Dual Operator Algebras and a Hereditary Property of Minimal Isometric Dilations

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1. Introduction

Let $\mathcal{H}$ be a separable, infinite dimensional, complex Hilbert space, and let $\mathcal{L}(\mathcal{H})$ be the algebra of all bounded linear operators on $\mathcal{H}$. A *dual algebra* is a subalgebra of $\mathcal{L}(\mathcal{H})$ that contains the identity operator $I_{\mathcal{H}}$ and is closed in the ultraweak operator topology on $\mathcal{L}(\mathcal{H})$. Note that the weak* topology on $\mathcal{L}(\mathcal{H})$ coincides with the ultraweak operator topology on $\mathcal{L}(\mathcal{H})$. The theory of dual algebras is closely related to the study of the classes $\mathbf{A}_{m,n}$ (to be defined below), where m and n are any cardinal numbers such that $1 \leq m, n \leq \aleph_0$. The structures of the classes $\mathbf{A}_{m,n}$ have been applied to the topics of invariant subspaces, dilation theory, and reflexivity (cf. [6]). In particular, the study of these classes has been focused in the last five years on sufficient conditions that a contraction $T \in \mathcal{L}(\mathcal{H})$ belongs to some $\mathbf{A}_{m,n}$. An abstract geometric criterion for membership in $\mathbf{A}_{\aleph_0,\aleph_0}$ was first given in [1]. In a sequel to this study, Brown–Chevreau–Exner–Pearcy (cf. [8], [11], [12], [13]) obtained some relationships between dual algebras and Fredholm theory, and established topological criteria for membership in $\mathbf{A}_{\aleph_0,\aleph_0}$ or $\mathbf{A}_{1,\aleph_0}$. Recently many authors have studied sufficient conditions for membership in the class $\mathbf{A}_{1,\aleph_0}$, $\mathbf{A}_{\aleph_0,\aleph_0}$, or $\mathbf{A}$ (cf. [10], [14], [15], [18]). In particular, in [11] Chevreau–Exner–Pearcy obtained some surprising and unexpected characterizations of the class $\mathbf{A}_{1,\aleph_0}$. As a sequel to these studies, in this note we define a certain hereditary property concerning the minimal isometric dilation of a contraction operator T in $\mathbf{A}$, namely property $(\tilde{\mathbf{H}})$, and show that $T \in \mathbf{A}(\mathcal{H})$ has property $(\tilde{\mathbf{H}})$ if and only if $T \in \mathbf{A}_{1,\aleph_0}$.

2. Notation and Preliminaries

The notation and terminology employed herein agree with that in [2], [6], and [19]. The class $\mathcal{C}_1(\mathcal{H})$ is the Banach space of trace-class operators on $\mathcal{H}$ equipped with the trace norm. The dual algebra $\mathcal{A}$ can be identified with the

Received September 7, 1990. Revision received September 9, 1991.
The second author was partially supported by a grant from TGRC-KOSEF and Korea Research Foundation, Ministry of Education, 1990–1991.
Michigan Math. J. 39 (1992).

dual space of $\mathcal{Q}_{\mathcal{A}} = \mathcal{C}_1(\mathcal{H})/{}^{\perp}\mathcal{A}$, where ${}^{\perp}\mathcal{A}$ is the preannihilator in $\mathcal{C}_1(\mathcal{H})$ of $\mathcal{A}$ under the pairing

$$(2.1) \qquad \langle T, [L]_{\mathcal{A}} \rangle = \operatorname{tr}(TL), \quad T \in \mathcal{A}, \ [L] \in \mathcal{Q}_{\mathcal{A}}.$$

We write $[L]$ for $[L]_{\mathcal{A}}$ when there is no possibility of confusion. If x and y are vectors in $\mathcal{H}$, we denote by $x \otimes y$ the rank 1 operator whose action is $(x \otimes y)(u) = (u, y)x$ for all u in $\mathcal{H}$.

