Minimizing the Cardinality of the Fixed Point Set for Self-Maps of Surfaces with Boundary

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1. Introduction

Among all maps homotopic to a given self-map of a compact polyhedron, there is an integer which gives the greatest lower bound for the number of fixed points. We develop a method for computing this bound when the polyhedron is a surface with boundary. A few applications of this method are also given.

Given a compact manifold of dimension n and a map $f: X \to X$, a problem of general interest is to obtain information about the fixed point set $\text{Fix}(f) = \{x \in X \mid f(x) = x\}$. As f can be approximated by a map having $\text{Fix}(f)$ finite, a natural problem is to find a lower bound for the cardinality of $\text{Fix}(f)$ where the lower bound is among maps homotopic to the given map f. Following conventional notation, let

$$MF[f] = \min\{\#\text{Fix}(g) \mid g \text{ is homotopic to } f\}.$$

The dependence on the space X is suppressed in this notation. We are interested in computing $MF[f]$ given the space X and the homotopy class of maps $[f]$.

Central to the solution of this problem is the Nielsen number $N(f)$, which is the number of essential fixed point classes of f; see either [B] or [J1] for the definition and properties of $N(f)$. Its importance is that it always gives a lower bound for $MF[f]$ and the following classical theorem due to Wecken [W].

THEOREM 1.0. *If X is a manifold of dimension greater than* 2, *then* $N(f) = MF[f]$.

In dimension 2, the above result holds if f is a homeomorphism, but in general the lower bound given by $N(f)$ cannot be achieved. Counterexamples appear in [J2], [J3], [K], and [Z]. In [K] the author gives an algorithm for computing $MF[f]$ in the case where X is a disk with two open holes removed—that is, the pair of pants. The aim of the present paper is to abstract

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the ideas in [K] and to give a geometric scheme, but not an algorithm, for computing $MF[f]$ for any self-map of a compact surface with boundary. This geometric method is quite different than the commutator calculus and braid group techniques used by Jiang [J4] to give an algebraic formulation for $MF[f]$. In Section 3 two factorization theorems for computing $MF[f]$ are presented. In Section 4 a few applications of these theorems are given, including the proof of the following embedding theorem.

THEOREM 1.1. *Let P and M be compact, connected surfaces each having nonempty boundary. Suppose that $i: P \to M$ is an embedding such that the induced map $i_*: H_1(P; Z) \to H_1(M; Z)$ is a monomorphism. Let $F: M \to M$ be a map such that $F(M)$ is contained in $i(P)$. If $f: P \to P$ is the induced map satisfying $i \circ f = F \circ i$, then $MF[F] = MF[f]$.*

To see that the hypothesis imposed in Theorem 1.1 regarding the embedding of P into M is necessary, consider the following example.

EXAMPLE 1.2. Let C_r denote the circle in the plane with center $(0, 0)$ and radius r. Let M be the annular region in the plane bounded by C_1 and C_4. Let B denote the circle with center $(2.5, 0)$ and radius $\frac{1}{2}$, and let P be a regular neighborhood of $C_2 \cup B$ in M. As the loop B bounds a disk in M, the inclusion of P into M violates the hypothesis in Theorem 1.1.

Fix orientations for the loops C_2 and B, which are based at the point $(2, 0) \equiv C_2 \cap B$. Let n be a positive integer and define a family of maps $f_n: P \to P$ by sending B to the point $(2, 0)$ and mapping C_2 to the loop $(BC_2B^{-1}C_2^{-1})^n BC_2$. Extend to a map F_n of M by mapping each component of $M \backslash P$ into P. Since $(F_n)_\#: \pi_1(M) \to \pi_1(M)$ is the identity homomorphism, F_n is homotopic to the identity map on M and so $MF[F_n] = 0$. On the other hand, Corollary 1.2 of [K] states that $MF[f_n] = 2n$.

