Extension Domains for A_p Weights

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1. Introduction

In this paper we determine conditions on a bounded domain D in $\mathbf{R}^N$ ($N \geq 2$) so that every A_p weight on D can be extended to an A_p weight on $\mathbf{R}^N$. We also give boundary conditions on a Jordan domain D in $\mathbf{R}^2$ so that D is an extension domain for A_p.

Let D be a connected open set in $\mathbf{R}^N$, $N \geq 2$. A positive, locally integrable function w on D is said to belong to the class $A_p(D)$, $1 < p < \infty$, if

$$\|w\|_p = \sup_{Q \in \mathfrak{F}_0} \left(\frac{1}{|Q|}\int_Q w\,dx\right)\left(\frac{1}{|Q|}\int_Q \left(\frac{1}{w}\right)^{1/(p-1)} dx\right)^{p-1} < \infty, \tag{1.1}$$

where $\mathfrak{F}_0$ denotes the set of all cubes contained in D. The class $A_p(D)$ has been extensively studied in the case $D = \mathbf{R}^N$; they are precisely the class of weights for which, for example, the Hardy–Littlewood maximal function

$$Mf(x) = \sup_{Q \ni x} \frac{1}{|Q|}\int_Q |f(t)|\,dt$$

satisfies

$$\int (Mf(x))^p w(x)\,dx \leq C_p \int |f(x)|^p w(x)\,dx.$$

There is a close connection between $A_p(D)$ and the space of functions of bounded mean oscillation, abbreviated BMO(D). We say $f \in \mathrm{BMO}(D)$ if

$$\sup_{Q \in \mathfrak{F}_0} \frac{1}{|Q|}\int_Q |f - f_Q| < \infty,$$

where $f_Q = (1/|Q|)\int_Q f\,dt$ denotes the average of f on Q. This connection is as follows. If $w \in A_p(D)$ then $\log w \in \mathrm{BMO}(D)$, while if $f \in \mathrm{BMO}(D)$ then (by the theorem of John and Nirenberg) $e^{\delta f} \in A_p(D)$ for some $\delta > 0$.

We say that the domain D is an *extension domain* for $A_p(D)$ if whenever $w \in A_p(D)$ there exists $W \in A_p(\mathbf{R}^N)$ such that $W = w$ a.e. on D. Extension domains for BMO are defined analogously and have been characterized in [6], where it is shown that D is an extension domain for BMO if and only if

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D is a uniform domain. Before defining a uniform domain, we introduce two metrics on D. For any points $x, y \in D$ we define

$$k(x,y) = \inf_{\gamma} \int_{\gamma} \frac{1}{\delta(x)}\, ds,$$

where the infimum is taken over all rectifiable arcs γ in D connecting x and y. (Here $\delta(x) = \operatorname{dist}(x, \partial D)$.) The metric k is called the *quasihyperbolic* metric on D. If $E_1, E_2 \subseteq D$ we define $k(E_1, E_2) = \inf_{x \in E_1, y \in E_2} k(x,y)$. In [4] it is shown that

$$k(x,y) \geq \left| \log \frac{\delta(x)}{\delta(y)} \right| \tag{1.2}$$

and

$$k(x,y) \geq \log\left(1 + \frac{|x-y|}{\delta(x)}\right), \quad x, y \in D,$$

and hence $k(x,y) \geq j(x,y)$ where

$$j(x,y) = \frac{1}{2} \log\left(1 + \frac{|x-y|}{\delta(x)}\right)\left(1 + \frac{|x-y|}{\delta(y)}\right).$$

A domain D is then said to be *uniform* if there exist constants a, b such that

$$k(x,y) \leq a j(x,y) + b, \quad x, y \in D.$$

(An equivalent description of uniform domains is given in [3].) The above metrics can be interpreted through a Whitney decomposition of D, by which we mean a decomposition of D into cubes $\{Q_j\}$ with $\bigcup Q_j = D$ satisfying:

$$Q_j^0 \cap Q_k^0 = \emptyset, \quad j \neq k;$$

$$\frac{1}{4} \leq \frac{l(Q_j)}{l(Q_k)} \leq 4 \quad \text{if } Q_j \cap Q_k \neq \emptyset;$$

$$1 \leq \frac{\operatorname{dist}(Q_j, \partial D)}{l(Q_j)} \leq 4\sqrt{N}, \quad j = 1, 2, \ldots .$$

We will denote the collection of Whitney subcubes of D by E. If $Q_j, Q_k \in E$ we say

$$Q_j = Q(0) \to Q(1) \to \cdots \to Q(m) = Q_k$$

is a Whitney chain connecting Q_j to Q_k if $Q(i) \in E$, $Q(i) \cap Q(i+1) \neq \emptyset$, and we define the length of this chain to be m. We denote the length of the shortest chain connecting Q_j and Q_k by $\tilde{k}(Q_j, Q_k)$; then it is easy to show k is equivalent to $\tilde{k}$ on E. Likewise, j is equivalent to $\tilde{j}$ on E, where

$$\tilde{j}(Q_j, Q_k) = \left| \log \frac{l(Q_j)}{l(Q_k)} \right| + \log\left\{ \frac{d(Q_j, Q_k)}{l(Q_j) + l(Q_k)} + 2 \right\}. \tag{1.3}$$

Here $d(\ ,\ )$ denotes the Euclidean metric.

The proof of Theorem 1 in [6] consists of relating the average of a BMO(D) function on small Whitney cubes to the average on some large Whitney cube, the differences in the averages being bounded by the lengths of the corresponding shortest Whitney chain which is then controlled by (1.3) in a uniform domain. This method does not work for A_p weights; the lengths of the Whitney chains are too long for the A_p condition, even in the case when D is the upper half-plane. Furthermore, easy examples show that not every uniform domain is an extension domain for A_p. To circumvent this we introduce certain large cubes into the domain, through which to connect the Whitney cubes. The precise statement of our theorem is as follows.

THEOREM I. *Let D be a bounded uniform domain in $\mathbf{R}^N$ satisfying the following property: There exist $M \in \mathbf{N}$, $a, b, r_0 > 0$, $K > 1$, and a positive bounded increasing function $\epsilon(r)$ with $\lim_{r\to 0} \epsilon(r) = 0$ such that, for all $0 < r \le r_0$ and $p \in \partial D$, there exist M cubes $Q_1, \ldots, Q_M \subseteq B(p, Kr)$ with $l(Q_i) \ge (1/K)r$ such that for all $x \in B(p,r) \cap D$*

$$\min_{1\le i\le M} k(x, Q_i) \le a \log\left(1 + \frac{\epsilon(r) r}{\delta(x)}\right) + b. \tag{1.4}$$

Then D is an extension domain for A_p.

