{2\delta}{c} + 2E - 4E\sin^2 t\right\}\cos^2 t \\ &\leq -4E\sin^2 t\cos^2 t \leq 0. \end{aligned}$$

We are going to show that m in Theorem 1 satisfies

$$m \geq \pi/9\sqrt{3}. \tag{2.15}$$

Define a function $Z: [-\pi/2, 0] \to \mathbf{R}^1$ by

$$Z(t) = \frac{1+\sin t - (4/\pi)(\cos t + t\sin t)}{\cos^4 t \sin t}.$$

Then

$$\lim_{t\to 0^-} Z(t) = \lim_{t\to -\pi/2+0} Z(t) = +\infty.$$

Therefore there exists some $\bar{t} \in (-\pi/2, 0)$ such that

$$m = Z(\bar{t}) \tag{2.16}$$

and

$$0 = Z'(\bar{t})$$
$$= \left.\frac{(4/\pi)\cos^3 t - \cos^4 t + 4\sin^2 t + 4\sin^4 t - (16/\pi)(\cos t\sin^2 t + t\sin^3 t)}{\cos^5 t\sin^2 t}\right|_{t=\bar{t}}. \tag{2.17}$$

(2.16) and (2.17) yield

$$m = Z(\bar{t}) = \frac{\cos\bar{t} - 4/\pi}{4\cos\bar{t}\sin^3\bar{t}} \geq -\frac{\pi}{48}\max\frac{1}{\cos^3\bar{t}\sin\bar{t}} = \frac{\pi}{9\sqrt{3}}.$$

Therefore (2.15) holds. So

$$U'(t) = U_1'(t) + U_2'(t)$$
$$= 2\left\{\frac{(4/\pi)(\cos t + t\sin t) - 1 - \sin^2 t}{\cos^3 t} + E\cos t\sin t\right\}$$
$$\geq 0,$$

by (2.15) for $t \in [-\pi/2, 0]$ and by $U_1'(t) \geq 0$ and $\sin t \geq 0$ for $t \in [0, \pi/2]$. □

LEMMA 5. *Let f be the first Dirichlet eigenfunction of* (0.1), *with $f > 0$ in Ω. Then there exists $\epsilon > 0$ such that the function* $\log f$ *is strictly concave in the ϵ-neighborhood of $\partial\Omega$.*

Proof. Choose a normal coordinate around $x_0 \in \partial\Omega$ such that $\partial_1 = \partial/\partial x_1$ is the outward unit normal vector field on $\partial\Omega$. Take a point $\bar{x}$ in Ω with small distance d to $\partial\Omega$, $d = \mathrm{dist}(\bar{x}, x_0) = \mathrm{dist}(\bar{x}, \partial\Omega)$. Then $f_1|_{x_0} < 0$, by the Hopf lemma, and $f_i|_{x_0} = 0$, $i \geq 2$, because $\partial/\partial x^i$ $(i \geq 2)$ are tangent vectors. Therefore,

$$f_1|_{\bar{x}} \sim c_1 d, \quad c_1 = -f_1(x_0) > 0.$$

and

$$f_i|_{\bar{x}} \sim O(d) \quad \text{when } i \geq 2.$$

Here "$\sim A/d^\alpha$" means "$=(A+o(1))/d^\alpha$" with $o(1) \to 0$ as $d \to 0$. Let $w = \log f$; then

$$w_i|_{\bar{x}} = (f_i/f)|_{\bar{x}} \quad \text{and}$$

$$w_{ij}|_{\bar{x}} = \{f_{ij}/f - f_i f_j/f^2 - \Gamma_{ij}^k f_k/f\}|_{\bar{x}},$$

where Γ_{ij}^k are the Christoffel symbols. Hence we obtain

$$w_{11}|_{\bar{x}} \sim O(1/d) - f_1^2/d^2|_{x_0}$$

and

$$w_{1i}|_{\bar{x}} \sim O(1/d), \quad i \geq 2.$$

When $i, j \geq 2$,

$$f_{ij}|_{\bar{x}} = \partial_i \partial_j f - (\nabla_{\partial i} \partial_j) f|_{\bar{x}} \sim O(d) - \nabla_{\partial i} \partial_j f|_{x_0}.$$

Now

$$\nabla_{\partial i} \partial_j f|_{x_0} = \langle \nabla_{\partial i} \partial_j, \partial_k \rangle \partial_k f|_{x_0} = \langle \nabla_{\partial i} \partial_j, \partial_1 \rangle f_1|_{x_0}$$

and

$$\langle \partial_1, \partial_j \rangle = 0, \quad j \geq 2,$$

where $\langle \cdot, \cdot \rangle$ is the scalar product on the tangent bundle of $\mathbf{S}^n$ or $\mathbf{R}^n$, and

$$0 = \partial_i \langle \partial_1, \partial_j \rangle = \langle \nabla_{\partial i} \partial_1, \partial_j \rangle + \langle \partial_1, \nabla_{\partial i} \partial_j \rangle$$
$$= h_{ij} + \langle \nabla_{\partial i} \partial_j, \partial_1 \rangle,$$

where $(h_{ij})_{n-1,n-1}$ is the second fundamental form of $\partial\Omega$ via ∂_1. Therefore,

$$f_{ij}|_{\bar{x}} \sim O(d) + h_{ij} f_1|_{x_0} \quad \text{and} \quad w_{ij} \sim h_{ij} f_1 \quad \text{for } i, j \geq 2.$$

Hence we obtain

$$(-w_{ij})_{n,n}|_{\bar{x}} \sim \begin{bmatrix} f_1^2/d^2 & O(1/d) \\ O(1/d) & (-h_{ij} f_1)_{n-1,n-1} \end{bmatrix}.$$

Noting that $(h_{ij})_{n-1,n-1}$ is positive definite, we have now completed the proof. □

FINAL REMARK. Theorems 1 and 2 are valid for Riemannian manifolds for which the conditions $\nabla^2(-\log f)(\partial_i, \partial_i) > 0$ and $\mathrm{Ric}(M) \geq 0$ are satisfied.

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