2. Definitions and Related Examples

Let D be a 2-dimensional disk and M the surface obtained by attaching 1-handles $H_1, \dots, H_k$ to D:

$$M = D \cup H_1 \cup H_2 \cup \cdots \cup H_k.$$

Let $\{A_{2i-1}, A_{2i}\}$, $1 \le i \le k$, denote the attaching region for H_i. Then each A_j is a proper arc in M. Let $A = \bigcup_{j=1}^{2k} A_j$. Choose $x_0 \in \operatorname{int} D$, and for $1 \le i \le k$ let w_i be a simple closed curve based at x_0 such that $w_i \cap w_j = x_0$ if $i \ne j$, $w_i \cap (M \backslash D) \subset H_i$, and w_i meets each of A_{2i-1} and A_{2i} transversally in one point. Let $W = w_1 \cup \cdots \cup w_k$; then we can naturally associate $\pi_1(M, x_0)$ with the free group generated by $\{w_1, \dots, w_k\}$.

Since our interest is in finding the minimal number of fixed points possible for a given map $f: M \to M$, it follows from Lemma 3.2 and Corollary 3.6 in [K] that we may always assume that $f(x_0) \notin A$, $f^{-1}(A)$ is a 1-dimensional manifold transverse to both W and A, that $\mathrm{Fix}(f)\cap f^{-1}(A)=\emptyset$, and that $f^{-1}(A)$ does not contain any inessential simple closed curves.

The following is some terminology regarding $f: M\to M$ that was introduced in [K] and will be used throughout this paper. The closure of a component U of $M\backslash(f^{-1}(A)\cup A)$ is a *critical region* for f if both $f(U)$ and U are contained in the same component of $M\backslash A$. Observe that (topologically) critical regions are just 2-dimensional disks and that the union of their interiors contains the fixed point set of f. The *index* of a critical region C is the topological fixed point index of $f\,|\,C$ (see [B] or [J1]).

Let α be an arc contained in $f^{-1}(A_i)$ with $\partial\alpha\cap A_i=\emptyset$ and $\alpha\cap A_j=\emptyset$ when $i\neq j$. The *order* of α, denoted $|\alpha|$, is the cardinality of $\{\alpha\cap A_i\}$. If α has the property that $|\alpha|\geq|\beta|$ given any curve $\beta\subset f^{-1}(A_i)$ satisfying $\alpha\cap\beta\neq\emptyset$ and $\beta\cap A_j=\emptyset$ when $i\neq j$, then we say that

$$\begin{cases} \alpha \text{ is a } \textit{turn} & \text{if } |\alpha| \text{ is even,} \\ \alpha \text{ is a } \textit{crossing} & \text{if } |\alpha| \text{ is odd.} \end{cases}$$

If the order of α is at most 2 then we say that α is *simple.* To visualize, let $\bar{\alpha}$ be the component of $f^{-1}(A)\cap(D\cup H_i)$ containing α. Then $\partial\bar{\alpha}$ is contained in $(A\backslash A_i)$ and rel endpoints, $\bar{\alpha}$ and A_i can be oriented so that their algebraic intersection number is 0 if α is a turn, 1 if it is a crossing. Observe that a turn can be isotoped off of A_i by an isotopy of M with support on $D\cup H_i$.

Now, by [K; Lemma 2.1], up to homotopy each critical region has at most one fixed point; one if its index is nonzero and none if its index is zero. With this in mind we let $\#\mathrm{Fix}(f)$ denote the number of critical regions having a nonzero index. The following example illustrates the computation of the index of a critical region using the terminology above.

EXAMPLE 2.1. Let C be a critical region for f which is contained in D and bounded by A_i, A_j and by curves $\alpha\subset f^{-1}(A_i)$ and $\beta\subset f^{-1}(A_j)$, $i\neq j$, as indicated in Figure 1. In this case, α and β contain (respectively) crossings α_0 and β_0. Identify D with $I\times I$ so that $A_i\subset\{0\}\times I$ and $A_j\subset\{1\}\times I$.