We note that in a uniform domain, the extra condition in Theorem I holds with $\epsilon(r)$ replaced by some $\epsilon > 0$. Indeed, the proof of Lemma 2.3 in [7] shows that $B(p,r)$ contains a Whitney cube Q with $l(Q) \ge (1/K)r$. It is straightforward to show that (1.4) holds for this cube.

The proof of Theorem I in Section 2 utilizes the following theorem of Wolff [9].

THEOREM 1. *Let $E \subseteq \mathbf{R}^N$, $|E| > 0$. Then, for $1 < p < \infty$, the following are equivalent:*

(1) *there exists $W \in A_p(\mathbf{R}^N)$ such that $W = w$ a.e. on E;*
(2) *there exists $\epsilon > 0$ such that*

$$\sup_{Q\in\mathfrak{F}} \left(\frac{1}{|Q|}\int_{Q\cap E} w^{1+\epsilon}\,dx\right)\left(\frac{1}{|Q|}\int_{Q\cap E}\left(\frac{1}{w^{1+\epsilon}}\right)^{1/(p-1)} dx\right)^{p-1} < \infty.$$

Here $\mathfrak{F}$ denotes the set of all cubes in $\mathbf{R}^N$. A proof of this result may be found in [5]. The proof of Theorem I consists of showing that to each $Q \in \mathfrak{F}$, Q small, there exists $Q' \in \mathfrak{F}_0$ with Q' approximately the same length as Q and such that

$$\int_{Q\cap D} w\,dx \le C\int_{Q'} w\,dx \quad \text{and} \quad \int_{Q\cap D}\left(\frac{1}{w}\right)^{1/(p-1)} dx \le C\int_{Q'}\left(\frac{1}{w}\right)^{1/(p-1)} dx.$$

In the case where $Q \in \mathfrak{F}$ is large, we subdivide Q into small cubes and apply the above to each of these small cubes.

In Section 3 of this paper we consider boundary conditions on a Jordan domain in $\mathbf{R}^2$ so that D is an extension domain for A_p. Let γ be a (closed) Jordan curve in $\mathbf{R}^2$ and let $\gamma(p_1,p_2)$ denote the subarc of γ of smaller Euclidean diameter connecting p_1 and p_2. We say that γ is *quasiconformal* if the quantity

$$\max_{p\in\gamma(p_1,p_2)} \frac{|p-p_1|+|p-p_2|}{|p_1-p_2|}, \quad p_1,p_2\in\gamma,$$

is bounded. In [6], it is shown that a Jordan domain D is uniform if and only if ∂D is quasiconformal. We say γ is *asymptotically conformal* if γ is quasiconformal and

$$\max_{p\in\gamma(p_1,p_2)} \frac{|p-p_1|+|p-p_2|}{|p_1-p_2|} \to 1 \quad as\ |p_1-p_2|\to 0.$$

Asymptotically conformal curves have been studied previously in [1] and [8]. The theorem we obtain as a direct consequence of Theorem I is the following.

THEOREM II. *Let D be the region interior to an asymptotically conformal curve γ in $\mathbf{R}^2$. Then D is an extension domain for A_p.*

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Let $w\in A_p(D)$. Then $w^{1+\epsilon}\in A_p(D)$ for some $\epsilon>0$ (the proof of this fact is entirely analogous to the proof of the corresponding result in $\mathbf{R}^N$; see [2, §VI.6]). Hence, according to Theorem 1, in order to prove Theorem I it suffices to establish

$$w\in A_p(D)\Rightarrow \sup_{Q\in\mathfrak{F}}\left(\frac{1}{|Q|}\int_{Q\cap D} w\,dx\right)\left(\frac{1}{|Q|}\int_{Q\cap D}\left(\frac{1}{w}\right)^{1/(p-1)}dx\right)^{p-1}<\infty.$$

Throughout the remainder of this section w will denote a fixed A_p weight on D, and C will denote various constants which will depend only on $\|w\|_p$, N, and the constants M,K,a,b,r_0 occurring in the statement of Theorem I. Furthermore, we will assume that $\operatorname{diam}(D)\le 1$.

LEMMA 1.

(1) *$w\in A_p(D)$ implies $(1/w)^{1/(p-1)}\in A_q(D)$, $q=p/(p-1)$.*

(2) *If $Q\in\mathfrak{F}_0$ then*

$$\frac{1}{|Q|}\int_Q w\,dx\le C\exp\left(\frac{1}{|Q|}\int_Q \log w\,dx\right).$$

(3) *If $Q_1,Q_2\in E$ then*

$$\frac{1}{|Q_1|}\int_{Q_1} w\,dx\le C\exp(Ck(Q_1,Q_2))\frac{1}{|Q_2|}\int_{Q_2} w\,dx.$$

(4) *There exists $0<\alpha<1$ such that if $Q\in\mathfrak{F}_0$ and E is a measurable subset of Q, then*

$$\int_E w\,dx \le C\left(\frac{|E|}{|Q|}\right)^{\alpha}\int_Q w\,dx.$$

(5) *If $Q_1, Q_2\in\mathfrak{F}_0$, $Q_2\subseteq Q_1$, then*

$$\frac{1}{|Q_1|}\int_{Q_1} w\,dx \le C\left(\frac{|Q_1|}{|Q_2|}\right)^{p-1}\frac{1}{|Q_2|}\int_{Q_2} w\,dx.$$

Proof. The proofs of (1), (2), and (4) are analogous to the proofs of the corresponding results in $\mathbf{R}^N$ (see [2, §VI.6]). (3) follows from (2) and Lemma 2.2 in [6]. To prove (5), we have

$$\begin{aligned}\frac{1}{|Q_1|}\int_{Q_1} w\,dx &\le C\bigg/\left(\frac{1}{|Q_1|}\int_{Q_1}\left(\frac{1}{w}\right)^{1/(p-1)}dx\right)^{p-1}\\ &\le C\bigg/\left(\frac{|Q_2|}{|Q_1|}\right)^{p-1}\left(\frac{1}{|Q_2|}\int_{Q_2}\left(\frac{1}{w}\right)^{1/(p-1)}dx\right)^{p-1}\\ &\le C\left(\frac{|Q_1|}{|Q_2|}\right)^{p-1}\left(\frac{1}{|Q_2|}\int_{Q_2} w\,dx\right)\end{aligned}$$

by Hölder's inequality. □

LEMMA 2. *Let $Q_1, Q_2\in\mathfrak{F}_0$ satisfy*

(1) $\dfrac{1}{C}\le\dfrac{\operatorname{dist}(Q_i,\partial D)}{l(Q_i)}\le C$, $i=1,2$;

(2) $k(Q_1,Q_2)\le C$.

