Zero Sets, Interpolating Sequences, and Cyclic Vectors for Dirichlet Spaces

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1. Introduction

The main result of this paper is a factorization theorem for square integrable functions in the Dirichlet space of a finitely connected region. By using this theorem, we transfer many properties of the Dirichlet space of the unit disk to the setting of any finitely connected region.

In particular, we transfer the corona theorem for the algebra of bounded functions in the Dirichlet space and the result that invertible functions in the Dirichlet space are cyclic. We then study the structure of zero sets and that of universal interpolating sequences for the Dirichlet spaces. We show that, in an appropriate sense, the universal interpolating sequences for any finitely connected region are exactly the finite unions of the conformal images of those for the unit disk. Motivated by this result, we present a sufficient condition of Rosenbaum for a sequence in the unit disk to be universal interpolating for the Dirichlet space. In contrast to the situation for the algebra of bounded analytic functions, no metric characterization of these sequences is known for the Dirichlet space of the disk. We remark that our methods are very general and can be applied to many other settings.

THE DIRICHLET SPACE. Let G be a region (open connected set) in the Riemann sphere. The *Dirichlet space* $D(G)$ is the Hilbert space of functions f analytic on G whose derivative f' is square integrable; that is,

$$\int_G |f'|^2\,dA < \infty.$$

Geometrically, this integral condition means that f maps G onto a region that has finite area (counting multiplicity). An inner product on $D(G)$ can be defined by fixing a "base point" w in G and letting

$$\langle f, g\rangle = f(w)\overline{g(w)} + \int_G f'\overline{g'}\,dA.$$

Two different base points give rise to two equivalent norms (see e.g. [5, Prop. 6]), and hence they induce the same topology on the space.

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[†]Allen L. Shields died on 16 September 1989. This paper is dedicated to him by the first author, with great respect.

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ADDITIVE AND MULTIPLICATIVE DECOMPOSITIONS. If Ω is a finitely connected region in the complex plane $\mathbf{C}$ and $K_1, \ldots, K_N$ are the bounded components of $\mathbf{C} \backslash \Omega$, then $\Omega_0 = \Omega \cup K_1 \cup \cdots \cup K_N$ is a simply connected region, and also $\Omega_i = (\mathbf{C} \cup \{\infty\}) \backslash K_i$ is a simply connected region when $1 \leq i \leq N$. The Cauchy integral formula provides a method of decomposing every function f analytic on Ω as a sum: $f = f_0 + f_1 + \cdots + f_N$, where each f_i is analytic on Ω_i and $f_k(\infty) = 0$ if $k \neq 0$. Moreover, there is only one way, up to the order of addition, to write such a sum.

In [7] we proved the following version of the additive decomposition theorem: *If f is in the Dirichlet space $D(\Omega)$, then each f_i is in $D(\Omega_i)$.* Since each Ω_i is a conformal image of the unit disk, this decomposition allowed us to transfer many properties of the Dirichlet space of the disk to the setting of finitely connected regions.

However, this additive decomposition is not adequate for the study of zero sets and interpolating sequences in a finitely connected setting. For this, we need the following *multiplicative* decomposition theorem (Theorem 3 below): *If the boundary of Ω consists of simple closed curves, then for every square integrable function f in $D(\Omega)$ there exist functions f_i in $D(\Omega_i)$ such that $f = f_0 f_1 \cdots f_N$.* For example, this theorem tells us that the zeros of f can be regarded as a finite union of the conformal images of some zero sets for the Dirichlet space of the disk. Shapiro and Shields [14] have given some conditions for a sequence in the disk to be a zero set for the Dirichlet space. However, no complete characterization of these zero sets is known. For the algebra of bounded analytic functions on the disk, there is a necessary and sufficient condition for a sequence to be a zero set (see e.g. Duren [8] or Garnett [10]).

SOME APPLICATIONS. If φ is a function analytic on a region G with the property that $\varphi \cdot D(G)$ is contained in $D(G)$, then φ is called a *multiplier*. A function f in $D(G)$ is *cyclic* if the set $\{\varphi f : \varphi \text{ is a multiplier}\}$ is dense in $D(G)$. If $f(w) = 0$ for a point w in G then this set cannot be dense and hence f is not cyclic. In the converse direction, Brown [4] proved that *if G is the disk and if both f and $1/f$ are in $D(G)$, then f is cyclic.* In the present paper, we use the multiplicative decomposition to extend Brown's result to any finitely connected region.

The second application concerns the algebra formed by all bounded functions in the Dirichlet space $D(G)$. The following is known as the corona theorem for the algebra: *If $g_1, \ldots, g_m$ are functions in the algebra satisfying $|g_1(z)| + \cdots + |g_m(z)| > \delta > 0$ for all z in G, then there are functions $h_1, \ldots, h_m$ in the algebra such that $g_1 h_1 + \cdots + g_m h_m = 1$.* When G is the disk, the corona theorem for this algebra was proved by Tolokonnikov [15] and independently by Nicolau [11]. We transfer the corona theorem to any finitely connected region.

INTERPOLATING SEQUENCES. We let l^∞ denote the Banach space of all bounded sequences and l^2 denote the Hilbert space of all square summable sequences. Suppose $\{z_n\}_1^\infty$ is a sequence of distinct points in a region G and that f is a bounded analytic function; then the sequence $\{f(z_n)\}$ is in l^∞. The sequence $\{z_n\}$ is said to be a *universal interpolating sequence* for the algebra of all bounded analytic functions on G if, for every sequence $\{\mu_n\}$ in l^∞, there is a function f in the algebra such that $f(z_n)=\mu_n$ for all n.

When G is the disk, Carleson [6] proved a necessary and sufficient condition for a sequence to be universal interpolating, and shortly thereafter Shapiro and Shields [13] gave a simpler proof. Recently, Berndtsson, Chang, and Lin [3] studied the analogue of Carleson's condition for the polydisk; they showed that the corresponding condition implies interpolation, but that the converse is false.

In [13] Carleson's result was generalized to the Hardy space H^p. Since H^p contains unbounded functions, the setting of its interpolation theory is different from that mentioned above. In this aspect, the Dirichlet space $D(G)$ is similar to H^p, and so we use the setting for H^p to study the interpolation problem of $D(G)$. This involves specifying, for every sequence $\{z_n\}$ of points in G, a natural positive weight sequence $\{w_n\}$ so that $\{f(z_n)w_n\}$ is a bounded sequence for each f in $D(G)$. A sequence $\{z_n\}$ is then a *universal interpolating sequence* for $D(G)$ if the following holds: *For every f in $D(G)$, the sequence $\{f(z_n)w_n\}$ is in l^2, and conversely every sequence in l^2 can be written in that form.*

We show that every universal interpolating sequence for the Dirichlet space of any finitely connected region is a finite union of the conformal images of those for the disk. Lastly, we present a sufficient condition of Rosenbaum for a sequence in the disk to be universal interpolating for the Dirichlet space.

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2. Multiplicative Decomposition

The main result of this section is a factorization theorem (Theorem 3 below) for square integrable functions in the Dirichlet space of any finitely connected region whose boundary consists of simple closed curves. As we shall see, any finitely connected region is conformally equivalent to a region where the factorization can be applied to every function in the Dirichlet space. Using these facts, we extend some results for the Dirichlet space of the unit disk to the setting of any finitely connected region.