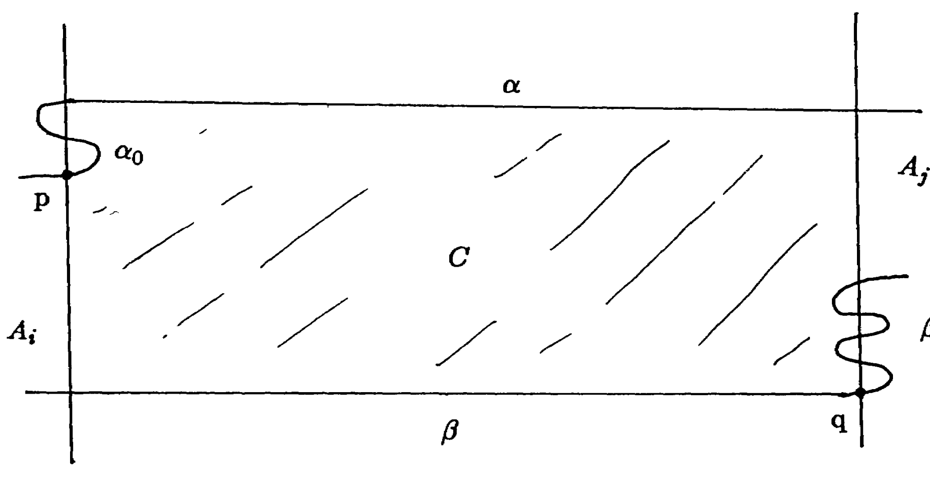

Figure 1

Following [K, §2], the index of C is completely determined by f acting on the set $S=\{\alpha\cap A_i\}\cup\{\beta\cap A_j\}$ as follows: Think of S as a cyclically ordered set on ∂C and label each $s\in S$ with $+$ if $f(s)>s$ or with $-$ if $f(s)<s$, using the given product structure on D. Now by successively cancelling adjacent $++$ or $--$ labeled pairs of points, reduce S to a set S' containing only alternating labeled points. Then the index of C is $\pm\frac{1}{2}(\#S')$.

As a consequence of Lemma 3.5 in [K], the labels for $\alpha\cap A_i$ alternate between $+$ and $-$. For if consecutive points had the same label then this lemma allows us to replace f by a new minimizing map g, with $g^{-1}(A)\cap A$ differing from $f^{-1}(A)\cap A$ only by the removal of these two points. Similarly, the labels for $\beta\cap A_j$ alternate sign. Hence the index of C is zero only when $|\alpha_0|=|\beta_0|$ and the labels for p and q agree. A special case is noteworthy as it turns out to be the generic case. If $|\alpha_0|=1=|\beta_0|$ then C meets A_i and A_j in arcs and, without loss, $f(C)$ is an arc joining the point $f(\alpha)\in A_i\backslash C$ to the point $f(\beta)\in A_j\backslash C$. If index$(C)=0$ then $f(C)\cap C=\emptyset$, otherwise index$(C)=\pm1$ and $f(C)$ meets C in a subarc of $f(C)$.

An arc or a simple closed curve γ in M is said to be *taut* if $\partial\gamma\cap A=\emptyset$, $\operatorname{int}\gamma$ meets $W\cup A$ transversally in a finite number of points, $A\cup\gamma$ does not contain any inessential simple closed curves, and for each i, $\operatorname{int}\gamma\cup(w_i\backslash x_0)$ does not contain any simple closed curves. This last condition keeps the geometric intersection of γ and W minimal up to isotopy (rel ∂) in $M\backslash x_0$. A map $f\colon M\to M$ is *taut* if each component of $f^{-1}(A)$ is a taut proper arc or a taut simple closed curve, $\mathrm{Fix}(f)\cap A=\emptyset$ and $f(M)\subset W$. It follows immediately from the definitions that if f is a taut map, then $f^{-1}(A)$ does not contain any turns and all crossings are simple.

The following two constructions of taut maps will be used throughout this paper. Since there is a bijection between $\mathrm{End}(\pi_1(M))$ and $[M,M]$, the first construction produces a representative map for the homotopy class corresponding to a given endomorphism. The second produces a map from more geometric data; in this case the endomorphism is not given explicitly.