DEFINITION 2.1 (cf. [6]). Suppose m and n are cardinal numbers such that $1 \leq m, n \leq \aleph_0$. A dual algebra $\mathcal{A}$ will be said to have property $(\mathbf{A}_{m,n})$ if every $m \times n$ system of simultaneous equations of the form

$$(2.2) \qquad [x_i \otimes y_j] = [L_{ij}], \quad 0 \leq i < m, \ 0 \leq j < n,$$

where $\{[L_{ij}]\}_{0 \leq i < m, 0 \leq j < n}$ is an arbitrary $m \times n$ array from $\mathcal{Q}_{\mathcal{A}}$, has a solution $\{x_i\}_{0 \leq i < m}, \{y_j\}_{0 \leq j < n}$ consisting of a pair of sequences of vectors from $\mathcal{H}$. We usually shorten $(\mathbf{A}_{n,n})$ to $(\mathbf{A}_n)$.

We write $\mathbf{D}$ for the open unit disc in the complex plane and $\mathbf{T}$ for the boundary of $\mathbf{D}$. The space $L^p = L^p(\mathbf{T})$, $1 \leq p \leq \infty$, is the usual Lebesgue function space relative to normalized Lebesgue measure m on $\mathbf{T}$. The space $H^p = H^p(\mathbf{T})$ is the usual Hardy space. It is well known (cf. [6]) that the space H^∞ is the Banach dual of L^1/H_0^1, where H_0^1 is the subspace of H^1 consisting of those functions whose zeroth Fourier coefficient vanishes.

A contraction operator T is *absolutely continuous* if in the canonical decomposition $T = T_1 \oplus T_2$, where T_1 is a unitary operator and T_2 is a completely nonunitary contraction, T_1 is either absolutely continuous or acts on the space (0). For T in $\mathcal{L}(\mathcal{H})$ we denote by $\mathcal{A}_T$ the (unital) dual algebra generated by T, and by Q_T the predual of $\mathcal{A}_T$. It is well known that an absolutely continuous contraction T has a Sz.-Nagy–Foiaş functional calculus $\Phi_T: H^\infty \to \mathcal{A}_T$ defined by $\Phi_T(f) = f(T)$ for each f in H^∞. (A full exposition may be found in [6].) We denote by $\mathbf{A} = \mathbf{A}(\mathcal{H})$ the class of all absolutely continuous contractions T in $\mathcal{L}(\mathcal{H})$ for which Φ_T is an isometry (in which case Φ_T maps H^∞ onto $\mathcal{A}_T$). For any m and n we denote by $\mathbf{A}_{m,n} = \mathbf{A}_{m,n}(\mathcal{H})$ the class of T in $\mathbf{A}$ for which $\mathcal{A}_T$ has property $(\mathbf{A}_{m,n})$.

For T in $\mathcal{L}(\mathcal{H})$ we let $\operatorname{Lat}(T)$ denote the lattice of subspaces invariant for T. If $\mathcal{M} \in \operatorname{Lat}(T)$ we write $T|\mathcal{M}$ for the restriction of T to $\mathcal{M}$. A subspace $\mathcal{K}$ is semi-invariant for T if there exist $\mathcal{M}$ and $\mathcal{N}$ in $\operatorname{Lat}(T)$ with $\mathcal{M} \supset \mathcal{N}$ such that $\mathcal{K} = \mathcal{M} \ominus \mathcal{N}$. If $\mathcal{K}$ is semi-invariant for T, we write $T_{\mathcal{K}} = P_{\mathcal{K}} T | \mathcal{K}$ for the compression of T to $\mathcal{K}$, where $P_{\mathcal{K}}$ is the orthogonal projection whose range is $\mathcal{K}$.

We say that an operator B is an *extension* of T if there exists $\mathcal{M}$ in $\operatorname{Lat}(B)$ such that $T = B|\mathcal{M}$; B is a *dilation* of T if there is a semi-invariant subspace $\mathcal{K}$ for B such that $T = B_{\mathcal{K}}$. It is well known from [19] that an absolutely continuous contraction T has a minimal isometric dilation and a minimal co-isometric extension, where minimality is defined in a natural way.