We begin by showing that, for the study of the Dirichlet space of a finitely connected region, we can assume without loss of generality that the region is bounded and has no isolated boundary point. This follows from the following three observations.

First, for any two conformally equivalent regions G_1 and G_2 in the Riemann sphere, the Dirichlet space $D(G_1)$ is unitarily equivalent to the Dirichlet space $D(G_2)$. To prove that, we let φ be a conformal map from G_1 onto G_2. Then the change-of-variable formula gives

$$\int_{G_2} |f'|^2 \, dA = \int_{G_1} |(f \circ \varphi)'|^2 \, dA.$$

If φ takes the base point that defines the norm of $D(G_1)$ to that of $D(G_2)$, then the composition map $f \to f \circ \varphi$ is an isometry from $D(G_2)$ onto $D(G_1)$.

Second, every isolated point, possibly ∞, of the complement of an open subset G in the Riemann sphere is a removable singularity of every function in $D(G)$. In view of the first observation, we need only to prove the case when G contains the unit disk Δ except the origin. If f is a function in $D(G)$ and $f(z) = \sum_{-\infty}^{\infty} a_n z^n$ for z in $\Delta \setminus \{0\}$, then for $0 < R < 1$ we have

$$\pi \sum n |a_n|^2 (1 - R^{2n}) = \int_{R < |z| < 1} |f'|^2 \, dA < \int_G |f'|^2 \, dA.$$

By taking $R \to 0$, we conclude that $a_n = 0$ if $n < 0$, and hence f is analytic at 0.

Third, the Dirichlet space of the whole complex plane $\mathbf{C}$, or of the whole Riemann sphere, contains only constant functions. This can be easily seen by computing the Dirichlet norm of the power series $\sum a_n z^n$.

For the study of the Dirichlet space of any finitely connected region Ω, by the last two observations we can assume that the complement of Ω is non-empty and does not have an isolated point. From the first observation we can further assume that ∞ is in one of the components of the complement of Ω; that is, Ω is bounded.

NOTATION. Throughout this paper, Ω always denotes a bounded finitely connected region with no isolated boundary point. We let $K_1, \ldots, K_N$ be all the bounded components of $\mathbf{C} \setminus \Omega$. We use Ω_0 to denote the region $\Omega \cup K_1 \cup \cdots \cup K_N$, and for $1 \le i \le N$ we use Ω_i to denote the region $(\mathbf{C} \cup \{\infty\}) \setminus K_i$.

We now prove a factorization result for all functions analytic on Ω, and then we return to the Dirichlet space $D(\Omega)$. From now on, if G is a subset of the Riemann sphere then ∂G denotes the boundary of G, and when G is open $H(G)$ denotes the space of all functions analytic on G. Note that our definitions imply that every function in $H(\Omega_k)$ is analytic at ∞ when $1 \le k \le N$.

PROPOSITION 1. *Suppose f is a function in $H(\Omega)$. Then there exist functions $f_0, f_1, \ldots, f_N$, with each f_k in $H(\Omega_k)$, such that $f = f_0 f_1 \cdots f_N$.*

Proof. We begin by fixing a nonzero function f in $H(\Omega)$ and assuming $N = 2$ for notational simplicity. Our method works for any N, but the notation is more complicated. The zeros (counting multiplicities) of f in Ω form a set, which can be written as a disjoint union of three subsets S_0, S_1, S_2, so that if S_i has any limit point then the point belongs to $\partial \Omega_i$.

Now we use the following fact: *Suppose G is a region, $\{m_n\}$ is a sequence of positive integers, and $\{a_n\}$ is a sequence of points in G which has no limit point in G; then there is an analytic function on G vanishing only at the points $\{a_n\}$, with multiplicities $\{m_n\}$.* So if $0 \le i \le 2$, then there is a function h_i in $H(\Omega_i)$ vanishing precisely at the points of S_i with the same multiplicities as f. Note that the function $f/h_0 h_1 h_2$ is analytic and nonvanishing on Ω. Thus, to continue the proof, we can assume that f is nonvanishing on Ω.

Let L_1 and L_2 be two nonintersecting line segments in Ω such that $\Omega\backslash(L_1 \cup L_2)$ is a simply connected region. On this region we can write $f(z) = |f(z)|e^{i\Theta(z)}$, where Θ is a real-valued continuous function.

Without loss of generality, we assume that L_1 is parallel to the real axis. Hence the function that takes w in L_1 to

$$\lim_{\theta\to 2\pi} \Theta(we^{i\theta}) - \lim_{\theta\to 0} \Theta(we^{i\theta})$$

is continuous, and thus the function must be a constant of the form $2n\pi$ for an integer n. Let ζ_1 be a point in K_1 and ζ_2 be a point in K_2. If m is an integer corresponding to L_2, chosen in the same way as n, then the function

$$g(z) = (z-\zeta_1)^{-n}(z-\zeta_2)^{-m} f(z)$$

can be written in the form $|g(z)|e^{i\psi(z)}$, where ψ is a real-valued function continuous on Ω. Hence, $\log g$ can be defined as an analytic function on Ω.

By the Cauchy integral formula, we can write $\log g = g_0 + g_1 + g_2$, with each g_i in $H(\Omega_i)$, and so

$$f = (z-\zeta_1)^n (z-\zeta_2)^m e^{g_0} e^{g_1} e^{g_2}.$$

If $n \geq 0$ then $(z-\zeta_1)^n e^{g_0}$ is in $H(\Omega_0)$. On the other hand, if $n < 0$ then $(z-\zeta_1)^n e^{g_1}$ is in $H(\Omega_1)$. By applying the same argument to the term $(z-\zeta_2)^m$, we finish the proof. □

For f in the Dirichlet space $D(\Omega)$, the factorization given by the last proposition does not guarantee that each factor f_k is in $D(\Omega_k)$. Our next theorem is that f_k can be so chosen under additional hypotheses. Before we prove the theorem, we need a definition and a lemma.

DEFINITION. Throughout this paper, for $0 \leq i \leq N$, we say that U is an *isolating neighborhood* of $\partial\Omega_i$ if U is an open subset of Ω in the form $U = V \cap \Omega$, where V is an open set containing $\partial\Omega_i$ such that the closure $\bar{V}$ of V has empty intersection with $\partial\Omega_k$ whenever $i \neq k$.

Suppose f is a function analytic on the unit disk Δ; it is easy to see that f is in $D(\Delta)$ if and only if f is in $D(R<|z|<1)$ for some positive $R<1$. By using a conformal map from Δ onto Ω_i, where $0 \leq i \leq N$, we transfer this property to $D(\Omega_i)$ as follows: Suppose U_i is an isolating neighborhood of $\partial\Omega_i$ and f is a function analytic on Ω_i; then f is in $D(\Omega_i)$ if and only if f is in $D(U_i)$. We shall use this remark repeatedly.

LEMMA 2. *Let f be a square integrable function in $D(\Omega)$. Suppose $f_0, \ldots, f_N$ are functions such that each f_i is analytic on Ω_i and $f = f_0 f_1 \cdots f_N$. Suppose that for each k, the function f_k does not vanish on $\partial\Omega_i$ whenever $i \neq k$. Then f_k is in $D(\Omega_k)$ for each k.*

Proof. By our assumption on the functions, each $\partial\Omega_i$ has an isolating neighborhood U_i such that f_k does not vanish on $\bar{U}_i$ whenever $i \neq k$.

