Rational Powers of Generators of Möbius Groups

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1. Introduction

In this paper we will be concerned with 2-generator Fuchsian groups, in particular groups of the first kind (see [4] or [1]; also, for 2-generator groups see e.g. [10] or [12]). We will use the following notation: The group of all Möbius self-maps of the upper half plane $\mathbf{H} = \{z \in \mathbf{C} \colon \mathrm{Im}(z) > 0\}$ will be denoted by $\mathrm{PSL}(2, \mathbf{R})$, and Möbius self-maps of the Riemann sphere will be denoted by $\mathrm{PSL}(2, \mathbf{C})$.

We need a definition of roots and rational powers of Möbius maps; this is motivated by the fact that, for each integer n, the nth power of g in $\mathrm{PSL}(2, \mathbf{R})$, written g^n, is its nth iterate. Thus we want an nth root of g to be a map in $\mathrm{PSL}(2, \mathbf{R})$ whose nth iterate is g; this will be denoted $g^{1/n}$ and will not in general be unique. We similarly define rational powers of g. We begin with the simplest case, namely rational powers of parabolic Möbius maps. Rational powers of hyperbolic maps are also easy to define but will not be needed here.

DEFINITION. Let g_0 be the parabolic Möbius map given by $g_0 \colon z \mapsto z+1$, and let g in $\mathrm{PSL}(2, \mathbf{R})$ also be parabolic. Thus there is a map f in $\mathrm{PSL}(2, \mathbf{C})$ with $g = f g_0 f^{-1}$. For each k in $\mathbf{Q}$ we define the kth *power* of g_0 to be $g_0^k \colon z \mapsto z+k$ and the kth power of g to be $g^k = f g_0^k f^{-1}$.

We remark that it is necessary for f to be in $\mathrm{PSL}(2, \mathbf{C})$ and not just in $\mathrm{PSL}(2, \mathbf{R})$. The above definition gives a unique kth power of a parabolic Möbius map for all rational k. Hyperbolic maps in $\mathrm{PSL}(2, \mathbf{R})$ also have unique rational powers. Rational powers of elliptic maps are not unique, and are defined as follows.

DEFINITION. Let g_0 be the elliptic Möbius self-map of the unit disc $g_0 \colon z \mapsto e^{i\theta} z / e^{-i\theta}$ where $0 < |\theta| \leq \pi/2$, and let g be an elliptic element of $\mathrm{PSL}(2, \mathbf{R})$ whose trace satisfies $\mathrm{tr}^2(g) = \mathrm{tr}^2(g_0) = 4\cos^2\theta$. Thus there is an f in $\mathrm{PSL}(2, \mathbf{C})$ with $g = f g_0 f^{-1}$. For each rational k we define a kth power of g_0 to be $g_0^k \colon z \mapsto e^{ki\theta} z / e^{-ki\theta}$ and a kth power of g to be $g^k = f g_0^k f^{-1}$.

Received October 17, 1990. Revision received May 31, 1991.
Michigan Math. J. 39 (1992).

If we had chosen a different interval for the argument of g_0, then we would have found a different argument of the kth power of g_0. When k is in $\mathbf{Z}$ these differ by an integer multiple of 2π and we obtain the same map as the kth power; but in general these give distinct kth powers. With the above choice of interval for θ, the only ambiguity occurs when $\theta = \pm\pi/2$. Both of these angles give the same map and a choice of kth powers which are inverses of each other.

The geometrical interpretation of rational powers of elliptics is as follows. For elliptic g in $\mathrm{PSL}(2,\mathbf{R})$ and rational $k<1$ we may write $g=\sigma_1\sigma_0$ and $g^k = \sigma_k\sigma_0$, where σ_j is reflection in the geodesic L_j. If θ is the angle between L_0 and L_1 at the fixed point of g, then the angle between L_0 and L_k is $k\theta$ at the fixed point of g, which is of course the fixed point of g^k.

This paper is devoted to the investigation of the group generated by rational powers of the generators of a 2-generator Fuchsian group. Suppose g and h in $\mathrm{PSL}(2,\mathbf{R})$ have distinct fixed points and generate a discrete group $\langle g,h\rangle$. For each pair of integers a and b the group $\langle g^a,h^b\rangle$ is a subgroup of $\langle g,h\rangle$ and hence is also discrete. We consider two problems: First, for which integers a and b is the group $\langle g^{1/a},h^{1/b}\rangle$ discrete; and, more generally, for which rationals t and u is $\langle g^t,h^u\rangle$ discrete?

These questions are most interesting when we assume that both groups are of the first kind. To see this, let g and h be as above, and let rationals t and u be chosen so that g^t and h^u are primitive and g^th^u and $g^{-t}h^u$ are both hyperbolic; then the group $\langle g^t,h^u\rangle$ is discrete and of the second kind (see Proposition 2.1). Such examples are very easy to construct. We therefore restrict our attention to groups of the first kind.

The case of two parabolics generating a Fuchsian group of the first kind was solved by Beardon in [2]. There he proved the following result.

THEOREM 1.1 (Theorem 1 of [2]). *Suppose that g and h in* $\mathrm{PSL}(2,\mathbf{R})$ *are parabolic with distinct fixed points, that the groups $\langle g,h\rangle$ and $\langle g^t,h^u\rangle$ are discrete and of the first kind, and that t and u are rational. Then, to within conjugacy, the pair $\langle g,h\rangle$ and $\langle g^t,h^u\rangle$ is one of the six cases:*

$$\left\langle z\mapsto z+\frac{c}{2},\, z\mapsto\frac{z}{2z+1}\right\rangle,\qquad \left\langle z\mapsto z+\frac{a}{2},\, z\mapsto\frac{z}{2z+1}\right\rangle,$$

where $t=a/c$, $u=1$, $a<c$, and $a,c\in\{1,2,3,4\}$. □

The purpose of this paper is to extend Beardon's result to other groups of the first kind. In Section 3 we list all conjugacy classes of parabolics g and elliptics h and all pairs of rationals t and u for which $\langle g,h\rangle$ and $\langle g^t,h^u\rangle$ are both Fuchsian groups of the first kind. In Section 4 we give two partial results for groups generated by two classes of elliptics. Namely, we list all pairs of integers a and b for which $\langle g,h\rangle$ and $\langle g^{1/a},h^{1/b}\rangle$ are discrete groups of the first kind with elliptic generators, and, for rational u, we list all Fuchsian groups $\langle g,h\rangle$ and $\langle g,h^u\rangle$ that are generated by elliptics and are of the first kind. The size of the computation prevents a solution of the general problem in this case. The method we shall use, which is developed in Section 2, is to reduce the problem to a result about (possibly cusped) hyperbolic triangles which have a side in common. The results in [8] and [10] give us conditions on the angles of these triangles. This in turn leads to an equation involving cosines of rational angles that may be solved by extending the results in [3].