3. Property ($\tilde{\mathbf{H}}$) and the Class $\mathbf{A}_{1,\aleph_0}$

Let T be a contraction operator in $\mathcal{L}(\mathcal{H})$. We denote by $B_T \in \mathcal{L}(\mathcal{K})$ a minimal isometric dilation of T. Then it follows from the Wold decomposition theorem (cf. [19, Thm. I.1.1]) that

$$B_T = U_T \oplus R_T, \tag{3.1}$$

where $U_T \in \mathcal{L}(\mathcal{U}_T)$ is a (forward) unilateral shift operator and $R_T \in \mathcal{L}(\mathcal{R}_T)$ is a unitary operator. Furthermore, it follows from (3.1) that

$$B_T^* = U_T^* \oplus R_T^* \tag{3.2}$$

is a minimal co-isometric extension of T^*.

Suppose $T \in \mathcal{L}(\mathcal{H})$ has an invariant subspace $\mathfrak{M}$ with $\mathfrak{M} \neq (0)$. Then a minimal isometric dilation $B_T \in \mathcal{L}(\mathcal{K})$ is an isometric dilation of $T \mid \mathfrak{M}$. Hence $T \mid \mathfrak{M}$ has a minimal isometric dilation $B_{T\mid\mathfrak{M}} \in \mathcal{L}(\tilde{\mathcal{K}})$ such that $\mathfrak{M} \subset \tilde{\mathcal{K}} \subset \mathcal{K}$ with $\tilde{\mathcal{K}}$ in $\operatorname{Lat}(B_T)$ and $B_{T\mid\mathfrak{M}} = B_T \mid \tilde{\mathcal{K}}$. We use this notation throughout the following definitions.

DEFINITION 3.1. Let T be a contraction operator in $\mathcal{L}(\mathcal{H})$. We say T has property (**H**) if, for any $\mathfrak{M} \in \operatorname{Lat}(T)$ with $\mathfrak{M} \neq (0)$, the minimal isometric dilation $B_{T\mid\mathfrak{M}} \in \mathcal{L}(\tilde{\mathcal{K}})$ of $T \mid \mathfrak{M}$ obtained as a restriction $B_T \mid \tilde{\mathcal{K}}$ with $\tilde{K} \in \operatorname{Lat}(B_T)$ satisfies $\mathcal{U}_{T\mid\mathfrak{M}} \subset \mathcal{U}_T$.

DEFINITION 3.2. A contraction operator $T \in \mathbf{A}$ has property ($\tilde{\mathbf{H}}$) if there exists $\mathfrak{M} \in \operatorname{Lat}(T)$ such that $T \mid \mathfrak{M} \in \mathbf{A}(\mathfrak{M})$ and $T \mid \mathfrak{M}$ has property (**H**).

If $T \in C_{\cdot 0}(\mathcal{H})$ (i.e., if $\|T^{*n}x\| \to 0$ for all $x \in \mathcal{H}$), then it is easy to show that T has property (**H**) (cf. [2, Cor. I.2.11]). Hence a unilateral shift U of any multiplicity has property (**H**). Let W be a bilateral shift of some multiplicity. It is easy to show that W does not have property (**H**) but does have property ($\tilde{\mathbf{H}}$).

Recall that a completely nonunitary contraction $T \in \mathcal{L}(\mathcal{H})$ is said to be of class C_0 if there exists $u \in H^\infty$, $u \not\equiv 0$, such that $u(T) = 0$.

PROPOSITION 3.3. *If U is a unilateral shift of multiplicity* 1, *then U^* does not have property* (**H**). *But $U^* \mid \mathfrak{M}$ has property* (**H**) *for any nontrivial invariant subspace $\mathfrak{M}$ for U^*.*

Proof. Let us take a nontrivial invariant subspace $\mathfrak{N}$ for U^*, and let $\tilde{T} = U^* \mid \mathfrak{N}$. Since $\tilde{T} \in C_0 \subset C_{\cdot 0}$, by [2, Cor. I.2.11] $B_{\tilde{T}}$ is a unilateral shift operator of multiplicity 1. But B_{U^*} is a bilateral shift of multiplicity 1; hence U^* cannot have property (**H**). For the second statement, let $\mathfrak{M}$ be a nontrivial invariant subspace for U^*. Again, $U^* \mid \mathfrak{M} \in C_{\cdot 0}$. Hence $U^* \mid \mathfrak{M}$ has property (**H**), and the proof is complete. □

The following is the main theorem in this paper. The study of $\mathbf{A}_{1,\aleph_0}$ in [11] used heavily the minimal co-isometric extension of an operator T. We give a characterization of the class founded instead on the minimal isometric dilation.