Then

$$\frac{1}{C}\int_{Q_2} w\,dx \le \int_{Q_1} w\,dx \le C\int_{Q_2} w\,dx.$$

Proof. Let Q_1', Q_2' be cubes in E that contain the centers of Q_1, Q_2 respectively. Then (1) implies

$$\frac{1}{C}\le\frac{l(Q_i')}{l(Q_i)}\le C,\quad i=1,2. \tag{2.1}$$

(1) also implies that there are at most C cubes in E that intersect each of Q_1 and Q_2. Together with (2), this easily implies $k(Q_1', Q_2')\le C$. In particular, by (1.3),

$$\frac{1}{C}\le\frac{l(Q_1')}{l(Q_2')}\le C$$

and hence

$$\frac{1}{C}\le\frac{l(Q_1)}{l(Q_2)}\le C. \tag{2.2}$$

Since Q_i' contains the center of Q_i, $Q_i \cap Q_i'$ contains a cube Q_i'' with $l(Q_i'') \geq (1/2\sqrt{N}) \min(l(Q_i), l(Q_i'))$. Together with (2.1) we also have $l(Q_i) \leq Cl(Q_i'')$ and $l(Q_i') \leq Cl(Q_i'')$, $i = 1, 2$. Hence, by Lemma 1(5),

$$\begin{aligned}
\frac{1}{|Q_1|}\int_{Q_1} w\,dx &\leq C\frac{1}{|Q_1''|}\int_{Q_1''} w\,dx \\
&\leq C\frac{|Q_1'|}{|Q_1''|}\frac{1}{|Q_1'|}\int_{Q_1'} w\,dx \\
&\leq C\frac{1}{|Q_1'|}\int_{Q_1'} w\,dx \\
&\leq C\frac{1}{|Q_2'|}\int_{Q_2'} w\,dx \quad \text{by Lemma 1(3)} \\
&\leq C\left(\frac{|Q_2'|}{|Q_2''|}\right)^{p-1}\frac{1}{|Q_2''|}\int_{Q_2''} w\,dx \quad \text{by Lemma 1(5)} \\
&\leq C\frac{1}{|Q_2''|}\int_{Q_2''} w\,dx \\
&\leq C\frac{1}{|Q_2|}\int_{Q_2} w\,dx.
\end{aligned}$$

Together with (2.2) we thus have

$$\int_{Q_1} w\,dx \leq C\int_{Q_2} w\,dx.$$

The proof of $\int_{Q_2} w\,dx \leq C\int_{Q_1} w\,dx$ is the same. □

We will also require the following lemma, the proof of which is contained in the proof of Lemma 2.8 in [6].

LEMMA 3. *There exists $k \in \mathbf{N}$ such that if $Q \in \mathfrak{F}$ then there exists a dyadic subcube Q' of Q with $l(Q') = 2^{-k} l(Q)$ such that* $\operatorname{dist}(Q', \partial D) \geq l(Q')$.

We now set various constants. Choose $m \in \mathbf{N}$ and $m > k$ with $2^{-m} < 1/32K^2$. Choose δ_1 so that $\epsilon(10KN2^m\delta_1) < 1/20KN$ and let $\delta_0 = \min(r_0/20KN, \delta_1, 1)$.

Now let $Q_0 \in \mathfrak{F}_0$. We want to establish

$$\left(\frac{1}{|Q_0|}\int_{Q_0\cap D} w\,dx\right)\left(\frac{1}{|Q_0|}\int_{Q_0\cap D}\left(\frac{1}{w}\right)^{1/(p-1)} dx\right)^{p-1} < C; \tag{2.3}$$

clearly it suffices to assume $0 < |Q_0 \cap D| < |Q_0|$, for otherwise (2.3) is trivial. We will first assume $l(Q_0) \leq \delta_0$. Let Q_0' denote the cube with the same center as Q_0 and of length $2K\sqrt{N}\,l(Q_0)$. Then Q_0' contains a cube $Q \in E$ with $l(Q) \geq (1/32K^2) l(Q_0')$. Indeed, if $p \in Q_0 \cap \partial D$ then $Q_0 \cap D \subseteq B(p, \sqrt{N}\,l(Q_0))$. The hypothesis of the theorem implies there exists $Q' \subseteq B(p, K\sqrt{N}\,l(Q_0)) \cap D$ with $l(Q') \geq (\sqrt{N}/K) l(Q_0)$; in particular, $Q' \subseteq Q_0'$. Now Q' contains a cube $Q \in E$ with $l(Q) \geq (1/16) l(Q')$. Hence $Q \subseteq Q_0'$ and $l(Q) \geq (1/32K^2) l(Q_0')$. We will denote Q_0' by Q_0 and note now that $l(Q_0) \leq 2K\sqrt{N}\,\delta_0$, and we will establish (2.3) for this cube. By redefining the grid of dyadic cubes we can assume that Q_0 is dyadic.