(1) Given an endomorphism ϕ of $\pi_1(M,x_0)$ by $w_i\mapsto\phi(w_i)$ (where $\phi(w_i)$ is a reduced word), there is a natural taut map $f_\phi\colon M\to M$ satisfying $f_\phi(D)=x_0$, and $f_\phi^{-1}(A)\cap H_i$ consists of $2\cdot[\mathrm{length}(\phi(w_i))]$ proper arcs in M each traversing w_i in a single point. Define f_ϕ so that the loop $f_\phi(w_i)$ corresponds to the word $\phi(w_i)$. (See [K, p. 82] for more detail.) Since each critical region has index ±1, we have

$$\#\mathrm{Fix}(f_\phi)=1+\sum_{i=1}^{k}(\text{number of appearances of } w_i^{\pm1} \text{ in } \phi(w_i)).$$

(2) Given the following data:

(i) A finite collection Γ of pairwise disjoint, taut proper arcs and simple closed curves;

(ii) a map $\mu\colon\Gamma\to W\cap A$;

(iii) a map $\rho\colon\{\text{components of } M\backslash\Gamma\}\to\{\text{components of } M\backslash A\}$;

(iv) for each component V of $M\backslash\Gamma$ and each curve $\gamma\in\Gamma$ which lies on ∂V, if $\rho(V)=H_i$ then $\mu(\gamma)\in\{A_{2i-1},A_{2i}\}$;

there exists a taut map $f_\Gamma: M\to M$ such that $\Gamma=\{f_\Gamma^{-1}(A)\}$.

The construction (1) above can be thought of as a special case of (2). Given the endomorphism ϕ, the proper arcs in $M\backslash D$ comprise all of Γ and the functions μ, ρ are determined by the words $\phi(w_1),\dots,\phi(w_n)$.

The constructions of the maps f_ϕ and f_Γ are certainly not unique, but we have the following lemma.

LEMMA 2.2. *Let $h: M\to M$ be a homeomorphism isotopic to the identity. Given (Γ,μ,ρ) and f_Γ as defined above, let $\Gamma'=h(\Gamma)$, $\mu'=\mu\circ h^{-1}$, and $\rho'=\rho\circ h^{-1}$. Then for any $f_{\Gamma'}$ defined corresponding to (Γ',μ',ρ'), $f_{\Gamma'}$ is homotopic to f_Γ.*

Proof. Since $f_\Gamma\circ h$ is homotopic to $f_{\Gamma'}$, we may assume that $\Gamma=\Gamma'$, $\mu=\mu'$, and $\rho=\rho'$. As W is a deformation retract of M, and since the induced map $(f|_W)$ on fundamental groups is completely determined by $\Gamma\cap W$ and μ, the result follows. □

The computation of #Fix(f) in construction (2), which depends on Γ, μ, and ρ, is illustrated in the next example.

EXAMPLE 2.3. The collection Γ, lying on the disk with three holes removed, consists of six proper arcs and two simple closed curves isotopic to w_3, as indicated in Figure 2. The μ-values (indices only) for the arcs are