THEOREM 3.4. *Suppose* $T \in \mathbf{A}(\mathcal{H})$. *Then the following statements are equivalent:*

(1) $T \in \mathbf{A}_{1,\aleph_0}(\mathcal{H})$;
(2) *there exists an invariant subspace* $\mathfrak{M}$ *for* T *such that* $T|\mathfrak{M} \in \mathbf{A} \cap C_{\cdot 0}$;
(3) T *has property* $(\tilde{\mathbf{H}})$.

The proof of Theorem 3.4 will appear in the next section.

4. Proof of the Main Theorem

Let T be a contraction in $\mathcal{L}(\mathcal{H})$ and suppose that $\mathfrak{M} \in \mathrm{Lat}(T)$. Recall (cf. [7], [11], and [17]) that $\mathfrak{M}$ is an *analytic invariant subspace* for T if there exists a nonzero conjugate analytic function $e: \lambda \to e_\lambda$ from $\mathbf{D}$ into $\mathfrak{M}$ such that $(T|\mathfrak{M}-\lambda)^* e_\lambda = 0$, $\lambda \in \mathbf{D}$. If in addition to those conditions the function e satisfies $\bigvee_{\lambda \in \mathbf{D}} e_\lambda = \mathfrak{M}$, then $\mathfrak{M}$ is said to be a *full analytic invariant subspace* for T. It follows from [11, Thm. 6.2] that if $T \in \mathbf{A}_{1,\aleph_0}(\mathcal{H})$ then T has a full analytic invariant subspace $\mathfrak{M}$. In particular, we have

$$\mathfrak{M} = \bigvee_{\lambda \in \mathbf{D}} \mathrm{Ker}(T|\mathfrak{M}-\lambda)^*. \tag{4.1}$$

The following lemma is [12, Prop. 2.8]. Recall that $T \in C_{0\cdot}$ if $\|T^n x\| \to 0$ for all x in $\mathcal{H}$.

LEMMA 4.1. *Let* T *be a contraction in* $\mathcal{L}(\mathcal{H})$. *Let* $\emptyset \neq \Lambda \subset \mathbf{D}$ *and let* M *be a nonempty set of natural numbers. Suppose*

$$\mathfrak{M} = \bigvee_{\substack{\lambda \in \Lambda \\ n \in M}} \mathrm{Ker}(T-\lambda)^n. \tag{4.2}$$

Then $T|\mathfrak{M} \in C_{0\cdot}$.

The following proposition shows (1) ⇒ (2), and follows easily from the above remarks and Lemma 4.1.

PROPOSITION 4.2. *If* $T \in \mathbf{A}_{1,\aleph_0}(\mathcal{H})$, *then there exists an invariant subspace* $\mathfrak{M}$ *for* T *such that* $T|\mathfrak{M} \in \mathbf{A} \cap C_{\cdot 0}$.

Let $T \in \mathbf{A}(\mathcal{H})$. Recall that there is, for each λ in $\mathbf{D}$, an element $[C_\lambda]$ of $\mathcal{Q}_T$ of norm 1 and satisfying, for all f in H^∞,

$$\langle f(T), [C_\lambda] \rangle = \tilde{f}(\lambda), \tag{4.3}$$

where $\tilde{f}$ is the analytic extension of f to $\mathbf{D}$ (see [6, §IV]).

The next lemma is a useful tool from [16]; see [6, proof of Thm. 6.6] for the sketch of an essentially similar result.

LEMMA 4.3. *If $T \in \mathbf{A}(\mathcal{H})$, then for any positive integer n there exist an invariant subspace $\mathfrak{M}_n$ for T and an orthonormal set $\{e_k^{(n)}\}_{k=1}^n$ in $\mathfrak{M}_n$ such that*

$$\text{(4.4a)} \qquad e_k^{(n)} \in \operatorname{Ker}(T|\mathfrak{M}_n)^{*k} \ominus \operatorname{Ker}(T|\mathfrak{M}_n)^{*k-1}$$

and

$$\text{(4.4b)} \qquad [e_k^{(n)} \otimes e_k^{(n)}]_T = [C_0]_T, \quad k = 1, 2, \ldots, n.$$

The following lemma is the key step in the proof that (3) ⇒ (1).