To show that f_0 is in $D(\Omega_0)$, it is enough to show that f_0' is square integrable on U_0. We let $g = f_1 f_2 \cdots f_N$ and write $f_0' = f'g^{-1} - fg^{-2}g'$. The first term on the right-hand side of the equation is square integrable on U_0 because g is bounded away from zero on U_0 and f' is square integrable on Ω. Since f is square integrable on Ω and g is analytic on a neighborhood of $\bar{U}_0$, the second term is also square integrable on U_0. The same argument shows that f_k is in $D(\Omega_k)$ for every positive integer $k \leq N$. □

We remark that the conclusion of Lemma 2 does not hold if we only require f_k to be analytic on $\Omega_k \backslash \{\infty\}$ when $k \neq 0$. For example, take Ω to be an annulus and set $f(z) = z$, $f_0(z) = 1$, and $f_1(z) = z$.

We are now ready to prove our main theorem. We will use this theorem to extend many properties of $D(\Delta)$ to $D(\Omega)$.

THEOREM 3. *Suppose $\partial\Omega_0, \ldots, \partial\Omega_N$ are simple closed curves. If f is a nonzero square integrable function in $D(\Omega)$, then there exist functions f_i in $D(\Omega_i)$ for $0 \leq i \leq N$ such that $f = f_0 f_1 \cdots f_N$. Furthermore, each f_i can be chosen so that it does not vanish on $\partial\Omega_k$ whenever $i \neq k$.*

Proof. By Proposition 1, there exist functions f_i in $H(\Omega_i)$ such that $f = f_0 f_1 \cdots f_N$. We rearrange the zeros of f as follows. Suppose $f_0(\alpha) = 0$ for a point α in $\partial\Omega_1$ and $f_2(\beta) = 0$ for a point β in $\partial\Omega_0$. Since $\partial\Omega_1$ is a simple closed curve, we can fix a point ζ_1 in $\mathbf{C} \backslash \overline{\Omega_1}$ and write f as

$$f = \frac{(z-\beta)(z-\zeta_1) f_0}{z-\alpha} \cdot \frac{(z-\alpha) f_1}{z-\zeta_1} \cdot \frac{f_2}{z-\beta} \cdot f_3 \cdots f_N.$$

The term $z - \zeta_1$ is introduced in the equation so that the second term on the right-hand side is still analytic at ∞. For $i \neq k$, the function f_i has at most finite number of zeros on $\partial\Omega_k$, and so by rearranging these zeros as above we can assume that f_i does not vanish on $\partial\Omega_k$. By Lemma 2, each f_i is in $D(\Omega_i)$. □

The technique of rearranging zeros in the last proof, together with Proposition 1, can be used to prove similar factorization results for other spaces of analytic functions. We shall use $H^\infty(G)$ to denote the algebra of all bounded analytic functions on a region G.

COROLLARY 4. *Suppose $\partial\Omega_0, \ldots, \partial\Omega_N$ are simple closed curves. If f is a function in $H^\infty(\Omega)$, then there exist functions f_i in $H^\infty(\Omega_i)$ for $0 \leq i \leq N$ such that $f = f_0 f_1 \cdots f_N$.*

COROLLARY 5. *Suppose $\partial\Omega_0, \ldots, \partial\Omega_N$ are simple closed curves. If f is a square integrable function in $H^\infty(\Omega) \cap D(\Omega)$, then there exist functions f_i in $H^\infty(\Omega_i) \cap D(\Omega_i)$ for $0 \leq i \leq N$ such that $f = f_0 f_1 \cdots f_N$.*

We have shown how to factorize functions f in $D(\Omega)$ under additional hypotheses on f and Ω. As we shall see, Ω is conformally equivalent to a region where the factorization can be applied to all functions in the Dirichlet space.

A *Poincaré region* G is a region that has the property that every function in $D(G)$ is square integrable on G. For example, by using the power series representation for analytic functions, it is easy to see that the unit disk and the annuli are Poincaré regions. Although the unit disk is conformally equivalent to any simply connected region, Axler and Shields [1] have given an example of a bounded simply connected region which is not a Poincaré region.

THEOREM 6. *The region Ω is conformally equivalent to a bounded finitely connected Poincaré region whose boundary consists of differentiable simple closed curves.*

Proof. Without loss of generality, we assume $N=2$ and so $\Omega=\Omega_0\backslash(K_1\cup K_2)$. Our proof works for any N, but the notation is more complicated.

Let φ_1 be a conformal map that takes Ω_1 onto the complement Δ^c of the unit disk Δ. Note that K_2 is contained in Ω_1 and $\varphi_1(K_2)$ is a simply connected compact subset of Δ^c. Thus, there is a conformal map φ_2 that takes the complement of $\varphi_1(K_2)$ onto Δ^c, and so φ_2 takes the boundary of $\varphi_1(\Omega_1)$, which is the unit circle, onto a differentiable simple closed curve. Now, let φ_0 be a conformal map that takes the bounded simply connected region inside $\varphi_2\circ\varphi_1(\partial\Omega_0)$ onto Δ. Hence $\varphi_0\circ\varphi_2\circ\varphi_1$ takes Ω onto a finitely connected region G whose boundary consists of simple closed curves.

From the above construction, the unit circle is a component of the boundary of G and it has an isolating neighborhood (as defined before Lemma 2) which is an annulus. Furthermore, for each bounded component of the complement of G, there is a conformal map that takes $\bar{\Delta}$ onto that component and takes a neighborhood of $\bar{\Delta}$ onto a neighborhood of that component. Thus, each component of the boundary of G has an isolating neighborhood U which is the image of a conformal map φ on an annulus W, and moreover φ extends to be conformal on an open set containing $\bar{W}$.

We claim that G is a Poincaré region. It is sufficient to prove that $U=\varphi(W)$ is a Poincaré region. For that, we let f be a function in $D(U)$. Hence the composite function $f\circ\varphi$ is in $D(W)$ and, since every annulus is a Poincaré region,

$$\int_W |f\circ\varphi|^2\,dA<\infty.$$

The fact that φ is conformal on a neighborhood of W implies that φ' is bounded on W, and hence

$$\int_U |f|^2\,dA=\int_W |f\circ\varphi|^2\,|\varphi'|^2\,dA<\infty. \qquad \square$$

Now we turn our attention to some applications of Theorems 3 and 6. For that, we need some definitions and elementary facts.

The *maximal ideal space* of a Banach algebra is the set of all multiplicative linear functionals on the algebra. With the supremum norm, $H^\infty(G)$ is a Banach algebra. We define the norm on $H^\infty(G)\cap D(G)$ by taking the sum of the norm of $H^\infty(G)$ and the norm of $D(G)$. With this norm, $H^\infty(G)\cap D(G)$ is also a Banach algebra. Let w be a point in G. For either algebra, the point evaluation functional that takes every function f to $f(w)$ is multiplicative. By identifying every point in G as a multiplicative linear functional, we can regard G as a subset of its maximal ideal space. It is well known that the following two statements are equivalent for both Banach algebras (see e.g. [10, p. 191]):

(1) In the weak* topology, G is dense in the maximal ideal space.

(2) If $f_1,\dots,f_n$ are functions in the Banach algebra and if

$$|f_1(z)|+\cdots+|f_n(z)|\geq\delta>0$$

for all z in G, then there exist $g_1,\dots,g_n$ in the Banach algebra such that $f_1g_1+\cdots+f_ng_n=1$.