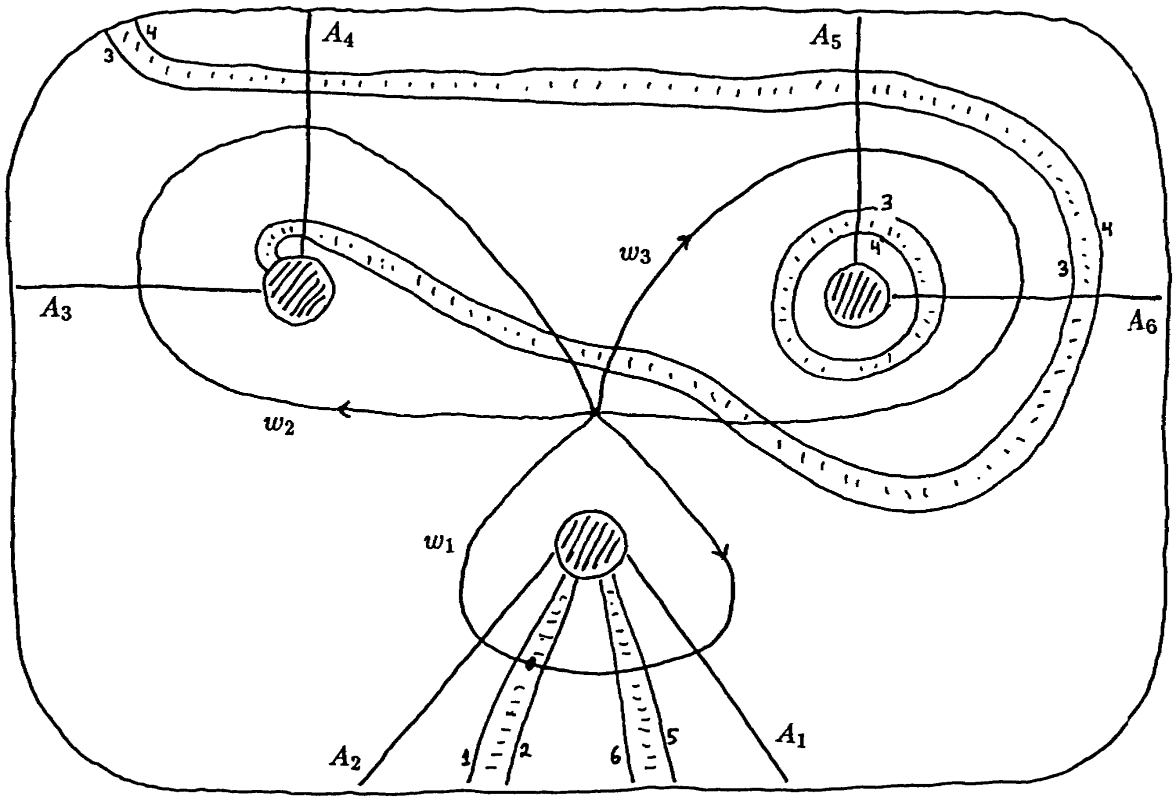

Figure 2

marked along the outer boundary curve. The μ-values for the circles are as labeled. For ρ-values, the region between 1 and 2 goes to H_1, between 3 and 4 goes to H_2, between 5 and 6 goes to H_3, and the rest are mapped to D. There are seven critical regions in all; one in H_1, two in H_2, and four in D. All have index 0 except the one in H_1; thus $\#\text{Fix}(f_\Gamma)=1$. On the level of a fundamental group, w_1 is sent to $w_3w_1^{-1}$, w_2 to w_2, and w_3 goes to the trivial word. Since the Lefschetz number of this map is nonzero, $MF[f_\phi]=1$.

EXAMPLE 2.4. This example illustrates the importance of the topology of the surface, in addition to the endomorphism ϕ acting on a fundamental group, in determining $MF[f_\phi]$. Suppose M is defined as above with $\text{rank}(H_1(M))=2$. Let m be a positive integer and define $\phi\colon \pi_1(M)\to\pi_1(M)$ by

$$w_1 \mapsto (w_2w_1w_2^{-1}w_1^{-1})^m w_2w_1,$$
$$w_2 \mapsto 1.$$

In [K] it was shown that if M is homeomorphic to a disk with two holes removed then $MF[f_\phi]=2m$. On the other hand, if M is topologically a punctured torus then one can construct a fixed point free map homotopic to f_ϕ. Roughly, the fixed point free map occurs because the word $w_2w_1w_2^{-1}w_1^{-1}$ corresponds to the boundary of the punctured torus.

Let $f\colon M\to M$ be given, taut except possibly having turns and nonsimple crossings. A critical region for f is said to be *exceptional* if it meets at least three components of A. Now suppose $\mathcal{L}=L_1\cup L_2$ is given, where each L_i consists of a nested collection (possibly empty) of simple closed curves contained in distinct critical regions for f. These critical regions are assumed to meet at least three components of A. Suppose further that ρ,μ are defined for $f^{-1}(A)\cup\mathcal{L}$, and that the innermost curve of L_i bounds a critical region E_i (its index must be zero). Let D_i be a disk containing L_i and not intersecting $f^{-1}(A)\cup A$. Let $\tilde f$ be defined by construction (2) above so that: $\tilde f=f$ on the complement of the union of all the exceptional critical regions; and, on each such region, $\tilde f$ has a fixed point, possibly having index 0.