LEMMA 4.4. *Suppose $T \in \mathbf{A}(\mathcal{H})$ has property* (**H**). *Then there exists a sequence $\{f_j\}_{j=1}^\infty$ of unit vectors in $\mathcal{H}$ satisfying*

$$\text{(4.5a)} \qquad [f_j \otimes f_j] = [C_0]_T, \quad j = 1, 2, \ldots,$$

and

$$\text{(4.5b)} \qquad \lim_j \|[f_j \otimes z]_T\| = 0 \quad \text{for all } z \in \mathcal{H}.$$

Proof. Let $B_T \in \mathcal{L}(\mathcal{K})$ be a minimal isometric dilation of T. Then B_T^* is a minimal co-isometric extension of T^*. Suppose $B_T^* = U_T^* \oplus R_T^*$, where $U_T \in \mathcal{L}(\mathcal{U})$ is a unilateral shift operator and $R_T \in \mathcal{L}(\mathcal{R})$ is a unitary operator. By Lemma 4.3 we may produce, for each positive integer n, a subspace $\mathfrak{M}_n \in \operatorname{Lat}(T)$ and an orthonormal set $\{e_k^{(n)}\}_{k=1}^n$ in $\mathfrak{M}_n$ such that

$$\text{(4.6a)} \qquad e_k^{(n)} \in \operatorname{Ker}(T|\mathfrak{M}_n)^{*k}, \quad k = 1, 2, \ldots, n,$$

and

$$\text{(4.6b)} \qquad [e_k^{(n)} \otimes e_k^{(n)}]_T = [C_0]_T, \quad k = 1, 2, \ldots, n.$$

Let $B_n \in \mathcal{L}(\tilde{\mathcal{K}}_n)$ be the minimal isometric dilation of $T|\mathfrak{M}_n$ obtained as $B_T|\tilde{\mathcal{K}}_n$ for some $\tilde{\mathcal{K}}_n$ in $\operatorname{Lat}(B_T)$. Then B_n^* is a minimal co-isometric extension of $(T|\mathfrak{M}_n)^*$. Suppose

$$\text{(4.7)} \qquad B_n^* = U_n^* \oplus R_n^*,$$

where $U_n \in \mathcal{L}(\mathcal{U}_n)$ is a unilateral shift operator and $R_n \in \mathcal{L}(\mathcal{R}_n)$ is a unitary operator. From (4.6a) and (4.7) it is easy to show that $e_k^{(n)} \in \mathcal{U}_n$, $k = 1, 2, \ldots, n$. Since T has property (**H**) we have $\mathcal{U}_n \subset \mathcal{U}$ for all n, and thus $e_k^{(n)} \in \mathcal{U}$ for all pairs k and n with n a positive integer and $1 \le k \le n$.

As in the proof of [6, Thm. 6.6], from the finite orthonormal sets $\{e_k^{(n)}\}_{k=1}^n$, $n = 1, 2, \ldots$, we may extract a sequence $\{f_j\}_{j=1}^\infty$ of unit vectors weakly convergent to zero and satisfying $[f_j \otimes f_j] = [C_0]_T$ for all j (where each f_j is some $e_k^{(n)}$). Briefly, if $\{w_i\}_{i=1}^\infty$ is an orthonormal basis for $\mathcal{H}$ then it suffices to choose the sequence $\{f_j\}_{j=1}^\infty$ from among the $e_k^{(n)}$ so that $|(w_i, f_j)| \le 1/\sqrt{j}$ for all $1 \le i \le j$. This may be achieved by considering, for $n = j^2$, the $\aleph_0 \times n$ rectangular array whose (i, k)th entry is $|(w_i, e_k^{(n)})|^2$, and noting that there must exist some column for which the sum of the first j entries is less than or equal to $1/j$.