Either of the two statements above is called the *corona theorem*. As indicated in the introduction, when G is the unit disk Δ, the corona theorem is known to be true for the algebra $H^\infty(\Delta)\cap D(\Delta)$. Now we extend the corona theorem to our region Ω.

PROPOSITION 7. *The maximal ideal space of $H^\infty(\Omega)\cap D(\Omega)$ contains Ω as a dense subset.*

Proof. As indicated at the beginning of this section, if two regions are conformally equivalent then their Dirichlet spaces are unitarily equivalent. Thus if statement (2) above holds for one region then it must hold also for the other region. This implies that the corona theorem holds for $H^\infty(\Omega_k)\cap D(\Omega_k)$ and also that, by Theorem 6, it is enough to prove our proposition for the case where Ω is a Poincaré region whose boundary consists of differentiable simple closed curves.

Fix a multiplicative linear functional λ on $H^\infty(\Omega)\cap D(\Omega)$. Then, for each nonnegative integer $k\leq N$, we can view $H^\infty(\Omega_k)\cap D(\Omega_k)$ as a subspace of $H^\infty(\Omega)\cap D(\Omega)$ and so λ is a multiplicative linear functional on $H^\infty(\Omega_k)\cap D(\Omega_k)$. Viewing Ω_k as a dense set in the maximal ideal space of $H^\infty(\Omega_k)\cap D(\Omega_k)$, we see that for each k there is a net of multiplicative linear functionals in Ω_k converging to λ.

We claim that all these nets, as nets of points in $\mathbf{C}$, converge to exactly one common point. To prove that, we need a fact from the standard case of the unit disk Δ: If a net in Δ, regarded as a net in the maximal ideal space, converges to a linear functional, then by applying the net to the position function z in $D(\Delta)$ we see that this net, regarded as a net of points in $\mathbf{C}$, converges to one point in $\bar{\Delta}$. Note that every conformal map from Δ to Ω_k extends to be continuous on the boundary, and so the net in Ω_k defined in the last paragraph converges to a point z_k in $\bar{\Omega}$. To finish our claim, we must show that all z_k are the same; without loss of generality, we need only show that $z_0 = z_1$. Fix a point ζ_1 in $\mathbf{C}\setminus\overline{\Omega_1}$. It is easy to see that $1/(z-\zeta_1)$ is a function in $D(\Omega_1)$ and $z-\zeta_1$ is in $D(\Omega_0)$. Since $1=(z-\zeta_1)(1/(z-\zeta_1))$, by applying the linear functional λ on both sides of the equation we obtain $z_0 = z_1$.

If z_0 is in Ω then we are done. Otherwise, z_0 belongs to $\partial\Omega_i$ for some i. Pick the net in Ω_i, denoted by $\{z_\alpha\}$, that converges to λ in the maximal ideal space of $H^\infty(\Omega_i)\cap D(\Omega_i)$. We can assume that each z_α is a point in Ω. For f in $H^\infty(\Omega)\cap D(\Omega)$, by Corollary 5 we can write $f=f_0 f_1 \cdots f_N$, where f_k is in $H^\infty(\Omega_k)\cap D(\Omega_k)$. Thus $f(z_\alpha)=f_0(z_\alpha)\cdots f_N(z_\alpha)\to\lambda(f)$. □

With a slight modification, the argument of the last proof, along with Corollary 4, can be used to extend the corona theorem for the algebra $H^\infty(\Delta)$ to $H^\infty(\Omega)$. There are many other ways to do that; see Fisher [9, Ch. 6].

COROLLARY 8. *The maximal ideal space of $H^\infty(\Omega)$ contains Ω as a dense subset.*

If Ω is a Poincaré region as in the statement of Theorem 6, then Theorem 3 helps us to factorize all functions in $D(\Omega)$. In the rest of this section we study this region in greater detail, and make greater use of the properties of its Dirichlet space. Before doing that, we give some related remarks.

If α is a point in a region G then, by the definition of the norm $\|\cdot\|_G$ of $D(G)$, the linear functional that takes f in $D(G)$ to $f(\alpha)$ is continuous. This linear functional is called the *point evaluation* for α. It is also easy to show that the functional $f\to f'(\alpha)$ is continuous on $D(G)$. Hence, by the principle of uniform boundedness, for any compact subset K of G there is a positive constant $C=C(K)$ such that if z is a point in K then both $|f(z)|$ and $|f'(z)|$ are less than $C\|f\|_G$.

With the above remarks, we are now ready to prove the next lemma, which can be regarded as the converse of Theorem 3. We shall use $\|\cdot\|$ to denote the norm of $D(\Omega)$, and $\|\cdot\|_i$ to denote the norm of $D(\Omega_i)$ for $0\le i\le N$.

LEMMA 9. *With Ω the same region as before, we suppose further that Ω is a Poincaré region. Then there is a positive constant C such that, if f_k is a function in $D(\Omega_k)$ for each nonnegative integer $k\le N$, then*

$$\|f_0\cdots f_N\|\le C\|f_0\|_0\cdots\|f_N\|_N.$$

Proof. For notational simplicity, we prove the lemma for the case $N=2$; the same argument will work for any N. For $i=0,1,2$ we let U_i be an isolating neighborhood (as defined before Lemma 2) of $\partial\Omega_i$.

Let $f=f_0 f_1 f_2$. Thus

$$f'=f_0' f_1 f_2+f_0 f_1' f_2+f_0 f_1 f_2'.$$

If we let $K=\Omega\setminus(U_0\cup U_1\cup U_2)$, then K is either empty or a compact subset of Ω. If K is nonempty then, by the remarks preceding the lemma, there is a positive constant C_0, independent of the functions, such that

$$\int_K |f'|^2\,dA \le C_0\|f_0\|_0^2\cdot\|f_1\|_1^2\cdot\|f_2\|_2^2.$$

To finish the proof, we must show the corresponding inequality with K replaced by U_i. Without loss of generality, we consider only the case of U_0. Since $\bar{U}_0$ is a compact subset of Ω_1, there is a constant $C_1 > 0$, independent of the functions, such that if z is in $\bar{U}_0$ then the values $|f_1(z)|$ and $|f_1'(z)|$ are less than $C_1\|f_1\|_1$. Similarly, there is a constant $C_2 > 0$ such that if z is in $\bar{U}_0$ then $|f_2(z)|$ and $|f_2'(z)|$ are less than $C_2\|f_2\|_2$. By our expression for f' in the last paragraph, we need only show that there is a constant $C_3 > 0$, independent of the functions, such that

$$\int_{U_0} |f_0|^2\,dA \le C_3\|f_0\|_0^2.$$

Since Ω is a Poincaré region and $D(\Omega_0)$ is contained in $D(\Omega)$, we see that every function in $D(\Omega_0)$ is square integrable on U_0. Now the existence of C_3 follows from an application of the closed graph theorem; or see [5, Prop. 6]. □

We say that a sequence of points in a region G is a *zero set* for $D(G)$ if there is a function in $D(G)$ vanishing precisely at the points (counting multiplicities) of the sequence. Note that a conformal map from a region G_1 onto a region G_2 takes a zero set for $D(G_1)$ to a zero set for $D(G_2)$. This implies that, for $0 \le k \le N$, a sequence of points in Ω_k is a zero set for $D(\Omega_k)$ if and only if it is the conformal image of a zero set for the Dirichlet space of the unit disk. By Theorem 6, we can assume, via a conformal map, that Ω satisfies the hypotheses of Theorem 3 and Lemma 9. So every zero set for $D(\Omega)$ can be written as a finite union $\bigcup s_k$, where s_k is a zero set for $D(\Omega_k)$. Conversely, any such union is a zero set for $D(\Omega)$.