That these are reasonable questions to ask is indicated by the following finiteness result, which generalises Lemma 1 of [7].

PROPOSITION 1.2. *Let g and h be Möbius maps with distinct fixed points, and suppose that $\langle g,h\rangle$ is discrete. Then the group $\langle g^{1/a},h^{1/b}\rangle$ is discrete for only finitely many pairs of natural numbers a and b.*

Proof. Suppose that the result is false; that is, suppose the group $\langle g^{1/a},h^{1/b}\rangle$ is discrete for infinitely many pairs of natural numbers a and b. Passing to a subsequence a_n, b_n of these pairs, we may assume that a_n+b_n tends to infinity as n tends to infinity, and passing to a further subsequence either a_n or b_n tends to infinity. So without loss of generality g^{1/a_n} tends to I in the matrix topology. Thus, as n tends to infinity, $\mathrm{tr}^2(g^{1/a_n})$ tends to 4 and also $\mathrm{tr}[g^{1/a_n},h^{1/b_n}]$ tends to 2; this can be seen by using the well-known expression

$$\mathrm{tr}[g,h]=\mathrm{tr}^2(g)+\mathrm{tr}^2(h)+\mathrm{tr}^2(gh)-2-\mathrm{tr}(g)\,\mathrm{tr}(h)\,\mathrm{tr}(gh) \tag{1.1}$$

and by observing that for each $\delta>0$ there is an n_0 for which $|\mathrm{tr}(g^{1/a_n})|<2+\delta$ and $|\mathrm{tr}(g^{1/a_n}h^{1/b_n})|<(1+\delta)|\mathrm{tr}(h^{1/b_n})|$for all $n>n_0$.

We now apply Jørgensen's inequality (see [6]) to the sequence of groups $\langle g^{1/a_n},h^{1/b_n}\rangle$. Then for all a_n, b_n we have

$$|\mathrm{tr}^2(g^{1/a_n})-4|+|\mathrm{tr}[g^{1/a_n},h^{1/b_n}]-2|\geq 1.$$

Since the left-hand side tends to zero as n tends to infinity we have a contradiction, proving the result. □

Finally, we make a few comments about groups with hyperbolic generators. There are two main types of difficulty in this case. First, when g and h are not hyperbolics with intersecting axes, we do not have simple criteria indicating when $\langle g,h\rangle$ is discrete, or even whether it is of the first or second kind. The best technique available is the algorithm given by Matelski in [10] and refined by Gilman and Maskit in [5]. The second difficulty is that the equations we need to solve involve terms of the form $\cosh(t\lambda)$ and $\cosh(u\mu)$. There are no general methods of solving such equations of hyperbolic functions analogous to the results for trigonometric functions proved by Conway and Jones in [3].

2. Rational Powers of Generators

For the rest of this paper we will be concerned with discrete subgroups of PSL(2, **R**) that are of the first kind and generated by nonhyperbolic Möbius maps g and h having distinct fixed points. Thus g and h each have a single fixed point in the closure of the hyperbolic plane and there is a unique geodesic L_0 which either passes through or is asymptotic to these points. We denote reflection across L_0 by σ_0. It is well known that maps in $\mathrm{PSL}(2, \mathbf{R})$ may be written as the composition of reflection across two geodesics. For nonhyperbolic maps these are either asymptotic to or intersect in the fixed point of the map. Hence we can find geodesics L_1 and K_1 such that $g = \sigma_1 \sigma_0$ and $h = \sigma_0 \tau_1$, where reflection across L_1 is denoted by σ_1 and reflection across K_1 by τ_1. We will suppose that there is a component $\mathfrak{J}$ of $\mathbf{H} - (L_0 \cup L_1 \cup K_1)$ whose internal angles at the fixed points of both g and h are in the range $[0, \pi/2]$, this is the same as requiring that $|\mathrm{tr}(gh)| \leq |\mathrm{tr}(g^{-1}h)|$; see Proposition 2.2.2 of [11]. This is no restriction, for if it is not satisfied then we may replace the generator g by its inverse. This does not change the group.

Consider the groups $\langle g^{1/a}, h^{1/b} \rangle$ for integers a and b. We will find conditions on a and b which imply that $\langle g^{1/a}, h^{1/b} \rangle$ is also discrete. A trivial example of this may be eliminated as follows. Suppose that g is elliptic but is not primitive; that is, suppose there is a power of g that is not the identity but whose trace has a larger modulus. For example, suppose that $|\mathrm{tr}(g)| = 2\cos(p\pi/q)$, where p and q are coprime with $1 < p < q/2$. It is clear that $|\mathrm{tr}(g^{1/p})| = 2\cos(\pi/q)$. However, as p and q are coprime, there is an integer r with $1 < r < q$ such that $1 \equiv pr \pmod{2q}$ and so $|\mathrm{tr}(g^r)| = 2\cos(\pi/q)$. Thus $g^{1/p}$ and g^r have the same trace and the same fixed point, which implies that $g^{1/p}$ is either g^r or g^{-r}. So $\langle g^{1/p}, h \rangle = \langle g^r, h \rangle$, a subgroup of $\langle g, h \rangle$, and hence is discrete. In fact $\langle g^r, h \rangle = \langle g, h \rangle$. In order to eliminate things of this sort happening, we will always assume that elliptic generators are primitive; that is, we assume the modulus of their trace is of the form $2\cos(\pi/n)$ for some $n = 2, 3, \ldots$. This is no restriction since replacing a generator of finite order by a power of itself that has the same order yields the same group.

We will use the following characterisation of groups of the first kind. This result implies that $\mathfrak{J}$ is a (possibly cusped) triangle when $\langle g, h \rangle$ is of the first kind.

PROPOSITION 2.1. *Let g and h in* $\mathrm{PSL}(2, \mathbf{R})$ *be primitive and nonhyperbolic with distinct fixed points. Suppose that the group $\langle g, h \rangle$ is discrete. Then $\langle g, h \rangle$ is of the first kind if and only if either gh or $g^{-1}h$ is nonhyperbolic.*

Proof. Without loss of generality we may assume that $|\mathrm{tr}(gh)| \leq |\mathrm{tr}(g^{-1}h)|$. Let $\mathfrak{J}$ be as above.