Let $\theta_i\in\{1,3\}$ be given. A taut arc α in M is a *merging arc for f* if it satisfies the following conditions.

(1) With $\partial\alpha=\{p_1,p_2\}$, if $L_i=\emptyset$ then $p_i\in\text{Fix}(\tilde f)$, otherwise $p_i\in\text{int}\,E_i$.
(2) For each $l\in L_i$, α meets l transversally in θ_i points.
(3) There is a regular neighborhood N of α such that, for each $l\in L_i$, $N\cap l$ consists of θ_i arcs and $f(\alpha)$ is homotopic to α rel$(\partial\alpha\cup(M\setminus N))$.
(4) α is transverse to, and does not meet any turns of, $f^{-1}(A)$. Here transverse means (in addition to topologically transverse) that if α_0 is a component of $(\alpha\setminus A)$, τ is a component of $(f^{-1}(A)\setminus A)$, and $\tau\cap\alpha_0$ is nonempty, then α_0 and τ meet in exactly one point and their closures do not intersect a common component of A.

The above definition is no doubt unwieldy, and is only used in its most general form in the proof of Theorem 3.2 and in Example 4.1. For convenience, and indeed it is the generic situation, the reader should first consider the case where $\mathcal{L}$ is empty so that α joins two fixed points of $\tilde{f}$. In this case α will be called a *simple merging arc.* By condition (3), it follows that $\partial\alpha$ belongs to the same fixed point class.

Suppose that $f: M \to M$ is a map and α a merging arc. We define a new map $g: M \to M$ as follows: Let $S_0 = \alpha \cap f^{-1}(A)$. Suppose $p, q \in S_0$ so that the interior of the subarc of α from p to q misses S_0 and both $f(p), f(q)$ are in the same A_j. Also, if α is not simple, then the subarc from p to q is not contained in $(D_1 \cup D_2)$. We alter $f^{-1}(A)$ as indicated in Figure 3.

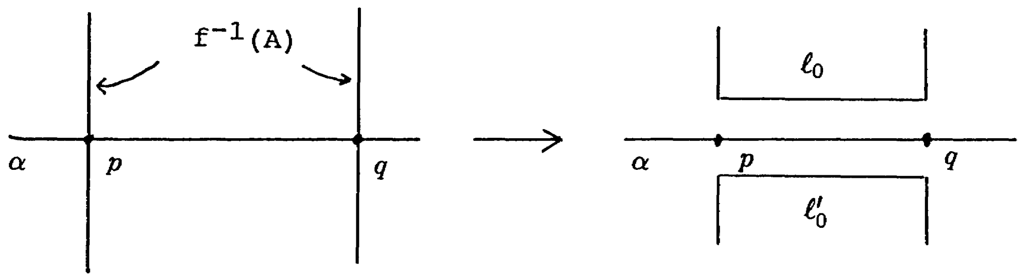

Figure 3

Thus, neighborhoods of p and q in $f^{-1}(A)$ are replaced by arcs l_0, l_0' which are parallel to α. Setting $S_1 = S_0 \setminus \{p, q\}$, we repeat the above procedure choosing l_1, l_1' disjoint from l_0, l_0'. Exhausting all such pairs, we end up with a set of points S_e on α and a family of curves $\{l_i, l_i'\}$. We obtain an intermediate map $\hat{g}: M \to M$ by letting Γ denote the taut collection of curves obtained by isotoping the $\{l_i, l_i'\}$ to taut curves and letting μ, ρ be naturally induced by f; then $\hat{g} = f_\Gamma$. Notice that since the curves in $f^{-1}(A)$ which did not meet α were already taut, they are unchanged in Γ.

Now, write $\alpha \cap A = \{q_1, \ldots, q_n\}$ ordered along α. Since $f(\alpha)$ is homotopic to α, from the above construction it follows that $S_e = \hat{g}^{-1}(A) = \{r_1, \ldots, r_n\}$ (using the same ordering along α) with $\hat{g}(r_i)$ and q_i contained in the same component of A. Namely, $\mu(L_i)$ where $r_i \in L_i \subset \hat{g}^{-1}(A)$.