Now we study the multipliers of the Dirichlet spaces. If φ is a function analytic on a region G and has the property that φf is in $D(G)$ for each f in $D(G)$, then φ is called a *multiplier*. Let $m(G)$ denote the algebra of all multipliers. Since $D(G)$ contains the constants, $m(G)$ is contained in $D(G)$.

As a corollary of Theorem 3 and Lemma 9 together, we now show that multipliers can be factorized in the same way as other spaces.

COROLLARY 10. *With Ω the same region as before, we suppose further that Ω is a Poincaré region whose boundary consists of simple closed curves. If φ is a function in $m(\Omega)$, then there exist functions φ_i in $m(\Omega_i)$ for $0 \le i \le N$ such that $\varphi = \varphi_0\varphi_1\cdots\varphi_N$.*

Proof. Suppose φ is a function in $m(\Omega)$. Then, by Theorem 3, we can write $\varphi = \varphi_0\varphi_1\cdots\varphi_N$, where φ_i is in $D(\Omega_i)$ and does not vanish on $\partial\Omega_j$ for $i \ne j$.

Let g be a function in $D(\Omega_0)$ and C be a constant such that $g + C$ does not vanish on $\partial\Omega_i$ for each positive integer $i \le N$. Since g is in $D(\Omega)$, the product $\varphi(g+C)$ is in $D(\Omega)$ and so $\varphi(g+C)$ is square integrable on Ω. Note that $\varphi(g+C) = \varphi_0(g+C)\varphi_1\cdots\varphi_N$. By Lemma 2, $\varphi_0(g+C)$ is in $D(\Omega_0)$ and so

$\varphi_0 g$ is in $D(\Omega_0)$. Thus φ_0 is in $m(\Omega_0)$. Similarly, one can show that φ_i is in $m(\Omega_i)$ for each positive integer $i \leq N$. □

Suppose G is a region. A function f in $D(G)$ is *cyclic* for $m(G)$ if the vector subspace $\{\varphi f : \varphi \in m(G)\}$ is dense in $D(G)$. We call the function f a *cyclic vector*. When G is the unit disk, the cyclic vectors have been studied extensively by Brown and Shields [5]. Now we show that the product of cyclic vectors of $D(\Omega_k)$ is a cyclic vector of $D(\Omega)$.

LEMMA 11. *With Ω the same region as before, we suppose further that Ω is a Poincaré region whose boundary consists of differentiable simple closed curves. For $0 \leq i \leq N$, suppose that f_i is a function in $D(\Omega_i)$ and is cyclic for $m(\Omega_i)$. Then $f = f_0 \cdots f_N$ is in $D(\Omega)$ and is cyclic for $m(\Omega)$.*

Proof. We prove the case $N=1$; our argument will work for any N. It follows from Theorem 3 and Lemma 9 that if φ_0 is in $m(\Omega_0)$ and φ_1 is in $m(\Omega_1)$ then $\varphi_0\varphi_1$ is in $m(\Omega)$. For every function g in $D(\Omega)$, by Theorem 3 there are functions g_i in $D(\Omega_i)$ such that $g = g_0 g_1$. Then by Lemma 9 there is a positive constant C, independent of the functions, such that

$$\begin{aligned}\|\varphi_0\varphi_1 f_0 f_1 - g\| &\leq \|\varphi_0\varphi_1 f_0 f_1 - g_0\varphi_1 f_1\| + \|g_0\varphi_1 f_1 - g_0 g_1\| \\ &\leq C\cdot\|\varphi_0 f_0 - g_0\|_0\cdot\|\varphi_1 f_1\|_1 + C\cdot\|g_0\|_0\cdot\|\varphi_1 f_1 - g_1\|_1\end{aligned}$$

We can choose φ_1 and then φ_0 so that the expression in the last line is as small as possible. □

Brown [4] has proved that every invertible function in the Dirichlet space of the unit disk is cyclic. This result can be generalized to Ω_k for $0 \leq k \leq N$ by using the following fact: If φ is a conformal map from a region G_1 onto a region G_2, then a function f in $D(G_2)$ is cyclic for $m(G_2)$ if and only if $f \circ \varphi$ is cyclic for $m(G_1)$. Brown's result can further be generalized to our region Ω, with the help of Lemma 11.

PROPOSITION 12. *If f is in $D(\Omega)$ and its reciprocal $1/f$ is also in $D(\Omega)$, then f is cyclic for $m(\Omega)$.*

Proof. In view of Theorem 6, it is sufficient to prove the proposition for the case where Ω is a Poincaré region and $\partial\Omega$ consists of differentiable simple closed curves.

We only prove the proposition for the case $N=1$; our argument will work for any N. Let f be a function in $D(\Omega)$ such that $1/f$ is in $D(\Omega)$. By Proposition 1 we factorize f as $f = f_0 f_1$, where f_k is in $H(\Omega_k)$ for each k. Note that f_0 and f_1 do not vanish on Ω. Furthermore, f_0 has at most a finite number of zeros in $\mathbf{C}\backslash\Omega_1$, and f_1 has at most a finite number of zeros in $\mathbf{C}\backslash\Omega_0$.

Now we make three observations. First, suppose $f_0(\alpha) = 0$ for a point α in $\mathbf{C}\backslash\Omega_1$, with multiplicity 1. If we fix a point ζ_1 in $\mathbf{C}\backslash\overline{\Omega_1}$ and write

$$\frac{f}{z-\zeta_1}=\frac{f_0}{z-\alpha}\cdot\frac{(z-\alpha)f_1}{z-\zeta_1},$$

then the first term on the right-hand side does not vanish at α and is in $H(\Omega_0)$, and the second term is analytic at ∞ and is in $H(\Omega_1)$.

Second, suppose β in $\mathbf{C}\backslash\Omega_0$ is a zero of f_1; then we can divide out the zero:

$$f=(z-\beta)f_0\cdot\frac{f_1}{z-\beta}.$$

Note that the first term on the right-hand side is in $H(\Omega_0)$ and the second term is in $H(\Omega_1)$.

Third, if $f_1(\infty)=0$ then we can find a positive integer k such that $(z-\zeta_1)^k f_1$ is analytic at ∞ and does not vanish at ∞. This can be shown by writing f_1 as a power series of $1/(z-\zeta_1)$ on a neighborhood of ∞.

From these three observations, we conclude that there is an integer k such that we can write

$$(z-\zeta_1)^k f=g_0 g_1,$$

where each g_k is analytic and nonvanishing on Ω_k.

Note that Ω is a bounded Poincaré region and $1/(z-\zeta_1)$ is bounded on Ω. It follows that if f is in $D(\Omega)$ then so are $(z-\zeta_1)f$ and $f/(z-\zeta_1)$. Thus $z-\zeta_1$ and $1/(z-\zeta_1)$ are in $m(\Omega)$. Hence $(z-\zeta_1)^k f$ is in $D(\Omega)$ and, by Lemma 2, each g_k is in $D(\Omega_k)$. By taking reciprocals on each side of the last displayed equation, and repeating the above argument, we see that $1/g_k$ is in $D(\Omega_k)$. Thus each g_k is cyclic for $m(\Omega_k)$.