We now apply Lemma 3 to obtain a dyadic decomposition of $Q_0 \cap D$. Let $G_0 = \{Q_0\}$ and subdivide Q_0 into 2^{mN} cubes $\{Q_j^1\}$ of equal length, and let

$$F_1 = \{Q_j^1 \subseteq Q \in E\};$$
$$G_1 = \{Q_j^1 : |Q_j^1 \cap D| > 0\}.$$

Since Q_0 contains a cube $Q \in E$ with $l(Q) \geq (1/32K^2)\, l(Q_0)$, $F_1 \neq \emptyset$. Thus

$$\sum\{|Q_j^1| : Q_j^1 \in G_1 \backslash F_1\} \leq (1 - 2^{mN}) |Q_0|.$$

Subdivide each $Q_j^1 \in G_1 \backslash F_1$ into 2^{mN} cubes $\{Q_j^1\}$ of equal length, and let

$$F_2 = \{Q_j^2 : Q_j^2 \subseteq Q \in E\};$$
$$G_2 = \{Q_j^2 : |Q_j^2 \cap D| > 0\}.$$

Then $F_2 \neq \emptyset$, since some cube $Q_j^1 \in F_1$ is adjacent to a cube $Q_k^1 \in G_1 \backslash F_1$. Because Q_j^1 is contained in a cube $Q \in E$ and each dyadic cube of length $\frac{1}{8} l(Q)$ adjacent to Q is contained in some $Q' \in E$, it follows that Q_k^1 contains a cube Q_j^2 that is contained in some $Q'' \in E$. In particular, $Q_j^2 \in F_2$. By Lemma 3, each $Q_j^1 \in G_1 \backslash F_1$ contains either a cube $Q_j^2 \in F_2$ or a cube $Q_j^2 \notin G_2$. Thus if $Q_k^1 \in G_1 \backslash F_1$ we have

$$\sum\{|Q_j^2| : Q_j^2 \subseteq Q_k^1,\ Q_j^2 \in G_2 \backslash F_2\} \leq (1 - 2^{mN}) |Q_k^1|$$

and hence

$$\sum\{|Q_j^2| : Q_j^2 \in G_2 \backslash F_2\} \leq (1 - 2^{mN})^2 |Q_0|.$$

Continue forming $\{Q_j^3\}, \{Q_j^4\}, \ldots, F_3, F_4, \ldots, G_3, G_4, \ldots$. A cube Q_j^n satisfies $l(Q_j^n) = 2^{-mn} l(Q_0)$ and $Q_j^n \subseteq Q_k^{n-1}$ for some $Q_k^{n-1} \in G_{n-1} \backslash F_{n-1}$. Define

$$F_n = \{Q_j^n : Q_j^n \subseteq Q \in E\};$$
$$G_n = \{Q_j^n : |Q_j^n \cap D| > 0\}.$$

The above arguments show that $F_n \neq \emptyset$ and

$$\sum\{|Q_j^n| : Q_j^n \in G_n \backslash F_n\} \leq (1 - 2^{mN})^n |Q_0|,$$

and hence $Q_0 \cap D = \bigcup_{n=1}^{\infty} \{Q : Q \in F_n\}$.

Now, for each $k \geq 0$, let $\lambda_k = 5\sqrt{N}\, 2^{-m(k-1)} l(Q_0)$ and define a sequence $\{n_k\}$ by $n_0 = 0$ and

$$n_k = n_{k-1} + \left[\frac{1}{m} \log_2\left(\frac{1}{\epsilon(\lambda_{n_{k-1}})}\right)\right] + 2, \quad k \geq 1, \tag{2.4}$$

where [ ] denotes the greatest integer function.

Now if $Q \in G_n$ then $Q \subseteq Q_1$ for some $Q_1 \in G_{n-1} \backslash F_{n-1}$. This implies

$$\begin{aligned} \operatorname{dist}(Q, \partial D) &\leq \sqrt{N}\, l(Q_1) + \operatorname{dist}(Q_1, \partial D) \\ &\leq (\sqrt{N} + 1)\, l(Q_1) \\ &\leq 2\sqrt{N}\, 2^{-m(n-1)} l(Q_0), \end{aligned}$$

and hence there exists $p \in \partial D$ such that $Q \cap D \subseteq B(p, \lambda_n)$. The hypothesis of Theorem I implies there exist M cubes $R_1(Q), \ldots, R_M(Q) \subseteq D \cap B(p, K\lambda_n)$ with $l(R_i(Q)) \geq K^{-1}\lambda_n$ such that if $x \in Q \cap D$ then

$$k(x, \cup R_i(Q)) \leq a \log\left(1 + \frac{\epsilon(\lambda_n)\lambda_n}{\delta(x)}\right) + b. \tag{2.5}$$

Now if $n_k < n \leq n_{k+1}$ then to each $Q_j^n \in F_n$ we will associate a cube $\tilde{Q}_j^n$, $\tilde{Q}_j^n \subseteq R_i(Q)$, for some $Q \in G_{n_k}$ and satisfying:

(1) $k(Q_j^n, \tilde{Q}_j^n) \leq C$;

(2) $(1/C)\, l(\tilde{Q}_j^n) \leq \operatorname{dist}(\tilde{Q}_j^n, \partial D) \leq C l(\tilde{Q}_j^n)$;

(3) if $Q \in G_{n_k}$ then each point $x \in R_i(Q)$ is contained in at most C cubes $\tilde{Q}_j^n$.

To obtain the cubes $\tilde{Q}_j^n$ we proceed as follows. Each $Q_j^n \in F_n$ is contained in some $Q \in G_{n_k}$. Furthermore, if $x \in Q_j^n$ then

$$\begin{aligned} \delta(x) &\geq 2^{-mn_{k+1}} l(Q_0) \\ &\geq \frac{2^{-3m}}{5\sqrt{N}} \epsilon(\lambda_{n_k})\lambda_{n_k} \quad \text{by (2.4)} \\ &= \eta_0 \epsilon(\lambda_{n_k})\lambda_{n_k}, \quad \text{say.} \end{aligned}$$

Thus by (2.5), for some cube $R_i(Q)$, $k(x, R_i(Q)) \leq a \log(1 + 1/\eta_0) + b = a\eta_1 + b$, say. Now let

$$p = \left[\log_2\left(e^{a\eta_1 + b}\sqrt{N}\, \frac{l(R_i(Q))}{l(Q_j^n)}\right)\right] + 2,$$

and note that since

$$l(R_i(Q)) \geq \frac{1}{K}\lambda_{n_k} \geq \frac{2^m}{K} l(Q_j^n) > l(Q_j^n)$$

we have $p \geq 2$. Subdivide $R_i(Q)$ into 2^{pN} subcubes $\{P_i\}$ of equal length. Choose one such cube for which $k(Q_j^n, P_i) \leq a\eta_1 + b$ and denote this cube by $\tilde{Q}_j^n$. Then clearly (1) holds. Furthermore, by the choice of p,

$$\frac{1}{4\sqrt{N}} e^{-(a\eta_1 + b)} l(Q_j^n) \leq l(\tilde{Q}_j^n) \leq \frac{1}{2\sqrt{N}} e^{-(a\eta_1 + b)} l(Q_j^n). \tag{2.6}$$