First, we suppose that gh is not hyperbolic. This means that $\mathfrak{J}$ is a (possibly cusped) triangle, that is, there is no arc of $\partial\mathbf{H}$ on the boundary of $\mathfrak{J}$. By the results of Knapp [8] and Matelski [10] we see that $\langle g, h \rangle$ is a triangle group and so is of the first kind; see Theorem 10.6.4 of [1] for a proof of this fact.

Secondly, we suppose that gh is hyperbolic. We observe that there is an arc of $\partial\mathbf{H}$ contained in the boundary of $\mathfrak{I}$; call this arc γ. Then by Klein's combination theorem (see §VII.A.13 of [9]) the interior of $\bar{\mathfrak{I}} \cup \sigma_1 \bar{\mathfrak{I}}$ is a fundamental polygon for $\langle g, h\rangle$. Thus the arc γ is not contained in the limit set of $\langle g, h\rangle$ and so this group is of the second kind.

This proves the result. □

We now perform a similar construction for the maps $g^{1/a}$ and $h^{1/b}$. As g and h are primitive, $g^{1/a}$ and $h^{1/b}$ are also primitive. They are well defined if g and h have order greater than 2. When either g or h has order 2, the ambiguity in the definition gives a choice between two roots that are inverses of each other. Observe that g and h cannot both have order 2 or else $\langle g,h\rangle$ would not be of the first kind. Because for each a in $\mathbf{N}$, the fixed point of $g^{1/a}$ is the same as that of g, we can find a geodesic L_a with $g^{1/a} = \sigma_a \sigma_0$ where σ_a is reflection across L_a. Similarly for each b in $\mathbf{N}$ there is a geodesic K_b with $h^{1/b} = \sigma_0 \tau_b$ where τ_b is reflection across K_b. Observe that this is consistent with the definitions of L_1 and K_1. By replacing $g^{1/a}$ by $g^{-1/a}$ if necessary, there is a (possibly cusped) triangle $\mathfrak{I}(a, b)$ bounded by arcs of L_0, L_a and K_b. This triangle is defined even if g or h has order two. The arc of L_0 whose endpoints are the fixed points of g and h is a boundary arc of both $\mathfrak{I}$ and $\mathfrak{I}(a, b)$. By combining this fact with information about the internal angles of these triangles, we may use hyperbolic trigonometry to solve the problem. This is done in the next two sections.

Consider the groups $\langle g^t, h^u\rangle$ for t and u in $\mathbf{Q}$. We want to determine which groups $\langle g^t, h^u\rangle$ are discrete and of the first kind. In order to eliminate the problems (outlined above) due to nonprimitive elliptic generators, we assume that g, h, g^t, and h^u are all primitive. To simplify this problem we will pass to new Möbius maps g_0 and h_0 in the following manner. Suppose that $t = a/c$ where a and c are coprime natural numbers. Let $g_0 = g^a$. When g and $g^{a/c}$ are both primitive elliptics, a divides the order of g and c divides the order of g^t. Thus g_0 is primitive; of course, it may be the identity. When g_0 is not the identity we construct $g_0^{1/a}$ and $g_0^{1/c}$ as before; these are also primitive. When g_0 is the identity we define $g_0^{1/a}$ and $g_0^{1/c}$ to be primitive elliptics of order a and c, respectively, that have the same fixed point as g. Observe that $g_0^{1/a}$ is either g or g^{-1} and $g_0^{1/c}$ is either g^t or g^{-t}.

Similarly, when $u = b/d$ with b and d coprime natural numbers, we let $h_0 = h^b$ and find $h_0^{1/b}$ and $h_0^{1/d}$.

We construct triangles $\mathfrak{I}(a, b)$ and $\mathfrak{I}(c, d)$ as before, each having internal angles at the fixed points of g and h which are in the range $[0, \pi/2]$. Observe that when either of the groups $\langle g_0^{1/a}, h_0^{1/b}\rangle$ or $\langle g_0^{1/c}, h_0^{1/d}\rangle$ is discrete then the group $\langle g_0, h_0\rangle$ is automatically discrete. We may restate the original problem as follows: When can we find natural numbers a, b, c, d so that the groups $\langle g_0^{1/a}, h_0^{1/b}\rangle$ and $\langle g_0^{1/c}, h_0^{1/d}\rangle$ are Fuchsian groups of the first kind? Again, this may be translated to a problem about the internal angles of $\mathfrak{I}(a, b)$ and $\mathfrak{I}(c, d)$ and thence solved using hyperbolic trigonometry.

3. Groups Generated by a Parabolic and an Elliptic Map

The goal for this section will be to prove an analogue of Theorem 1.1 for parabolic-elliptic discrete groups. Let g in PSL(2, **R**) be parabolic and h in PSL(2, **R**) be primitive elliptic of order n, generating a discrete group $\langle g, h\rangle$ of the first kind. We construct a cusped triangle $\mathfrak{I}$ as in Section 2 which has internal angles 0 and π/n at the fixed points of g and h, respectively. Let θ be the other internal angle of $\mathfrak{I}$.

DEFINITION. Let g and h be Möbius self-maps of **H** with g parabolic and h primitive elliptic of order n. Suppose that θ is as above. Following Definition 10.6.1 of [1], we will say that $\langle g, h\rangle$ is of *type* $(0, \pi/n, \theta)$. We shall use this ordering of the angles.

The following theorem, which gives conditions on when the group $\langle g, h\rangle$ is discrete, may be found in [10], but the proof is sufficiently short to merit inclusion.

THEOREM 3.1. *Let g in* PSL(2, **R**) *be parabolic and h in* PSL(2, **R**) *be primitive elliptic. Then the group $\langle g, h\rangle$ is discrete if and only if gh and $g^{-1}h$ are primitive.*

Proof. Let $\mathfrak{I}$ be as above. Suppose that $|\mathrm{tr}(gh)| \leq |\mathrm{tr}(g^{-1}h)|$. Thus $g^{-1}h$ is hyperbolic and hence primitive. When gh is primitive, the group $\langle \sigma_0, \sigma_1, \tau_1\rangle$ is discrete and has fundamental domain $\mathfrak{I}$. This group has index 2 in $\langle g, h\rangle$, which is also discrete.