It follows from the definition of a cyclic vector that f is cyclic for $m(\Omega)$ if and only if $(z-\zeta_1)^k f$ is cyclic for $m(\Omega)$. Now our result follows directly from Lemma 11. □

3. Universal Interpolating Sequences

In this section we study the interpolating sequences for $D(\Omega)$. As indicated in the introduction, the interpolation problem for the algebra of bounded analytic functions is different from that for $D(\Omega)$, which contains unbounded functions.

Critical to the formulation of the interpolating problem for $D(\Omega)$ is the notion of a reproducing kernel. As mentioned in the previous section, for every point a in a region G, the point evaluation that takes f to $f(a)$ is a continuous linear functional on $D(G)$. Thus there is a function k_a in $D(G)$ such that

$$g(a)=\langle g,k_a\rangle$$

for all g in $D(G)$. We call k_a the *reproducing kernel* for the point a.

For every sequence $\{z_n\}_1^\infty$ of points in G, we define a linear operator T on $D(G)$ by

$$Tf = \left\{ \frac{f(z_n)}{\|k_{z_n}\|} \right\}_1^\infty,$$

where $\|\cdot\|$ denotes the norm of $D(G)$. Since $|f(z_n)| \le \|k_{z_n}\| \cdot \|f\|$, T is a continuous map from $D(G)$ into l^∞. If Tf is in l^2 for each f in $D(G)$, then by the closed graph theorem T is a continuous map into l^2. If l^2 is contained in $TD(G)$ then there exists a positive constant M such that, for each sequence $\{c_n\}_1^\infty$ in l^2, there is a function f in $D(G)$ satisfying $\|f\| < M(\sum |c_n|^2)^{1/2}$ and $Tf = \{c_n\}$; one may show this by another application of the closed graph theorem.

Following the interpolation theory for H^2 in [13], we call $\{z_n\}$ a *universal interpolating sequence* for $D(G)$ when $TD(G) = l^2$. From this definition, we see that a universal interpolating sequence consists of distinct points and has no limit point in G.

Let ϕ be a conformal map from a region G_1 onto a region G_2. As indicated at the beginning of the previous section, we can assume that the composition map $f \to f \circ \phi$ defines an isometry from $D(G_2)$ onto $D(G_1)$. Hence if $\phi(\alpha) = \beta$ then the $D(G_1)$ norm of k_α equals to the $D(G_2)$ norm of k_β. It follows that a sequence $\{z_n\}$ in G_1 is a universal interpolating sequence for $D(G_1)$ if and only if $\{\phi(z_n)\}$ is a universal interpolating sequence for $D(G_2)$.

Now we focus our attention on our finitely connected region Ω. We claim that every subset of a zero set for $D(\Omega)$ is again a zero set. We first consider the case for the unit disk Δ. Suppose $f(z) = \sum a_n z^n$ is a function in $D(\Delta)$, with $f(0) = 0$; then

$$\int_\Delta \left| \left(\frac{f}{z} \right)' \right|^2 dA = \pi \sum_1^\infty n |a_{n+1}|^2 \le \pi \sum_1^\infty n |a_n|^2 = \int_\Delta |f'|^2 \, dA.$$

More generally, if α is a point in Δ and $\varphi : \Delta \to \Delta$ is the conformal map defined by $\varphi(z) = (z - \alpha)/(1 - \bar{\alpha} z)$, then the order of the above inequality is preserved when we replace f by $f \circ \varphi$ and f/z by $f \circ \varphi / \varphi$ simultaneously. Hence, if g in $D(\Delta)$ vanishes at all the zeros (counting multiplicities) of a Blaschke product B, then g/B is also in $D(\Delta)$. It follows from the last inequality displayed above that $D(\Delta)$ is contained in the Hardy space H^2, and so a zero set for $D(\Delta)$ is a zero set for H^2. Thus our claim is true if $\Omega = \Delta$. By using an argument involving a conformal map, we see that our claim holds for Ω_k, where $0 \le k \le N$. With these results in hand, our claim follows from Theorem 3, Theorem 6, and Lemma 9.

We claim that every universal interpolating sequence $\{z_n\}_1^\infty$ for $D(\Omega)$ is a zero set. To prove that, we can assume (via a conformal map) that Ω is a Poincaré region; it follows directly that z is a multiplier. Let f be a function in $D(\Omega)$, with $f(z_1) = 1$ and $f(z_n) = 0$ for $n \ge 2$. The nonzero function $g(z) = (z - z_1) f(z)$ vanishes at every z_n. By dividing out all other zeros from g, we obtain a function that vanishes precisely at $\{z_n\}$.

Now it is easy to see that if α is a point in Ω and $\alpha \ne z_n$ for all n, then $\{\alpha\} \cup \{z_n\}_1^\infty$ is also a universal interpolating sequence. Similarly, we can also remove a point.

We are dealing with interpolating sequences for different regions, so we must distinguish different reproducing kernels for one point w in Ω. Throughout this section, we use k_w and k_w^i to denote the reproducing kernel in $D(\Omega)$ and the reproducing kernel in $D(\Omega_i)$ respectively. As in the previous section, we use $\|\cdot\|$ to denote the norm for $D(\Omega)$ and $\|\cdot\|_i$ to denote the norm of $D(\Omega_i)$, where $0 \le i \le N$. Although the two norms $\|k_w\|$ and $\|k_w^i\|_i$ are different, the next lemma shows that they are equivalent for w near $\partial\Omega_i$.

LEMMA 13. *There exist C_1, C_2, and $\delta > 0$ such that, if $0 \le j \le N$ and w is a point in Ω satisfying* $\operatorname{dist}(w, \partial\Omega_j) < \delta$, *then*

$$C_2\|k_w\| \le \|k_w^j\|_j \le C_1\|k_w\|.$$

Proof. Without loss of generality, we only need to prove the case $j = 0$.

We can consider $D(\Omega_0)$ as a closed subspace of $D(\Omega)$. Hence, by the closed graph theorem, there is a constant $C_1 > 0$ such that $\|f\| \le C_1\|f\|_0$ for all f in $D(\Omega_0)$. For any point w in Ω, by the definition of a reproducing kernel we have

$$\|k_w^0\|_0^2 = k_w^0(w) \le C_1\|k_w^0\|_0 \cdot \|k_w\|.$$

For the other direction, we use the additive decomposition indicated in the introduction: *Every function h in $D(\Omega)$ can be written uniquely as $h = h_0 + h_1 + \cdots + h_N$, with each h_i in $D(\Omega_i)$ and $h_k(\infty) = 0$ when $k \ne 0$.* For $0 \le i \le N$, the linear operator $P_i : D(\Omega) \to D(\Omega_i)$ defined by $P_i h = h_i$ is continuous, by the closed graph theorem. For all f in $D(\Omega)$ we write $f = P_0 f + \cdots + P_N f$, and hence for w in Ω we have

$$|f(w)| \le \|f\| \sum_{i=0}^{N} \|P_i\|_{\text{op}} \cdot \|k_w^i\|_i$$

where $\|P_i\|_{\text{op}}$ is the operator norm of P_i. Letting $f = k_w/\|k_w\|$, we obtain

$$\|k_w\| - \sum_{i=1}^{N} \|P_i\|_{\text{op}} \cdot \|k_w^i\|_i \le \|P_0\|_{\text{op}} \cdot \|k_w^0\|_0.$$