To prove (2), let $x_1 \in \tilde{Q}_j^n$, $y \in Q_j^n$ be such that $k(x_1, y) \leq a\eta_1 + b$. Then by (1.2)

$$e^{-(a\eta_1 + b)} \leq \frac{\delta(x_1)}{\delta(y)} \leq e^{a\eta_1 + b},$$

and hence if $x \in \tilde{Q}_j^n$ is such that $\delta(x) = \operatorname{dist}(\tilde{Q}_j^n, \partial D)$ then

$$\begin{aligned} \delta(x) &\leq \delta(x_1) \\ &\leq e^{a\eta_1 + b}\delta(y) \\ &\leq C l(Q_j^n) \\ &\leq C l(\tilde{Q}_j^n) \quad \text{by (2.6).} \end{aligned}$$

Also,

$$\begin{aligned}\delta(x) &\ge \delta(x_1) - \sqrt{N}\, l(\tilde{Q}_j^n) \\ &\ge e^{-(a\eta_1+b)}\delta(y) - \sqrt{N}\, l(\tilde{Q}_j^n) \\ &\ge e^{-(a\eta_1+b)} l(Q_j^n) - \sqrt{N}\, l(\tilde{Q}_j^n) \\ &\ge \sqrt{N}\, l(\tilde{Q}_j^n) \quad \text{by (2.6)},\end{aligned}$$

and (2) now follows.

To prove (3), suppose $Q_1 \in F_m$, $Q_2 \in F_n$, $n \ge m > n_k$, and $\tilde{Q}_2 \cap \tilde{Q}_1 \ne \emptyset$. Then by (1) and (2), $k(Q_1, Q_2) \le C$. Thus if $Q_1', Q_2' \in E$ contain Q_1, Q_2 respectively then $k(Q_1', Q_2') \le C$. Hence by (1.3) we have $d(Q_1', Q_2') \le C(l(Q_1') + l(Q_2'))$ and $1/C \le l(Q_2')/l(Q_1') \le C$. Then since $l(Q_i') \le Cl(Q_i)$, $i = 1, 2$, it follows that $l(Q_2) \le Cl(Q_1)$ and $d(Q_1, Q_2) \le Cl(Q_2)$ and (3) follows easily.

Properties (1) and (2) and Lemma 2 imply

$$\int_{Q_j^n} w\,dx \le C \int_{\tilde{Q}_j^n} w\,dx.$$

Together with (3) this implies

$$\sum_{n=n_k+1}^{n_{k+1}} \sum_{Q_j^n \in F_n} \int_{Q_j^n} w\,dx \le C \sum_{Q \in G_{n_k}} \sum_{i=1}^{M} \int_{R_i(Q)} w\,dx. \tag{2.7}$$

We now claim that, for each $Q \in G_{n_k}$,

$$\sum_{i=1}^{M} \int_{R_i(Q)} w\,dx \le C \int_{R_1(Q)} w\,dx. \tag{2.8}$$

To prove this, we note that if $Q_i \in E$ contains the center of $R_i(Q)$ then

$$\frac{1}{C} \le \frac{l(R_i(Q))}{l(Q_i)} \le C$$

and hence

$$\frac{1}{C} \le \frac{l(Q_1)}{l(Q_i)} \le C, \tag{2.9}$$

since $l(R_i(Q)) \approx l(R_1(Q))$. Furthermore, $Q_i \cap R_i(Q)$ contains a cube Q_i' with $l(Q_i') \ge (1/C)\, l(R_i(Q))$. Thus, by Lemma 1(5)

$$\int_{R_i(Q)} w\,dx \le C \int_{Q_i'} w\,dx \le C \int_{Q_i} w\,dx,$$

and similarly

$$\int_{Q_i} w\,dx \le C \int_{R_i(Q)} w\,dx. \tag{2.10}$$

Since the cubes $R_i(Q)$ are contained in a sphere of radius $\le Cl(R_1(Q))$, we have

$$\operatorname{dist}(Q_i, Q_1) \le Cl(R_1(Q)) \le Cl(Q_1). \tag{2.11}$$

(2.9) and (2.11) imply $k(Q_i, Q_1) \le C$. Hence by Lemma 1(3) and (2.10) we have

$$\int_{Q_i} w\,dx \le C\int_{Q_1} w\,dx \le C\int_{R_1(Q)} w\,dx,$$

and (2.8) follows. Thus (2.7) reduces to

$$\sum_{n=n_k+1}^{n_{k+1}} \sum_{Q_j^n \in F_n} \int_{Q_j^n} w\,dx \le C \sum_{Q\in G_{n_k}} \int_{R_1(Q)} w\,dx. \tag{2.12}$$

Now to each cube $R_1(Q)$, $Q\in G_{n_k}$, $k\ge 1$, we will associate a cube $\tilde{R}_1(Q)$ satisfying the following properties:

(1) $\tilde{R}_1(Q)\subseteq R_i(Q_1)$ for some $1\le i\le M$, $Q_1\in G_{n_{k-1}}$;
(2) $\int_{R_1(Q)} w\,dx \le C\int_{\tilde{R}_1(Q)} w\,dx$;
(3) if $Q_1\in G_{n_{k-1}}$ then each point $x\in R_i(Q_1)$ is contained in at most C cubes $\tilde{R}_1(Q)$, $Q\in G_{n_k}$.
(4) For each $Q_1\in G_{n_{k-1}}$, $1\le i\le M$, let

$$E_i(Q_1)=\bigcup_{Q\in G_{n_k}}\{\tilde{R}_1(Q):\tilde{R}_1(Q)\subseteq R_i(Q_1)\}.$$

Then $|E_i(Q_1)|\le \epsilon_k|R_i(Q_1)|$, where $\{\epsilon_k\}$ is a positive decreasing sequence with $\lim_{k\to\infty}\epsilon_k=0$.