Suppose $\langle g, h\rangle$ is discrete and that gh is not primitive and so is elliptic. Let $\tau_0\tau_1$ be a power of gh which is primitive, where τ_0 is reflection across a geodesic K_0 passing through the fixed point of gh. Let $\mathfrak{I}'$ be the triangle contained in $\mathfrak{I}$ that is bounded by arcs of σ_0, τ_0, and τ_1. Then $\mathfrak{I}'$ must contain a fundamental domain of $\langle \sigma_0, \sigma_1, \tau_1\rangle$. Since $\mathfrak{I}$ has finite area it must be tiled by finitely many copies of the fundamental domain, but $\mathfrak{I}$ is not compact. This gives a contradiction, proving the result. □

We remark that this theorem implies that the angle θ defined above is either 0 or else π/r for some integer $r \geq 2$.

With g and h as above, we begin by finding all natural numbers a and b for which the group $\langle g^{1/a}, h^{1/b}\rangle$ is discrete and of the first kind. As in the previous section, we construct the cusped triangle $\mathfrak{I}(a, b)$. This has internal angles 0 and π/bn at the fixed points of g and h, respectively. Let ϕ be the other internal angle of $\mathfrak{I}(a, b)$. Thus $\langle g^{1/a}, h^{1/b}\rangle$ is of type $(0, \pi/bn, \phi)$.

Conjugating the group if necessary, we assume that g fixes ∞ and h fixes i, giving

$$g: z \mapsto z + t, \qquad h: z \to \left(z\cos\frac{\pi}{n} + \sin\frac{\pi}{n}\right)\Big/\left(-z\sin\frac{\pi}{n} + \cos\frac{\pi}{n}\right),$$

and so $\mathrm{tr}[g,h]=2+4t^2\sin^2(\pi/n)$. By (1.1) we also have that $\mathrm{tr}[g,h]=2+4(\cos(\pi/n)+\cos\theta)^2$. Combining these expressions, we obtain $t\sin(\pi/n)=\cos(\pi/n)+\cos\theta$. Also,

$$g^{1/a}: z\mapsto z+\frac{t}{a}, \qquad h^{1/b}: z\mapsto\left(z\cos\frac{\pi}{bn}+\sin\frac{\pi}{bn}\right)\Big/\left(-z\sin\frac{\pi}{bn}+\cos\frac{\pi}{bn}\right).$$

Similarly, these give $(t/a)\sin(\pi/bn)=\cos(\pi/bn)+\cos\phi$.

Eliminating t from this pair of equations yields

$$\frac{\cos\dfrac{\pi}{n}+\cos\theta}{\sin\dfrac{\pi}{n}}=\frac{a\left(\cos\dfrac{\pi}{bn}+\cos\phi\right)}{\sin\dfrac{\pi}{bn}}. \tag{3.1}$$

We are now able to prove the following theorem.

THEOREM 3.2. *Let g and h be in* PSL(2, **R**) *with g parabolic and h primitive elliptic of order n. Let a and b be natural numbers that are not both* 1. *Then $\langle g,h\rangle$ and $\langle g^{1/a},h^{1/b}\rangle$ are both discrete groups of the first kind if and only if they appear on the following list:*

(i) *$\langle g,h\rangle$ is of type $(0,\pi/n,0)$ and $\langle g,h^{1/2}\rangle$ is of type $(0,\pi/2n,\pi/2)$ for some $n\geq 2$,*
(ii) *$\langle g,h\rangle$ is of type $(0,\pi/n,\pi/n)$ and $\langle g^{1/2},h\rangle$ is of type $(0,\pi/n,\pi/2)$ for some $n\geq 3$,*
(iii) *$\langle g,h\rangle$ is of type $(0,\pi/2,0)$ and $\langle g^{1/2},h\rangle$ is of type $(0,\pi/2,\pi/3)$,*
(iv) *$\langle g,h\rangle$ is of type $(0,\pi/3,0)$ and $\langle g^{1/3},h\rangle$ is of type $(0,\pi/3,\pi/2)$.*

Proof. Using the notation established above, by equation (3.1) we have a relation between n, a, b, θ, and ϕ. By considering all admissible values of these parameters we obtain all solutions. We split the proof into four cases as follows.

First, suppose that $b\geq 2$. Then (3.1) and the fact that $\cos\phi\geq 0$ give

$$\frac{\cos\dfrac{\pi}{n}+\cos\theta}{\sin\dfrac{\pi}{n}}=\frac{a\left(\cos\dfrac{\pi}{bn}+\cos\phi\right)}{\sin\dfrac{\pi}{bn}}\geq\frac{a\cos\dfrac{\pi}{2n}}{\sin\dfrac{\pi}{2n}}$$

$$=\frac{2a\cos^2\dfrac{\pi}{2n}}{2\cos\dfrac{\pi}{2n}\sin\dfrac{\pi}{2n}}=\frac{a\left(\cos\dfrac{\pi}{n}+1\right)}{\sin\dfrac{\pi}{n}}.$$

Since the right-hand side is at least as big as the left-hand side, we have equality throughout. We obtain the solution $a=1$, $b=2$, $\theta=0$, and $\varphi=\pi/2$, which holds for all n; this is given in part (i).

Secondly, suppose that $a \geq 2$, $b=1$, and $\cos\phi > 0$ so $\cos\phi \geq 1/2$. Then (3.1) gives

$$\frac{\cos\frac{\pi}{n}+\cos\theta}{\sin\frac{\pi}{n}} = \frac{a\left(\cos\frac{\pi}{bn}+\cos\phi\right)}{\sin\frac{\pi}{bn}} \geq \frac{2\left(\cos\frac{\pi}{n}+\cos\phi\right)}{\sin\frac{\pi}{n}}$$

$$\geq \frac{2\cos\frac{\pi}{n}+1}{\sin\frac{\pi}{n}} \geq \frac{2\cos\frac{\pi}{n}+\cos\theta}{\sin\frac{\pi}{n}}.$$

Again we have equality throughout, and so $n=2$, $a=2$, $\theta=0$, and $\phi=\pi/3$. This is the solution given in part (iii).