As w approaches $\partial\Omega_0$, both $\|k_w\|$ and $\|k_w^0\|_0$ go to ∞; this is because Ω_0 is conformally equivalent to the disk Δ and there are unbounded functions in $D(\Delta)$. If $1 \le i \le N$ then $\|k_w^i\|_i$ stays bounded as w approaches $\partial\Omega_0$, by the principle of uniform boundedness. Hence there exists $\delta > 0$ such that if $\operatorname{dist}(w, \partial\Omega_0) < \delta$ then

$$\sum_{i=1}^{N} \|P_i\|_{\text{op}} \cdot \|k_w^i\|_i \le \frac{1}{2}\|k_w\|.$$

Combining the last two displayed inequalities, we finish the proof. □

For a universal interpolating sequence $\{z_n\}_1^\infty$ for $D(\Omega)$ whose accumulation points belong to $\partial\Omega_0$, we define some related linear operators. As before, we write k_n^0 to denote $k_{z_n}^0$. For each integer $m \ge 1$, we define a bounded linear operator $T_m : D(\Omega) \to l^2$ by

$$T_m f = \left\{ \frac{f(z_n)}{\|k_n^0\|_0} \right\}_m^\infty .$$

Note that $T_m f$ is in l^2 because of Lemma 13.

Let $E_m = \{f \in D(\Omega): f(z_n) = 0 \text{ for } n \geq m\}$ and let $E_m^\perp$ be the orthogonal complement of E_m in $D(\Omega)$. The operator T_m induces a bounded invertible operator $\tilde{T}_m: E_m^\perp \to l^2$ by restricting T_m to $E_m^\perp$. As before, we use $\|\cdot\|_{\text{op}}$ to denote the operator norm; we use $\|\cdot\|_*$ to denote the l^2 norm.

We remark that $\{\|\tilde{T}_n^{-1}\|_{\text{op}}\}$ is a nonincreasing sequence. To prove this we need only show that $\|\tilde{T}_2^{-1}\|_{\text{op}} \leq \|\tilde{T}_1^{-1}\|_{\text{op}}$. For $k = 1, 2$ we let $Q_k: D(\Omega) \to D(\Omega)$ be the orthogonal projection onto $E_k^\perp$. Note that $T_2 Q_2 f = T_2 f$ for all f in $D(\Omega)$, and so by the definition of an operator norm we have

$$\begin{aligned} \|\tilde{T}_2^{-1}\|_{\text{op}} &= \sup\{\|f\|: f \in E_2^\perp \text{ and } \|T_2 f\|_* = 1\} \\ &= \sup\{\|Q_2 f\|: f \in D(\Omega) \text{ and } \|T_2 f\|_* = 1\}. \end{aligned}$$

If f is in $D(\Omega)$ and $f(z_1) \neq 0$, then there exists a function g in $E_2 \backslash E_1$ such that $f(z_1) + g(z_1) = 0$. Thus $T_2(f+g) = T_2 f$ and $Q_2(f+g) = Q_2 f$. It follows that

$$\|\tilde{T}_2^{-1}\|_{\text{op}} = \sup\{\|Q_2 f\|: f \in D(\Omega) \text{ satisfies } \|T_2 f\|_* = 1 \text{ and } f(z_1) = 0\}.$$

Since $E_2^\perp \subset E_1^\perp$ and $\|Q_2 f\| \leq \|Q_1 f\|$ for all f in $D(\Omega)$,

$$\begin{aligned} \|\tilde{T}_2^{-1}\|_{\text{op}} &\leq \sup\{\|Q_1 f\|: f \in D(\Omega) \text{ and } \|T_1 f\|_* = 1\} \\ &= \|\tilde{T}_1^{-1}\|_{\text{op}}. \end{aligned}$$

This completes the proof of our remark. With this remark, we are ready to give an operator-theoretic proof for the following complex analytic result.

PROPOSITION 14. *Let S be a universal interpolating sequence for $D(\Omega)$. Suppose $S = \bigcup_{i=0}^N S_i$, where each S_i is either empty or has accumulation points all of which belong to $\partial\Omega_i$. Then each of the nonempty sets S_i is a universal interpolating sequence for $D(\Omega_i)$.*

Proof. Without loss of generality, we assume that S_0 is nonempty and prove that $S_0 = \{z_n\}_1^\infty$ is a universal interpolating sequence for $D(\Omega_0)$. For notational simplicity we assume $N = 2$; the argument we give will work for any N.

It is clear that T_1 takes $D(\Omega_0)$ into l^2, so we must show that T_1 takes $D(\Omega_0)$ onto l^2. It is enough to show that T_m is onto l^2 for some m. We consider the continuous linear operators $P_i: D(\Omega) \to D(\Omega_i)$ as defined in the proof of Lemma 13. By the principle of uniform boundedness, there exists a positive constant C such that for all f in $D(\Omega)$ and all z in $\overline{S_0}$,

$$|(P_1 f + P_2 f)(z)| \leq C\|f\|.$$

For $m \geq 1$, define $\tilde{S}_m: E_m^\perp \to l^2$ by the equation $\tilde{S}_m f = T_m P_0 f$. It is clear that $\tilde{S}_m$ is a bounded linear operator. For all f in $E_m^\perp$,

$$\|\tilde{T}_m f - \tilde{S}_m f\|_* = \|T_m(P_1+P_2)f\|_*$$
$$\leq C\|f\|\left(\sum_{n=m}^{\infty} \|k_n^0\|_0^{-2}\right)^{1/2}.$$

The summation above is finite since it is the square of $\|T_m 1\|_*$. Since $\{\|\tilde{T}_n^{-1}\|_{\text{op}}\}$ is a nonincreasing sequence, we can choose m large enough so that

$$\|\tilde{T}_m - \tilde{S}_m\|_{\text{op}} \leq \|\tilde{T}_m^{-1}\|_{\text{op}}^{-1}.$$

This implies that $\tilde{S}_m$ is also invertible and, in particular, that $\tilde{S}_m$ is onto l^2. By the definition of $\tilde{S}_m$, we conclude that T_m maps $D(\Omega_0)$ onto l^2. □

The converse of the last proposition holds. We now take a look at the simplest case. Suppose S is a sequence of points in Ω and that S is a universal interpolating sequence for the Dirichlet space $D(\Omega_k)$ for some k. By Lemma 13 and the fact that $D(\Omega_k)$ contains the constants, the sequence $\{1/\|k_z\|: z \in S\}$ is in l^2. Using the same notation as in the proof of Lemma 13, we write $f = P_0 f + \cdots + P_N f$ for any function f in $D(\Omega)$. Since the function $P_j f$ is bounded on S when $j \neq k$, the sequence $\{f(z)/\|k_z\|: z \in S\}$ is in l^2. In addition, $D(\Omega_k)$ is contained in $D(\Omega)$, and so S is a universal interpolating sequence for $D(\Omega)$. We generalize this observation as follows.

PROPOSITION 15. *Suppose $S_0, S_1, \ldots, S_N$ are sequences of points in Ω. If each S_i is a universal interpolating sequence for $D(\Omega_i)$, then $S = \bigcup_{i=0}^N S_i$ is a universal interpolating sequence for $D(\Omega)$.*

Proof. If f is in $D(\Omega)$ then, by the remark preceding our proposition, the sequence $\{f(z)/\|k_z\|: z \in S\}$ is in l^2. We must show that every l^2 sequence can be written in that form.