To obtian the cube $\tilde{R}_1(Q)$ we proceed as follows. Fix $R_1(Q)$, $Q\in G_{n_k}$. Let $Q_1\in E$ contain the center of $R_1(Q)$. Then

$$\begin{aligned} l(Q_1) &\le \operatorname{dist}(Q_1,\partial D)\\ &\le \sqrt{N}\,l(R_1(Q))+\operatorname{dist}(R_1(Q),\partial D)\\ &\le K\sqrt{N}\lambda_{n_k}+K\lambda_{n_k}\\ &\le 20KN2^m2^{-mn_k}l(Q_0)\\ &< 2^{-mn_{k-1}}l(Q_0), \end{aligned}\tag{2.13}$$

where the last inequality follows from (2.4) and the assumption $\epsilon(\lambda_{n_{k-1}})<1/20KN$. Also,

$$\begin{aligned} l(Q_1) &\ge \frac{1}{16}l(R_1(Q))\\ &\ge \frac{1}{16K}l(Q)\\ &\ge 2^{-m(n_k+N_1)}l(Q_0) \end{aligned}\tag{2.14}$$

for some $N_1\in \mathbf{N}$. Now Q_1 contains a cube $Q_2\in F_n$ with $l(Q_2)\ge 2^{-m}l(Q_1)$. By (2.13) and (2.14) we have $n_{k-1}<n<n_k+N_1+1$. Now the same construction that was used to obtain cubes $\tilde{Q}_j^n$ when $Q_j^n\in F_n$, $n_{k-1}<n\le n_k$, can be used to obtain a cube $\tilde{Q}_2$ contained in some $R_i(Q)$, $Q\in G_{n_{k-1}}$, and satisfying the following properties:

(i) $(1/C)l(\tilde{Q}_2)\le \operatorname{dist}(\tilde{Q}_2,\partial D)\le Cl(\tilde{Q}_2)$;
(ii) $k(Q_2,\tilde{Q}_2)\le C$.

We note that (i) and (ii) imply

(iii) $\dfrac{1}{C}\le\dfrac{l(Q_2)}{l(\tilde{Q}_2)}\le C$ and hence $\dfrac{1}{C}\le\dfrac{l(R_1(Q))}{l(\tilde{Q}_2)}\le C.$

We define $\tilde{R}_1(Q)$ to be $\tilde{Q}_2$, and it remains to verify properties (2)–(4). To prove (2), we first note that $Q_1 \cap R_1(Q)$ contains a cube Q_1' with $l(Q_1') \geq (1/C)\min(l(R_1(Q)), l(Q_1))$. Thus by Lemma 1(5),

$$\begin{aligned}\int_{R_1(Q)} w\,dx &\leq C\int_{Q_1'} w\,dx \\ &\leq C\int_{Q_1} w\,dx \\ &\leq C\int_{Q_2} w\,dx \\ &\leq C\int_{\tilde{Q}_2} w\,dx = C\int_{\tilde{R}_1(Q)} w\,dx,\end{aligned}$$

where the last inequality follows from properties (i) and (ii) and Lemma 2, and (2) now follows.

To prove (3), we will show that if $Q_1, Q_2 \in G_{n_k}$ and $\tilde{R}_1(Q_1) \cap \tilde{R}_1(Q_2) \neq \emptyset$ then $d(Q_1, Q_2) \leq Cl(Q_1)$, from which (3) will follow. For $i = 1, 2$ let $Q_i' \in E$ be the cube considered above that contains the center of $R_1(Q_i)$. Then $l(Q_i') \leq Cl(Q_i)$ and $k(Q_i', \tilde{R}_1(Q_i)) \leq C$ (by (ii)), and $d(Q_i', Q_i) \leq Cl(Q_i)$ (since $R_1(Q_i)$ is contained in a sphere of radius $\leq Cl(Q_i)$). Together with (i) this implies $k(Q_1', Q_2') \leq C$ and hence $d(Q_1', Q_2') \leq C(l(Q_1') + l(Q_2')) \leq Cl(Q_1') \leq Cl(Q_1)$. Thus $d(Q_1, Q_2) \leq Cl(Q_1)$.

To prove (4), we note that if $x \in E_i(Q_1)$ then $x \in \tilde{R}_1(Q)$ for some $Q \in G_{n_k}$. Thus, by (i),

$$\begin{aligned}\delta(x) &\leq Cl(\tilde{R}_1(Q)) \\ &\leq l(R_1(Q)) \quad \text{by (iii)} \\ &\leq Cl(Q).\end{aligned}$$

Hence

$$\begin{aligned}|E_i(Q_1)| &\leq Cl(Q)(l(R_i(Q_1)))^{N-1} \\ &\leq C\left(\frac{l(Q)}{l(Q_1)}\right)|R_i(Q_1)| \\ &\leq C2^{m(n_{k-1}-n_k)}|R_i(Q_1)| \\ &\leq C\epsilon(\lambda_{n_{k-1}})|R_i(Q_1)| \quad \text{by (2.4)} \\ &= \epsilon_k|R_i(Q_1)|,\end{aligned}$$

where $\epsilon_k \to 0$ as $k \to \infty$.

Returning to (2.12), we have

$$\begin{aligned}\sum_{n=n_k+1}^{n_{k+1}} \sum_{Q_j^n \in F_n} \int_{Q_j^n} w\,dx &\leq C \sum_{Q \in G_{n_k}} \int_{R_1(Q)} w\,dx \\ &\leq C \sum_{Q \in G_{n_k}} \int_{\tilde{R}_1(Q)} w\,dx \quad \text{by (2)} \\ &\leq C \sum_{Q_1 \in G_{n_{k-2}}} \sum_{i=1}^{M} \int_{E_i(Q_1)} w\,dx \quad \text{by (3)}\end{aligned}$$

$$\leq C \sum_{Q_1 \in G_{n_{k-1}}} \sum_{i=1}^{M} \left(\frac{|E_i(Q_1)|}{|R_i(Q_1)|}\right)^{\alpha} \int_{R_i(Q_1)} w\,dx \quad \text{by Lemma 1(4)}$$

$$\leq C\epsilon_k^{\alpha} \sum_{Q_1 \in G_{n_{k-1}}} \sum_{i=1}^{M} \int_{R_i(Q_1)} w\,dx \quad \text{by (4)}$$

$$\leq C\epsilon_k^{\alpha} \sum_{Q_1 \in G_{n_{k-1}}} \int_{R_1(Q_1)} w\,dx \quad \text{by (2.8).}$$

Proceeding by induction we have

$$\sum_{n=n_k+1}^{n_{k+1}} \sum_{Q_j^n \in F_n} \int_{Q_j^n} w\,dx \leq C^k \left(\prod_{j=1}^{k} \epsilon_j\right)^{\alpha} \int_{R_1(Q_0)} w\,dx.$$