Thirdly, suppose that $n\geq 3$, $a\geq 3$, $b=1$, and $\cos\phi=0$. As $\phi=\pi/2$ we have $\pi/n>\pi/2$; otherwise the triangle $\mathfrak{T}(a,1)$ has zero area, and so $n\geq 3$. Then (3.1) gives

$$\frac{\cos\frac{\pi}{n}+\cos\theta}{\sin\frac{\pi}{n}} = \frac{a\cos\frac{\pi}{n}}{\sin\frac{\pi}{n}} \geq \frac{3\cos\frac{\pi}{n}}{\sin\frac{\pi}{n}} \geq \frac{\cos\frac{\pi}{n}+1}{\sin\frac{\pi}{n}}.$$

Thus $n=3$, $a=3$, and $\theta=0$, which gives the solution in part (iv).

Finally, suppose that $a=2$, $b=1$, and $\cos\phi=0$. As above, $n\geq 3$; (3.1) then yields

$$\frac{\cos\frac{\pi}{n}+\cos\theta}{\sin\frac{\pi}{n}} = \frac{2\cos\frac{\pi}{n}}{\sin\frac{\pi}{n}} = \frac{\cos\frac{\pi}{n}+\cos\frac{\pi}{n}}{\sin\frac{\pi}{n}}.$$

So $\theta=\pi/n$, giving the solution in part (ii).

We have exhausted all possible values for n, a, b, and ϕ, so these are the only possible solutions. We can readily see that each gives a solution by constructing the fundamental domains for the relevant groups, as in Section 2. This proves the theorem. □

We now wish to prove an analogous theorem for rational powers of generators. We make the simplification discussed in Section 2; that is, we will find all natural numbers a, b, c, d with a, c coprime and b, d coprime for which the groups $\langle g_0^{1/a}, h_0^{1/b}\rangle$ and $\langle g_0^{1/c}, h_0^{1/d}\rangle$ are discrete and of the first kind. Thus $\langle g_0^{1/a}, h_0^{1/b}\rangle$ is of type $(0, \pi/bn, \theta)$ and $\langle g_0^{1/c}, h_0^{1/d}\rangle$ is of type $(0, \pi/dn, \phi)$ for some angles θ and ϕ in $\{0\}\cup\{(\pi/r): r\geq 2\}$.

Repeating the construction of equation (3.1) for these groups, we obtain

$$\frac{a\left(\cos\dfrac{\pi}{bn}+\cos\theta\right)}{\sin\dfrac{\pi}{bn}}=\frac{c\left(\cos\dfrac{\pi}{dn}+\cos\phi\right)}{\sin\dfrac{\pi}{dn}}. \tag{3.2}$$

This will enable us to prove the following theorem. However, the proof is sufficiently long and tedious that we confine our attention to two lemmas whose proofs are fairly short and illustrate the ideas contained in the proof of the theorem. Details of the rest of the proof may be found in [11].

THEOREM 3.3. *Let a, b, c, d be natural numbers with a, c coprime and b, d coprime. Also suppose that either $b=d$ and $a<c$ or else $b<d$. Let g and h be in* PSL(2, **R**) *with g parabolic, h primitive elliptic of order bn and $h^{b/d}$ primitive elliptic of order dn. The groups $\langle g, h\rangle$ and $\langle g^{a/c}, h^{b/d}\rangle$ are discrete and of the first kind if and only if they appear on the following list:*

(i) $\langle g, h\rangle$ *is of type* $(0, \pi/n, 0)$ *and* $\langle g, h^{1/2}\rangle$ *is of type* $(0, \pi/2n, \pi/2)$ *for some* $n\geq 2$,
(ii) $\langle g, h\rangle$ *is of type* $(0, \pi/n, 0)$ *and* $\langle g^2, h^{1/2}\rangle$ *is of type* $(0, \pi/2n, \pi/2n)$ *for some* $n\geq 2$,
(iii) $\langle g, h\rangle$ *is of type* $(0, \pi/n, \pi/n)$ *and* $\langle g^{1/2}, h\rangle$ *is of type* $(0, \pi/n, \pi/2)$ *for some* $n\geq 3$,
(iv) $\langle g, h\rangle$ *is of type* $(0, \pi/3, 0)$ *and* $\langle g^{1/3}, h\rangle$ *is of type* $(0, \pi/3, \pi/2)$,
(v) $\langle g, h\rangle$ *is of type* $(0, \pi/3, 0)$ *and* $\langle g^{2/3}, h\rangle$ *is of type* $(0, \pi/3, \pi/3)$,
(vi) $\langle g, h\rangle$ *is of type* $(0, \pi/3, \pi/3)$ *and* $\langle g^{3/2}, h^{1/2}\rangle$ *is of type* $(0, \pi/6, \pi/2)$,
(vii) $\langle g, h\rangle$ *is of type* $(0, \pi/3, \pi/3)$ *and* $\langle g^{3}, h^{1/2}\rangle$ *is of type* $(0, \pi/6, \pi/6)$,
(viii) $\langle g, h\rangle$ *is of type* $(0, \pi/3, \pi/2)$ *and* $\langle g^{3}, h^{1/2}\rangle$ *is of type* $(0, \pi/6, \pi/2)$,
(ix) $\langle g, h\rangle$ *is of type* $(0, \pi/3, \pi/2)$ *and* $\langle g^{6}, h^{1/2}\rangle$ *is of type* $(0, \pi/6, \pi/6)$,
(x) $\langle g, h\rangle$ *is of type* $(0, \pi/2, 0)$ *and* $\langle g^{1/2}, h\rangle$ *is of type* $(0, \pi/2, \pi/3)$,
(xi) $\langle g, h\rangle$ *is of type* $(0, \pi/2, \pi/3)$ *and* $\langle g^{2}, h^{1/2}\rangle$ *is of type* $(0, \pi/4, \pi/2)$,
(xii) $\langle g, h\rangle$ *is of type* $(0, \pi/2, \pi/3)$ *and* $\langle g^{4}, h^{1/2}\rangle$ *is of type* $(0, \pi/4, \pi/4)$,
(xiii) $\langle g, h\rangle$ *is of type* $(0, \pi/2, \pi/6)$ *and* $\langle g^{2/3}, h^{2/3}\rangle$ *is of type* $(0, \pi/3, \pi/2)$,
(xiv) $\langle g, h\rangle$ *is of type* $(0, \pi/2, \pi/6)$ *and* $\langle g^{4/3}, h^{2/3}\rangle$ *is of type* $(0, \pi/3, \pi/3)$,
(xv) $\langle g, h\rangle$ *is of type* $(0, \pi/2, \pi/6)$ *and* $\langle g^{2}, h^{2/3}\rangle$ *is of type* $(0, \pi/3, 0)$,
(xvi) $\langle g, h\rangle$ *is of type* $(0, \pi/2, \pi/6)$ *and* $\langle g^{2}, h^{1/3}\rangle$ *is of type* $(0, \pi/6, \pi/2)$,
(xvii) $\langle g, h\rangle$ *is of type* $(0, \pi/2, \pi/6)$ *and* $\langle g^{4}, h^{1/3}\rangle$ *is of type* $(0, \pi/6, \pi/6)$.