Let $P_{\mathcal{U}}$ be the orthogonal projection from $\mathcal{K}$ onto $\mathcal{U}$. Observe finally that for each j, and for any $z \in \mathcal{K}$,

$$\begin{aligned} \|[f_j \otimes z]_T\| &= \|[z \otimes f_j]_{T^*}\| \\ &= \|[z \otimes f_j]_{B_T^*}\| \quad \text{since } \mathcal{K} \in \operatorname{Lat}(B_T^*) \\ &= \|[P_{\mathcal{U}} z \otimes f_j]_{B_T^*}\|, \end{aligned} \tag{4.8}$$

where the last equality follows from $f_j \in \mathcal{U}$ and $\mathcal{U}$ reducing for B_T^*. But since $f_j \in \mathcal{U}$ we also have that $\|[P_{\mathcal{U}} z \otimes f_j]_{B_T^*}\| = \|[P_{\mathcal{U}} z \otimes f_j]_{U_T^*}\|$. Then, since $\{f_j\}_{j=1}^{\infty}$ is weakly convergent to zero, we have

$$\lim_j \|[P_{\mathcal{U}} z \otimes f_j]_{U_T^*}\| = 0, \tag{4.9}$$

using $U_T^* \in C_{0\cdot}$ and citing [12, Prop. 2.7]. Thus from (4.8) we have

$$\lim_j \|[f_j \otimes z]_T\| = 0, \tag{4.10}$$

and since each f_j is some $e_k^{(n)}$ we have, for each j,

$$[f_j \otimes f_j] = [C_0]_T \tag{4.11}$$

from (4.6b). Therefore the sequence $\{f_j\}_{j=1}^{\infty}$ satisfies (4.5a) and (4.5b) as desired. □

Suppose $\mathcal{A} \subset \mathcal{L}(\mathcal{K})$ is a dual algebra and $0 \le \theta < \gamma \le 1$. As in [13], we denote by $\mathcal{E}_\theta^r(\mathcal{A})$ the set of all $[L]$ in $\mathcal{Q}_{\mathcal{A}}$ for which there exist sequences $\{x_i\}_{i=1}^{\infty}$ and $\{y_i\}_{i=1}^{\infty}$ from the closed unit ball of $\mathcal{K}$ satisfying

$$\limsup_{i \to \infty} \|[L] - [x_i \otimes y_i]\| \le \theta \tag{4.12a}$$

and

$$\|[x_i \otimes z]\| \to 0 \ (i \to \infty), \quad \text{for all } z \in \mathcal{K}. \tag{4.12b}$$

The dual algebra $\mathcal{A}$ is said to have property $E_{\theta,\gamma}^r$ (for some $0 \le \theta < \gamma \le 1$) if the smallest closed absolutely convex set containing $\mathcal{E}_\theta^r(\mathcal{A})$ contains the closed ball $B_{0,\gamma}$ of radius γ centered at the origin in $\mathcal{Q}_{\mathcal{A}}$:

$$\overline{\operatorname{aco}}(\mathcal{E}_0^r(\mathcal{A})) \supset \{[L] \in \mathcal{Q}_{\mathcal{A}} : \|[L]\| \le \gamma\}. \tag{4.13}$$

It follows from [11, Thm. 6.2] that $T \in \mathbf{A}_{1,\aleph_0}$ if and only if $T \in \mathbf{A}$ and $\mathcal{A}_T$ has property $E_{0,1}^r$.

Now we are ready to show that (3) ⇒ (1).

PROPOSITION 4.5. *Suppose $T \in \mathbf{A}(\mathcal{K})$. If T has property* $(\tilde{\mathbf{H}})$, *then $T \in \mathbf{A}_{1,\aleph_0}(\mathcal{K})$.*

Proof. Without loss of generality we may assume that T has property $(\mathbf{H})$. For according to [16, Lemma 3.14], if for some $\mathcal{M} \in \operatorname{Lat}(T)$ we have $T|\mathcal{M} \in \mathbf{A}_{1,\aleph_0}$, then $T \in \mathbf{A}_{1,\aleph_0}$. Furthermore, by [11, Thm. 6.2] it is sufficient to show that $\mathcal{A}_T$ has property $E_{0,1}^r$. Moreover, according to [6, Prop. 1.21] it is enough to show that for each λ in $\mathbf{D}$ we have $[C_\lambda]$ in $\mathcal{E}_0^r(\mathcal{Q}_T)$. Finally, as in the proof of [6, Prop. 6.1], it is sufficient to construct a sequence $\{x_j\}_{j=1}^\infty$ of unit vectors from $\mathcal{H}$ such that