Hence

$$\int_{Q_0 \cap D} w\,dx = \sum_{k=0}^{\infty} \sum_{n=n_k+1}^{n_{k+1}} \sum_{Q_j^n \in F_n} \int_{Q_j^n} w\,dx$$

$$\leq \sum_{n=1}^{n_1} \sum_{Q_j^n \in F_n} \int_{Q_j^n} w\,dx + \sum_{k=1}^{\infty} C^k \left(\prod_{j=1}^{k} \epsilon_j\right)^{\alpha} \int_{R_1(Q_0)} w\,dx$$

$$\leq C \int_{R_1(Q_0)} w\,dx + C \int_{R_1(Q_0)} w\,dx \quad \text{since } \epsilon_k \to 0$$

$$\leq C \int_{R_1(Q_0)} w\,dx.$$

Applying the same arguments as above to the weight $(1/w)^{1/(p-1)}$ yields

$$\int_{Q_0 \cap D} \left(\frac{1}{w}\right)^{1/(p-1)} dx \leq C \int_{R_1(Q_0)} \left(\frac{1}{w}\right)^{1/(p-1)} dx.$$

Hence, since $1/C \leq l(R_1(Q_0))/l(Q_0) \leq C$ we have

$$\left(\frac{1}{|Q_0|} \int_{Q_0 \cap D} w\,dx\right) \left(\frac{1}{|Q_0|} \int_{Q_0 \cap D} \left(\frac{1}{w}\right)^{1/(p-1)} dx\right)^{p-1} \leq C,$$

and (2.3) in the case $l(Q_0) \leq \delta_0$ now follows.

Suppose now that $Q_0 \in \mathfrak{F}$, $l(Q_0) > \delta_0$, and $0 < |Q_0 \cap D| < |Q_0|$. Let m be the first positive integer for which $2^{-m} l(Q_0) \leq \delta_0$, and subdivide Q_0 into 2^{mN} cubes $Q_1, Q_2, \ldots$ of equal length. Since $\operatorname{diam}(D) \leq 1$, there exist at most M of these cubes which intersect D, where $M < (2/\delta_0)^N$. By relabelling we can assume these cubes are $Q_1, \ldots, Q_M$. By what we established above, there exist cubes $Q_1', \ldots, Q_M'$ contained in D such that

$$\text{(2.15)} \qquad \frac{1}{C} \leq \frac{l(Q_i')}{l(Q_i)} \leq C, \quad 1 \leq i \leq M,$$

and

$$\text{(2.16)} \qquad \int_{Q_i \cap D} w\,dx \leq C \int_{Q_i'} w\,dx, \quad 1 \leq i \leq M.$$

Furthermore, if $Q_i'' \in E$ contains the center of Q_i' then

(2.17)
$$\frac{1}{C} \leq \frac{l(Q_i'')}{l(Q_i')} \leq C$$

and

(2.18)
$$\int_{Q_i'} w\,dx \leq C \int_{Q_i''} w\,dx.$$

(2.16) and (2.18) imply

$$\int_{Q_0 \cap D} w\,dx = \sum_{i=1}^{M} \int_{Q_i \cap D} w\,dx$$
$$\leq C \sum_{i=1}^{M} \int_{Q_i''} w\,dx.$$

Also, (2.15) and (2.17) imply

$$\frac{1}{C} \leq \frac{l(Q_i'')}{l(Q_1'')} \leq C.$$

Together with the fact that $\operatorname{diam}(D) \leq 1$ this implies $k(Q_1'', Q_i'') \leq C$. Thus, by Lemma 1(3) we have

$$\sum_{i=1}^{M} \int_{Q_i''} w\,dx \leq C \int_{Q_1''} w\,dx$$

Hence

$$\int_{Q_0 \cap D} w\,dx \leq C \int_{Q_1''} w\,dx$$

and similarly, applying the above arguments to the weight $(1/w)^{1/(p-1)}$, we have

$$\int_{Q_0 \cap D} \left(\frac{1}{w}\right)^{1/(p-1)} dx \leq C \int_{Q_1''} \left(\frac{1}{w}\right)^{1/(p-1)} dx.$$

Since $l(Q_1'') \leq C l(Q_0)$ we have

$$\left(\frac{1}{|Q_0|} \int_{Q_0 \cap D} w\,dx\right)\left(\frac{1}{|Q_0|} \int_{Q_0 \cap D} \left(\frac{1}{w}\right)^{1/(p-1)} dx\right)^{p-1}$$
$$\leq C\left(\frac{1}{|Q_1''|} \int_{Q_1''} w\,dx\right)\left(\frac{1}{|Q_1''|} \int_{Q_1''} \left(\frac{1}{w}\right)^{1/(p-1)} dx\right)^{p-1}$$
$$\leq C,$$

and this establishes (2.3) in the case $l(Q_0) > \delta_0$ and completes the proof of Theorem I. □

REMARKS. (1) If in the statement of Theorem I all cubes are assumed to have sides parallel to the axes, and if we define the A_p condition by (1.1) where we now take the supremum over all cubes $Q \subseteq D$ with sides parallel to the axes, then D will be an extension domain for this class of A_p weights; the proof is the same as the proof above.

(2) We give an example to illustrate that Theorem I is sharp. Let $\{\alpha_n\}_{n\geq 0}$ be a positive decreasing sequence with $\alpha_n \leq 2^{-n}$. Let $s_n = \sum_{k=0}^{n} \alpha_k$, $D_n = \{(x,y): s_n \leq x \leq s_{n+1}, 0 \leq y \leq 2^{-n}\}$, $D = \operatorname{int}(\bigcup_{n\geq 0} D_n)$. It is straightforward to show that if $\lim_{n\to\infty} \alpha_n 2^{-n} = 0$ then D satisfies the hypothesis of Theorem I and hence will be an extension domain for A_p. Suppose $\lim_{n\to\infty} \alpha_n 2^{-n} > 0$. Then it is clear that any cube contained in D can intersect at most C rectangles D_n. Hence $w = \sum (1/|D_n|)\chi_{D_n}$ belongs to $A_p(D)$ for all $p > 1$ while $\int_D w\,dx = \infty$, and so D will not be an extension domain for A_p.