The first of the two lemmas deals with the case where $b=d$, that is, $b=d=1$ and $a<c$.

LEMMA 3.4. *Let g and h be in* PSL(2, **R**) *with g parabolic and h primitive elliptic of order n. Let a and c be coprime natural numbers with $a<c$. Then $\langle g,h\rangle$ and $\langle g^{a/c},h\rangle$ are discrete groups of the first kind if and only if they appear on the following list:*

(i) *$\langle g,h\rangle$ is of type $(0,\pi/n,\pi/n)$ and $\langle g^{1/2},h\rangle$ is of type $(0,\pi/n,\pi/2)$ for some $n\geq 3$,*
(ii) *$\langle g,h\rangle$ is of type $(0,\pi/2,0)$ and $\langle g^{1/2},h\rangle$ is of type $(0,\pi/2,\pi/3)$,*
(iii) *$\langle g,h\rangle$ is of type $(0,\pi/3,0)$ and $\langle g^{1/3},h\rangle$ is of type $(0,\pi/3,\pi/2)$,*
(iv) *$\langle g,h\rangle$ is of type $(0,\pi/3,0)$ and $\langle g^{2/3},h\rangle$ is of type $(0,\pi/3,\pi/3)$.*

Proof. The list gives groups with the desired properties by constructing the fundamental domains. It suffices to show that these are the only such groups. Let θ and ϕ be as above. Then by equation (3.2) we have

$$\frac{a\left(\cos\dfrac{\pi}{n}+\cos\theta\right)}{\sin\dfrac{\pi}{n}}=\frac{c\left(\cos\dfrac{\pi}{n}+\cos\phi\right)}{\sin\dfrac{\pi}{n}}.$$

This gives $0=a\cos\theta-c\cos\phi+(a-c)\cos(\pi/n)$.

We analyze this equation by using Theorem 6 of [3]. There are no vanishing sums of length 3, since this would imply that $a, c, a-c$ would all be ± 1. Thus all the cosines are rational multiples of one another, and so are either taken from the set $\{0,\pi/2,\pi/3\}$ or from the set $\{(\pi/n,\pi/2): n>3\}$. By considering the area of triangles $\mathfrak{T}(a,1)$ and $\mathfrak{T}(c,1)$, we see that $\theta<\phi$.

In the first case we have $n=2$ or 3, $\theta=0$ or $\pi/3$, and $\phi=\pi/3$ or $\pi/2$. If $n=2$ then $\phi\neq\pi/2$; otherwise the triangles $\mathfrak{T}(a,1)$ and $\mathfrak{T}(c,1)$ would have zero area. Solving for a and c in all of the above cases we obtain the results (ii), (iii), and (iv), together with the result (i) for $n=3$.

In the second case we must have $\theta=\pi/n$ and $\phi=\pi/2$. Solving for a and c we obtain the result (i) for $n>3$. □

The second lemma deals with the case where $bn=2$ and $dn>2$.

LEMMA 3.5. *Let g and h be in* PSL(2, **R**) *with g parabolic and h elliptic of order 2. Let a, b, c, d be natural numbers with a and c coprime, $b=1$ or 2, and $d>b$ with b and d coprime. The groups $\langle g,h\rangle$ and $\langle g^{a/c},h^{b/d}\rangle$ are discrete and of the first kind if and only if they appear on the following list:*

(i) *$\langle g,h\rangle$ is of type $(0,\pi/2,0)$ and $\langle g,h^{1/2}\rangle$ is of type $(0,\pi/4,\pi/2)$,*
(ii) *$\langle g,h\rangle$ is of type $(0,\pi/2,0)$ and $\langle g^2,h^{1/2}\rangle$ is of type $(0,\pi/4,\pi/4)$,*
(iii) *$\langle g,h\rangle$ is of type $(0,\pi/2,\pi/3)$ and $\langle g^2,h^{1/2}\rangle$ is of type $(0,\pi/4,\pi/2)$,*
(iv) *$\langle g,h\rangle$ is of type $(0,\pi/2,\pi/3)$ and $\langle g^4,h^{1/2}\rangle$ is of type $(0,\pi/4,\pi/4)$,*
(v) *$\langle g,h\rangle$ is of type $(0,\pi/2,\pi/6)$ and $\langle g^2,h^{2/3}\rangle$ is of type $(0,\pi/3,0)$,*

(vi) $\langle g, h\rangle$ *is of type* $(0, \pi/2, \pi/6)$ *and* $\langle g^{4/3}, h^{2/3}\rangle$ *is of type* $(0, \pi/3, \pi/3)$,
(vii) $\langle g, h\rangle$ *is of type* $(0, \pi/2, \pi/6)$ *and* $\langle g^{2/3}, h^{2/3}\rangle$ *is of type* $(0, \pi/3, \pi/2)$,
(viii) $\langle g, h\rangle$ *is of type* $(0, \pi/2, \pi/6)$ *and* $\langle g^{2}, h^{1/3}\rangle$ *is of type* $(0, \pi/6, \pi/2)$,
(ix) $\langle g, h\rangle$ *is of type* $(0, \pi/2, \pi/6)$ *and* $\langle g^{4}, h^{1/3}\rangle$ *is of type* $(0, \pi/6, \pi/6)$.

Proof. Again we may see that the list consists of groups of the desired form by constructing the fundamental domains. We show that these are the only solutions.

Using the notation already established, equation (3.2) gives

$$a\cos\theta = \frac{c\left(\cos\dfrac{\pi}{dn} + \cos\phi\right)}{\sin\dfrac{\pi}{dn}}.$$

Hence

$$a\cos(\pi/2-\theta-\pi/dn) + a\cos(\pi/2+\theta-\pi/dn) = 2c\cos(\pi/dn) + 2c\cos\phi.$$

Since $bn = 2$ we have θ in $\{0\}\cup\{(\pi/r): r\geq 3\}$ and ϕ in $\{0\}\cup\{(\pi/s): s\geq 2\}$.