(4.14a) $$[x_k \otimes x_k]_T = [C_0]_T, \quad k = 1, 2, \ldots,$$

and

(4.14b) $$\lim_k \|[x_k \otimes z]\| = 0 \quad \text{for all } z \in \mathcal{H}.$$

But the existence of such a sequence is exactly the conclusion of Lemma 4.4, and the proof is complete. □

Since we have observed before that [2, Cor. I.2.11] shows (2) ⇒ (3), the proof of Theorem 3.4 is complete.

REMARK. One might consider a property $(\mathbf{H}_*)$ analogous to $(\mathbf{H})$ but concerning instead the shift parts of the minimal co-isometric extensions of T and $T|\mathfrak{M}$. It is not too hard to show, however, that every contraction has this property $(\mathbf{H}_*)$ (the reason is essentially that a unitary operator is very far from being $C_{0\cdot}$).

ACKNOWLEDGMENTS. The authors wish to thank Professor Carl Pearcy for his support and encouragement, and the referee of an earlier version of this paper for valuable comments.

References

1. C. Apostol, H. Bercovici, C. Foiaş, and C. Pearcy, *Invariant subspaces, dilation theory, and the structure of the predual of a dual algebra.* I, J. Funct. Anal. 63 (1985), 369–404.
2. H. Bercovici, *Operator theory and arithmetic in H^∞*, Math. Surveys Monographs, 26, Amer. Math. Soc., Providence, RI, 1988.
3. ———, *Factorization theorems and the structure of operators on Hilbert space,* Ann. of Math. (2) 128 (1988), 399–413.
4. ———, *A contribution to the structure theory of the class* **A**, J. Funct. Anal. 78 (1988), 197–207.
5. H. Bercovici, C. Foiaş, and C. Pearcy, *Dilation theory and systems of simultaneous equations in the predual of an operator algebra.* I, Michigan Math. J. 30 (1983), 335–354.
6. ———, *Dual algebras with applications to invariant subspaces and dilation theory,* CBMS Regional Conf. Ser. in Math., 56, Amer. Math. Soc., Providence, RI, 1985.
7. S. Brown, *Full analytic subspaces for contractions with rich spectrum,* Pacific J. Math. 132 (1988), 1–10.
8. S. Brown, B. Chevreau, and C. Pearcy, *On the structure of contraction operators.* II, J. Funct. Anal. 76 (1988), 30–55.
9. A. Brown and C. Pearcy, *Introduction to operator theory. I, Elements of functional analysis,* Springer, New York, 1977.

10. B. Chevreau, *Sur les contractions à calcul fontionnel isométrique.* II, J. Operator Theory 20 (1988), 269–293.
11. B. Chevreau, G. Exner, and C. Pearcy, *On the structure of contraction operators.* III, Michigan Math. J. 36 (1989), 29–62.
12. B. Chevreau and C. Pearcy, *On the structure of contraction operators with applications to invariant subspaces,* J. Funct. Anal. 67 (1986), 360–379.
13. ———, *On the structure of contraction operators.* I, J. Funct. Anal. 76 (1988), 1–29.
14. G. Exner and P. Sullivan, *Normal operator and the classes* $\mathbf{A}_n$, J. Operator Theory 19 (1988), 81–94.
15. I. Jung, *Dual operator algebras and the classes* $\mathbf{A}_{m,n}$. I, J. Operator Theory (to appear).
16. ———, *Dual operator algebras and the classes* $\mathbf{A}_{m,n}$, Thesis, Univ. of Michigan, 1989.
17. R. Olin and J. Thompson, *Algebras of subnormal operators,* J. Funct. Anal. 37 (1980), 271–301.
18. P. Sullivan, *Subnormal operators in* $\mathbf{A}$, J. Operator Theory 18 (1987), 237–248.
19. B. Sz.-Nagy and C. Foiaş, *Harmonic analysis of operators on the Hilbert space,* North-Holland, Amsterdam, 1970.

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