We claim that it is sufficient to consider the case when Ω is a Poincaré region whose boundary consists of differentiable simple closed curves. To show this, let φ be a conformal map from Ω onto a region G. By the remark preceding our proposition, each S_k is a universal interpolating sequence for $D(\Omega)$ and so $\varphi(S_k)$ is a universal interpolating sequence for $D(G)$. By Proposition 14, all these $\varphi(S_k)$ satisfy the hypothesis of our proposition, with Ω replaced by G. If we can prove that $\bigcup \varphi(S_k)$ is a universal interpolating sequence for $D(G)$, then it follows that $\bigcup S_k$ is a universal interpolating sequence for $D(\Omega)$.

If $i \neq j$ than $S_i \cap S_j$ contains at most a finite number of points. Since both the hypothesis and the conclusion of our proposition do not depend on these points, we assume that $S_i \cap S_j$ is an empty set. For notational simplicity, we only prove the proposition for the case $N = 2$. The same argument will work for any N.

Since every universal interpolating sequence is a zero set, there are nonzero functions f in $D(\Omega_1)$ and g in $D(\Omega_2)$ such that $f = 0$ on S_1 and $g = 0$ on S_2. By dividing out all (finitely many) zeros in $\overline{S_0}$ from f and g, we may assume that f and g are bounded away from zero on $\overline{S_0}$.

Let $\{\alpha_n\}_1^\infty$ be a sequence in l^2 and let $S_0=\{z_n\}_1^\infty$. For notational simplicity, we write k_n and k_n^0 to denote k_{z_n} and $k_{z_n}^0$, respectively. By our assumption on S_0, along with Lemma 13, there is a function ψ in $D(\Omega_0)$ such that

$$\frac{\psi(z_n)}{\|k_n^0\|_0}=\frac{\alpha_n\|k_n\|}{f(z_n)g(z_n)\|k_n^0\|_0}.$$

By Lemma 9, the function $h=\psi fg$ is in $D(\Omega)$. We observe that $h=0$ on $S_1\cup S_2$, and that $h(z_n)=\alpha_n\|k_n\|$; that is, h interpolates $\{\alpha_n\}$ on S_0. Suppose $\{\beta_n\},\{\gamma_n\}$ are two sequences in l^2. By using the same argument as above, we can find h_1 in $D(\Omega)$ such that h_1 vanishes on $S_0\cup S_2$ and h_1 interpolates $\{\beta_n\}$ on S_1. Similarly, there is a function h_2 in $D(\Omega)$ such that h_2 vanishes on $S_0\cup S_1$ and interpolates $\{\gamma_n\}$ on S_2. Hence, the function $h+h_1+h_2$ is in $D(\Omega)$ and it interpolates $\{\alpha_n\}\cup(\beta_n\}\cup\{\gamma_n\}$ on S. □

A sequence of points in the unit disk Δ is a universal interpolating sequence for $D(\Delta)$ if and only if it is transformed, under a conformal map, into a universal interpolating sequence for $D(\Omega_i)$. The last two propositions together imply that, in an appropriate sense, the universal interpolating sequences for $D(\Omega)$ are exactly the finite unions of the conformal images of those for the unit disk.

For the space of bounded analytic functions $H^\infty(\Omega)$, the universal interpolating sequences have the same composition structure and can be described in terms of Green's functions; see [9, p. 161]. We remark that for this formulation, our argument for Proposition 15 works, but that for Proposition 14 does not.

For either $H^\infty(\Omega)$ or $D(\Omega)$, the composition structure allows us to construct a universal interpolating sequence in Ω by using those in Δ. There is a necessary and sufficient condition for a sequence in Δ to be a universal interpolating sequence for $H^\infty(\Delta)$. We shall show a sufficient condition for a sequence in Δ to be a universal interpolating sequence for the Dirichlet space, similar to that for $H^\infty(\Delta)$. It is not known whether such a condition is necessary. Nevertheless, that condition can be used for construction.

For the rest of this paper, we focus our attention on the Dirichlet space $D(\Delta)$. We define an inner product on $D(\Delta)$ by

$$\langle f,g\rangle=\sum_0^\infty(n+1)\hat{f}(n)\overline{\hat{g}(n)},$$

where $f(z)=\sum\hat{f}(n)z^n$ and $g(z)=\sum\hat{g}(n)z^n$. The norm induced by this inner product is equivalent to the norm defined in the introduction, but it is more convenient to use. The corresponding reproducing kernel $k_w(z)$ for a point w in Δ is given by

$$k_w(z)=\frac{1}{\bar{w}z}\log\frac{1}{1-\bar{w}z}=\sum_0^\infty\frac{\bar{w}^nz^n}{n+1}.$$

The following theorem was discovered by Rosenbaum [12] in 1965, but never published. He also studied the problem of interpolating l^2 sequences using Dirichlet functions and their higher order derivatives. To keep our discussion complete, we include his proof here also.

THEOREM 16. *Let $\{z_n\}_1^\infty$ be a sequence of points in the unit disk Δ. If*

$$\limsup \frac{\log(1-|z_n|)}{\log(1-|z_{n+1}|)} < \frac{1}{9}$$

then the sequence is a universal interpolating sequence for $D(\Delta)$.

Proof. For notational simplicity, in this proof we write k_j to denote k_{z_j}. Let A be the infinite matrix $[A_{nm}]$ where $A_{nm} = \langle k_n/\|k_n\|, k_m/\|k_m\|\rangle$. By [13, Thm. 3], we need only show that A is an invertible operator on l^2. (One may also refer to a result of Bari [2].)

Note that $A_{nn} = 1$, and that

$$|\langle k_n, k_m\rangle|^2 = \left|\frac{1}{\bar{z}_n z_m}\right|^2 \left(\log^2\left|\frac{1}{1-\bar{z}_n z_m}\right| + \operatorname{Arg}^2 \frac{1}{1-\bar{z}_n z_m}\right).$$

It follows that, for all $\epsilon > 0$, we can choose an integer $j = j(\epsilon)$ such that if $m, n > j$ then

$$\frac{1}{\|k_n\|^2} \le (1+\epsilon)|z_n|^2 \frac{-1}{\log(1-|z_n|)},$$

and

$$|\langle k_n, k_m\rangle|^2 \le \left|\frac{1}{\bar{z}_n z_m}\right|^2 (1+\epsilon)\log^2 \frac{1}{1-|z_n|}.$$

By definition $|A_{nm}| = |A_{mn}|$, and if $n, m > j$ then

$$|A_{nm}|^2 \le (1+\epsilon)^3 \frac{\log(1-|z_n|)}{\log(1-|z_m|)}.$$

Our hypothesis implies that we could have chosen ϵ small enough so that if $n, m > j$ then $|A_{nm}| \le \mu^{|n-m|}$, where $\mu < 1/3$. Now we estimate the operator norm of the difference of A and the identity operator I:

$$\begin{aligned} \|A - I\|_{\text{op}} &\le \sup_n \sum_{n \ne m} |A_{nm}| \\ &\le 2 \sum_{n=1}^{\infty} \mu^n \\ &< 1. \end{aligned}$$

Hence A is invertible. □

The constant 1/9 in the theorem may not be sharp. We conclude this paper by raising the following question.

QUESTION. Can we replace 1/9 by 1 in Theorem 16?

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