The only solution given by Theorem 6 of [3] has $dn = 4$ and $\theta = \pi/3$ and is of the form

$$0 = \cos\psi - \cos(\pi/3-\psi) - \cos(\pi/3+\psi).$$

Examining the possibilities for ϕ yields (iii) and (iv).

We are left with the case where all the cosines are rational multiples of at least one of the others, that is, the angles are either taken from the set $\{0, \pi/2, \pi/3\}$ or from the set $\{(\pi/n, \pi/2): n > 3\}$.

In the first case we have $dn = 3$ and so $\theta = \pi/6$. Considering all the possibilities for ϕ gives (v), (vi), and (vii).

In the second case we either have $\theta = \pi/dn$ or $\theta = 0$; these give rise to $dn = 6$ and $dn = 4$, respectively. When $dn = 6$ this gives solutions (viii) and (ix). When $dn = 4$ this gives solutions (i) and (ii). □

We find the rest of the solutions in a similar manner. This is done by extending Theorem 6 of [3] to include all vanishing sums of length 6; using these, we match the values of the angles in these sums with those coming from equation (3.2). This is a very long and tedious calculation that differs from the above calculations only in scale. Therefore we do not give the details but refer interested readers to [11].

4. Groups Generated by Two Elliptic Maps

In this section we look at discrete subgroups of PSL(2, **R**) that are of the first kind and are generated by two elliptics with distinct fixed points. We explore the question of when the group generated by rational powers of these maps is discrete.

Consider pairs of primitive elliptics g and h in PSL(2, **R**) with distinct fixed points and which generate a discrete group of the first kind. Let the order of g be m and the order of h be n. As before, we construct the triangle $\mathfrak{T}$ having internal angles of π/m and π/n at the fixed points of g and h, respectively. We let the other internal angle be θ. We observe that m and n cannot both be 2.

DEFINITION. Let g and h be primitive elliptic Möbius self-maps of **H** of orders m and n, respectively, with distinct fixed points. Let θ be as above. Following Definition 10.6.1 of [1], we will say that $\langle g, h\rangle$ is of type $(\pi/m, \pi/n, \theta)$. We use this ordering of the angles.

The following theorem tells us for which values of θ the group $\langle g, h\rangle$ is discrete and of the first kind.

THEOREM 4.1 (Knapp [8]). *Let g and h in* PSL(2, **R**) *be primitive elliptic Möbius maps with distinct fixed points of order m and n, respectively. Let θ be the angle defined above. The group $\langle g, h\rangle$ is discrete and of the first kind if and only if one of the following occurs:*

(i) $\theta = 0$, *that is, either gh or $g^{-1}h$ is parabolic;*
(ii) $\theta = \pi/l$, *where* $1/l + 1/m + 1/n < 1$, *that is, either gh or $g^{-1}h$ is primitive elliptic;*
(iii) $\theta = 2\pi/l$, *when $m = n$ and where* $1/l + 1/m < 1/2$;
(iv) $\theta = 2\pi/n$, *when $m = 2$ and $n \geq 7$;*
(v) $\theta = 3\pi/n$, *when $m = 3$ and $n \geq 7$;*
(vi) $\theta = 4\pi/n$, *when $m = n \geq 7$; or*
(vii) $\theta = 2\pi/7$, *when $m = 3$ and $n = 7$.*

Proof. That either gh or $g^{-1}h$ is nonhyperbolic follows from Proposition 2.1. The rest was proved by Knapp in [8]. □

We now give an analogue of Theorem 3.2. Let $\langle g, h\rangle$ be of type $(\pi/m, \pi/n, \theta)$ and let $\langle g^{1/a}, h^{1/b}\rangle$ be of type $(\pi/am, \pi/bn, \phi)$. Since $\mathfrak{T}$ and $\mathfrak{T}(a, b)$ are triangles with a common side, the following equation may be derived using the second hyperbolic cosine rule (see §7.12 of [1]):

$$\frac{\cos\dfrac{\pi}{m}\cos\dfrac{\pi}{n} + \cos\theta}{\sin\dfrac{\pi}{m}\sin\dfrac{\pi}{n}} = \frac{\cos\dfrac{\pi}{am}\cos\dfrac{\pi}{bn} + \cos\phi}{\sin\dfrac{\pi}{am}\sin\dfrac{\pi}{bn}}. \tag{4.1}$$

We are now able to prove the following theorem.

THEOREM 4.2. *Let g and h in* PSL(2, **R**) *be primitive elliptics of orders m and n, respectively, with distinct fixed points. Let a and b be natural numbers* *that are not both* 1 *with* $a \le b$. *The groups* $\langle g, h\rangle$ *and* $\langle g^{1/a}, h^{1/b}\rangle$ *are discrete and of the first kind if and only if they appear in the following list:*

(i) $\langle g, h\rangle$ *is of type* $(\pi/m, \pi/n, \pi/m)$ *and* $\langle g, h^{1/2}\rangle$ *is of type* $(\pi/m, \pi/2n, \pi/2)$ *for some* m *and* n *with* $1/m+1/2n<1/2$,

(ii) $\langle g, h\rangle$ *is of type* $(\pi/2, \pi/n, \pi/2n)$ *and* $\langle g, h^{1/2}\rangle$ *is of type* $(\pi/2, \pi/2n, \pi/3)$ *for some* $n \ge 4$,

(iii) $\langle g, h\rangle$ *is of type* $(\pi/2n, \pi/n, \pi/2)$ *and* $\langle g, h^{1/2}\rangle$ *is of type* $(\pi/2n, \pi/2n, 2\pi/3)$ *for some* $n \ge 4$,

(iv) $\langle g, h\rangle$ *is of type* $(\pi/3, \pi/n, \pi/3n)$ *and* $\langle g, h^{1/3}\rangle$ *is of type* $(\pi/3, \pi/3n, \pi/2)$ *for some* $n \ge 3$,

(v) $\langle g, h\rangle$ *is of type* $(\pi/3n, \pi/n, \pi/3)$ *and* $\langle g, h^{1/3}\rangle$ *is of type* $(\pi/3n, \pi/3n, 2\pi/3)$ *for some* $n \ge 3$,

(vi) $\langle g, h\rangle$ *is of type* $(\pi/4n, \pi/n, \pi/4n)$ *and* $\langle g, h^{1/4}\rangle$ *is of type* $(\pi/4n, \pi/4n, 2\pi/3)$ *for some* $n \ge 2$.