3

In this section we prove Theorem II. We will assume $\operatorname{diam}(D) \geq 1$. Denote by

$$\mu(\delta) = \sup_{|p_1 - p_2| < \delta} \left(\max_{p_3 \in \gamma(p_1, p_2)} \frac{|p_1 - p_3| + |p_2 - p_3|}{|p_1 - p_2|} \right) - 1.$$

Choose $r_0 < 1/48$ and such that $\mu(8r_0) < 1/32$.

If $r < r_0$ we define $\rho(r)$ to be the smallest positive number such that, whenever l_1, l_2, l_3 are three sides of a triangle which satisfy $\rho(r) r \leq l_i \leq 8r$, we have

$$l_i + l_j \geq (1 + \mu(8r)) l_k, \quad i \neq j \neq k. \tag{3.1}$$

(We assume r_0 is sufficiently small so that $\rho(r) < \frac{1}{8}$.) Let

$$\epsilon(r) = 6(\rho(r) + 4\mu(r))r$$

and note that $\lim_{r\to 0} \epsilon(r) = 0$. Now whenever $p \in \partial D$ and $r < r_0$ we will construct a cube $Q \subseteq D \cap B(p, Cr)$ such that if $x \in B(p, r) \cap D$ then there exists $y \in Q$ such that $\delta(y) \geq \delta(x)$ and $|x - y| \leq \epsilon(r) r$. Then, since D is uniform, we have

$$\begin{aligned} k(x, Q) &\leq k(x, y) \\ &\leq a \log\left(1 + \frac{|x-y|}{\delta(x)}\right) + b \\ &\leq a \log\left(1 + \frac{\epsilon(r) r}{\delta(x)}\right) + b \end{aligned}$$

and so Theorem II will follow from Theorem I. To obtain the cube Q satisfying the above, we proceed as follows. Fix an orientation of γ so that D lies to the left of γ and traverse γ from p in this direction. Let p_1, p_2 be the first and last points in $\gamma \cap B(p, 2r)$. Then clearly $p \in \gamma(p_1, p_2)$. We also note that

$$\begin{aligned} |p_1 - p_2| &\geq \frac{|p - p_1| + |p - p_2|}{1 + \mu(4r)} \\ &\geq \frac{4r}{1 + \mu(4r)} \\ &> 3r. \end{aligned}$$

Let $l(p_1, p_2)$ denote the line segment connecting p_1, p_2 and let Q_1, Q_2 be the two cubes with sides coinciding with $l(p_1, p_2)$. Translate each cube by an amount $\rho(r)r$ in a direction perpendicular to $l(p_1, p_2)$ so that p_1 and p_2 are not contained in the resulting cubes, which we again denote by Q_1, Q_2. We claim $Q_i \cap \gamma = \emptyset$, $i = 1, 2$. We will prove this for $i = 1$; the case $i = 2$ is the same. Suppose γ intersects Q_1 in the point p_3. Then p_1, p_2, p_3 are the vertices of a triangle, the side lengths of which satisfy (3.1). Hence $p_i \notin \gamma(p_j, p_k)$, $i \neq j \neq k$. This means that

$$\gamma = \gamma(p_1, p_2) \cup \gamma(p_2, p_3) \cup \gamma(p_1, p_3). \tag{3.2}$$

Now if $q_1, q_2 \in \gamma(p_i, p_j)$ then

$$\begin{aligned} |q_1 - q_2| &\leq \tfrac{1}{2}\{|q_1 - p_i| + |q_1 - p_j| + |q_2 - p_i| + |q_2 - p_j|\} \\ &\leq (1 + \mu(|p_i - p_j|))|p_i - p_j| \\ &< 16r < \tfrac{1}{3}. \end{aligned}$$

Thus $\operatorname{diam}(\gamma(p_i, p_j)) < \frac{1}{3}$ and hence, by (3.2), $\operatorname{diam}(\gamma) < \frac{2}{3}$ which contradicts the assumption $\operatorname{diam}(\gamma) \geq 1$. Hence $Q_1 \cap \gamma = \emptyset$ and similarly $Q_2 \cap \gamma = \emptyset$. Let Q be the Q_i that lies to the left of γ, so that $Q \subseteq D$.

Now if $x \in B(p, r) \cap D$ and $\delta(x) > (\rho(r) + 4\mu(r))r$ then we claim $x \in Q$. Indeed, if $y \in \partial D$ and $|y - x| = \delta(x)$ then $|y - p| \leq 2r$ and hence $y \in \gamma(p_1, p_2)$. A calculation then gives $\operatorname{dist}(y, l(p_1, p_2)) \leq 4\mu(r)r$ and hence $\operatorname{dist}(x, l(p_1, p_2)) \geq \rho(r)r$, which implies $x \in Q$. Suppose $\delta(x) \leq (\rho(r) + 4\mu(r))r$. Then

$$\operatorname{dist}(x, l(p_1, p_2)) \leq (\rho(r) + 8\mu(r))r$$

which implies

$$\begin{aligned} \operatorname{dist}(x, Q) &\leq (2\rho(r) + 8\mu(r))r \\ &< \tfrac{1}{2} l(Q). \end{aligned}$$

Hence there exists $y \in Q$ such that $\delta(y) \geq \delta(x)$ and

$$\begin{aligned} |y - x| &\leq 3(2\rho(r) + 8\mu(r))r \\ &\leq \epsilon(r)r, \end{aligned}$$

and this completes the proof of Theorem II. □

References

1. J. Becker and Ch. Pommerenke, *Über die quasikonforme Fortsetzung schlichter Funktionen,* Math. Z. 161 (1978), 69–80.
2. J. Garnett, *Bounded analytic functions,* Academic Press, New York, 1981.
3. F. Gehring and B. Osgood, *Uniform domains and the quasi-hyperbolic metric,* J. Analyse Math. 36 (1979), 50–74.
4. F. Gehring and B. Palka, *Quasiconformally homogeneous domains,* J. Analyse Math. 30 (1976), 172–199.
5. P. Holden, *Extension theorems for functions of vanishing mean oscillation,* Pacific J. Math. (to appear).

6. P. Jones, *Extension theorems for BMO,* Indiana Univ. Math. J. 29 (1979), 41–66.
7. ———, *Quasiconformal mappings and extendability of functions in Sobolev spaces,* Acta. Math. 147 (1981), 71–88.
8. Ch. Pommerenke, *On univalent functions, Bloch functions and VMOA,* Math. Ann. 236 (1978), 199–208.
9. T. Wolff, *Restrictions of A_p weights* (preprint).

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