We alter $\hat{g}^{-1}(A)$ as indicated in Figure 4. Here a neighborhood of r_i in $\hat{g}^{-1}(A)$ is pulled along α until $\mu(L_i) \cap q_i$ is traversed, forming a turn at

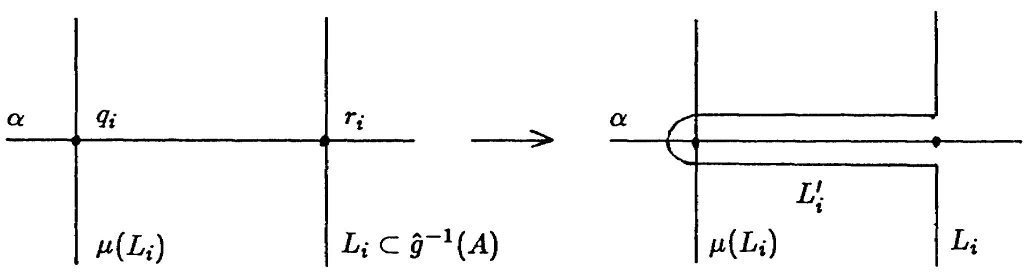

Figure 4

$\mu(L_i)$. Let L_i' denote this new curve. The turn is the subarc of L_i' which meets $\mu(L_i)$ in an even number of points and meets no other component of A.

Define $g: M \to M$ by requiring that $g(L_i') \subset \mu(L_i)$ and that g agrees with $\hat{g}$ outside a neighborhood of α. We say that *g is obtained from f by merging along α*, and write $f \xrightarrow{\alpha} g$. Certainly g is not uniquely determined, but by virtue of the construction and the proof of Lemma 2.2 we have the following proposition.

PROPOSITION 2.5. *If $f \xrightarrow{\alpha} g$ then f and g are homotopic maps.*

Suppose $f \xrightarrow{\alpha} g$ is given. If g has a merging arc β we can merge along β to obtain a map h. This gives

$$f \xrightarrow{\alpha} g \xrightarrow{\beta} h \quad \text{which we write as } f \xrightarrow{(\alpha,\beta)} h.$$

In general, $f \xrightarrow{\Lambda} g$ where $\Lambda = (\lambda_1, \ldots, \lambda_r)$ means that there exist maps f_i: $M \to M$ with $f_0 = f$, $f_r = g$, and $f_{i-1} \xrightarrow{\lambda_i} f_i$. We call Λ a *merging sequence* for f, and note that Λ is an ordered sequence since $\partial\lambda_i$ may not be in $\mathrm{Fix}(f_{i-2})$. We say Λ is *disjoint* if $\mathring{\lambda}_i \cap \mathring{\lambda}_j = \emptyset$ for each $i \neq j$. If in addition each λ_i is simple, we say that Λ is *simple.*

3. The Factorization Theorems

In manifolds of dimension 3 or more, the parallel notion of merging along an arc α would be a homotopy, with support in a neighborhood of α, which takes two fixed points and coalesces them to form a single fixed point (or cancels the pair) without introducing any other fixed points. In dimension 2 we can think of merging along α as coalescing $\partial\alpha$, but it is quite possible that new fixed points are introduced (by creating critical regions with nonzero index), and others may be cancelled out, in the process. Nonetheless, Theorem 3.1 will allow us to find the minimal number of fixed points for a given map by this process, but only after a certain taut map is given. First, one additional construction is needed.

Suppose $f: M \to M$ is given so that $f^{-1}(A)$ is taut except for having turns and crossings. Define a map $f_{\min}: M \to M$ associated to f by requiring that

(i) $f_{\min}^{-1}(A)$ and $f^{-1}(A)$ differ only in the order and labeling (as in Example 2.1) of the turns and crossings, and
(ii) $f_{\min}$ has the maximal number of critical regions with index 0 among all maps satisfying (i).

After the proof of Theorem 3.1, a method is described for obtaining $f_{\min}$ from f.