REMARK. The triangles given by this theorem are the same as those given by parts (iii), (iv), (v), and (vi) of Knapp's theorem, Theorem 4.1.

Proof. This is proved using equation (4.1) in the same way that Theorem 3.2 is proved using equation (3.1). □

We will now explore an analogue to Theorem 3.3. By a similar construction to that given above, we have the group $\langle g, h\rangle = \langle g_0^{1/a}, h_0^{1/b}\rangle$ being of type $(\pi/am, \pi/bn, \theta)$ and the group $\langle g^t, h^u\rangle = \langle g_0^{1/c}, h_0^{1/d}\rangle$ being of type $(\pi/cm, \pi/dn, \phi)$. Again we use the second hyperbolic cosine rule to obtain

$$(4.2) \qquad \frac{\cos\dfrac{\pi}{am}\cos\dfrac{\pi}{bn}+\cos\theta}{\sin\dfrac{\pi}{am}\sin\dfrac{\pi}{bn}} = \frac{\cos\dfrac{\pi}{cm}\cos\dfrac{\pi}{dn}+\cos\phi}{\sin\dfrac{\pi}{cm}\sin\dfrac{\pi}{dn}}.$$

This may be rearranged to give a vanishing sum involving twelve cosines, which may in principle be solved using an extension to Theorem 6 of [3]. In practice, the length of these calculations would make solving this equation completely impractical. We may partially solve it by assigning values to some of the parameters. In particular, the problem may be solved completely in the case $a=c=1$ using a technique similar to that used to prove Theorem 3.3. This may be found in [11], but as it is long and tedious we do not include it here.

THEOREM 4.3. *Let* b *and* d *be coprime natural numbers with* $b<d$. *Let* g *and* h *in* PSL(2, **R**) *be primitive and elliptic with distinct fixed points and of orders* m *and* bn, *respectively. Moreover, suppose that* $h^{b/d}$ *is primitive with order* dn. *The groups* $\langle g, h\rangle$ *and* $\langle g, h^{b/d}\rangle$ *are discrete and of the first kind if and only if they appear on the following list:*

(i) $\langle g, h\rangle$ *is of type* $(\pi/m, \pi/n, \pi/m)$ *and* $\langle g, h^{1/2}\rangle$ *is of type* $(\pi/m, \pi/2n, \pi/2)$ *for some* m *and* n *with* $1/m+1/2n<1/2$,

(ii) *$\langle g,h\rangle$ is of type $(\pi/2, \pi/n, \pi/2n)$ and $\langle g, h^{1/2}\rangle$ is of type $(\pi/2, \pi/2n, \pi/3)$ for some $n \geq 4$,*
(iii) *$\langle g,h\rangle$ is of type $(\pi/2n, \pi/n, \pi/2)$ and $\langle g, h^{1/2}\rangle$ is of type $(\pi/2n, \pi/2n, 2\pi/3)$ for some $n \geq 4$,*
(iv) *$\langle g,h\rangle$ is of type $(\pi/3, \pi/n, \pi/3n)$ and $\langle g, h^{1/3}\rangle$ is of type $(\pi/3, \pi/3n, \pi/2)$ for some $n \geq 3$,*
(v) *$\langle g,h\rangle$ is of type $(\pi/3n, \pi/n, \pi/3)$ and $\langle g, h^{1/3}\rangle$ is of type $(\pi/3n, \pi/3n, 2\pi/3)$ for some $n \geq 3$,*
(vi) *$\langle g,h\rangle$ is of type $(\pi/3, \pi/2n, \pi/6n)$ and $\langle g, h^{2/3}\rangle$ is of type $(\pi/3, \pi/3n, \pi/3)$ for some $n \geq 2$,*
(vii) *$\langle g,h\rangle$ is of type $(\pi/6n, \pi/2n, \pi/3)$ and $\langle g, h^{2/3}\rangle$ is of type $(\pi/6n, \pi/3n, \pi/2)$ for some $n \geq 2$,*
(viii) *$\langle g,h\rangle$ is of type $(\pi/4n, \pi/n, \pi/4n)$ and $\langle g, h^{1/4}\rangle$ is of type $(\pi/4n, \pi/4n, 2\pi/3)$ for some $n \geq 2$,*
(ix) *$\langle g,h\rangle$ is of type $(\pi/12n, \pi/3n, \pi/12n)$ and $\langle g, h^{3/4}\rangle$ is of type $(\pi/12n, \pi/4n, \pi/3)$ for some $n \geq 1$,*
(x) *$\langle g,h\rangle$ is of type $(\pi/12n', \pi/15n', \pi/20n')$ and $\langle g, h^{3/4}\rangle$ is of type $(\pi/12n', \pi/20n', \pi/3)$ for some $n = 5n' \geq 5$,*
(xi) *$\langle g,h\rangle$ is of type $(\pi/2, \pi/4, 0)$ and $\langle g, h^{2/3}\rangle$ is of type $(\pi/2, \pi/6, \pi/4)$,*
(xii) *$\langle g,h\rangle$ is of type $(\pi/2, \pi/6, \pi/8)$ and $\langle g, h^{3/4}\rangle$ is of type $(\pi/2, \pi/8, \pi/4)$,*
(xiii) *$\langle g,h\rangle$ is of type $(\pi/2, \pi/5, \pi/10)$ and $\langle g, h^{5/6}\rangle$ is of type $(\pi/2, \pi/6, \pi/5)$,*
(xiv) *$\langle g,h\rangle$ is of type $(\pi/2, \pi/10, \pi/12)$ and $\langle g, h^{5/6}\rangle$ is of type $(\pi/2, \pi/12, \pi/5)$.*

REMARK. In part (i), if m and n tend to infinity then g and h (respectively) become parabolic, so we obtain Theorem 3.2(i) and (iii). Letting both m and n tend to infinity gives part of Beardon's theorem, Theorem 1.1.

In parts (ii), (iv), and (vi), if n tends to infinity then h becomes parabolic, so we obtain Theorem 3.3(x), (iv), and (v), respectively.

In parts (iii), (v), (vii), (viii), (ix), and (x), if we let n tend to infinity then g and h become parabolic, so we obtain Beardon's theorem, Theorem 1.1. Observe that parts (ix) and (x) give the same solution from Theorem 1.1 whereas the others each give distinct solutions.

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