Our goal is to find a fixed point minimizing map in a prescribed homotopy class of self-maps of M, which we may assume is determined by a given endomorphism ϕ of the fundamental group. This in turn determines the map f_ϕ, constructed in Section 2, in the homotopy class corresponding to ϕ. The idea is to find a map f so that the related map $f_{\min}$ defined above is a minimizing map. The following theorem states that such a map f is the result of merging along some merging sequence which begins with a taut map. Thus, the fixed point minimization problem is reduced to the consideration of taut maps (construction (2) in Section 2) and merging sequences associated to these maps.

THEOREM 3.1. *Given an endomorphism ϕ of $\pi_1(M, x_0)$ there exists a taut map $g: M \to M$, homotopic to f_ϕ, and a simple merging sequence $g \xrightarrow{\Lambda} f$ such that* $\#\text{Fix}(f_{\min}) = MF[f_\phi]$. *Moreover, if h is a map homotopic to f_ϕ having $MF[f_\phi]$ fixed points and such that $h^{-1}(A)$ is a* 1*-dimensional proper submanifold of M, then g may be chosen so that $g^{-1}(A)$ is isotopic, by an isotopy which is the identity on ∂M, to $h^{-1}(A)$.*

Proof. Let $h: M \to M$ be a map having $MF[f_\phi]$ fixed points. Besides the assumptions concerning minimizing maps given at the beginning of Section 2, we also assume that h satisfies the following condition (*) which is a consequence of Lemmas 3.4–3.6 in [K]: Suppose τ is a subarc of $A_i \backslash h^{-1}(A)$ with endpoints in $h^{-1}(A)$, and that ζ is a subarc of $h^{-1}(A)$ with the same endpoints. If $\tau \cup \zeta$ forms an inessential simple closed curve, then ζ is a subarc of $h^{-1}(A_i)$ and it only meets A in its endpoints.

A graph $G(h)$ is embedded in M as follows: (i) The vertices of $G(h)$ are obtained by choosing a point in the interior of each critical region for h, a fixed point if one exists; and (ii) as each turn or crossing meets the boundary of exactly two critical regions, we connect the corresponding vertices of $G(h)$ with an arc that intersects the turn or crossing in exactly one point, is contained in the two critical regions, and is disjoint from all other edges. Note that vertices of valence 3 or more correspond to exceptional critical regions.

Let λ be a nonempty maximal subgraph of $G(h)$ defined so that each edge of λ meets a turn, and for each interior vertex the corresponding critical region is not an exceptional one. If λ is a simple closed curve, then apply the first half of the proof of [K, Lemma 4.5] to pull the turns meeting λ off of A without changing the indices of the critical regions. This does not increase the number of fixed points nor does it change the isotopy type (rel ∂M) of the collection of curves. Thus, we may assume that each such λ is an arc.

Let K_0 and K_1 denote the critical regions corresponding to the endpoints of λ. By the maximality of λ and condition (*), all of the turns of $h^{-1}(A)$ meeting λ can be pulled taut without crossing over the endpoints of λ. Let $h': M \to M$ be the map obtained by pulling these turns taut (but not over $\partial\lambda$) and let K_0', K_1' be the analogous critical regions for h' containing $\partial\lambda$. Observe that if $\text{index}(h', K_i') = 0$, then the intermediate map $\bar{h}$ obtained by pulling only the turn meeting K_i off of A has $\text{index}(\bar{h}, K_i') = 0$ and so $\#\text{Fix}(\bar{h}) \leq \#\text{Fix}(h)$. In this case, reduce $\bar{h}$ by using Lemmas 3.4–3.6 of [K] so that condition (*) is satisfied. As this involves replacing a given curve by a new curve which is isotopic rel ∂M to the original, plus an inessential simple closed curve, the isotopy type of the collection of curves is unchanged. Hence, we may assume that h has the property that index$(h', K_i') \neq 0$ for $i = 0, 1$. Thus, $\partial\lambda \subset \mathrm{Fix}(h')$. Reverse this process to get that the arc λ is a simple merging arc for h', so $h' \xrightarrow{\lambda} h$.

Suppose α is another such subgraph and suppose further that $\partial\alpha \cap \partial\lambda \neq \emptyset$. If K_0 is a critical region containing a point in common and if h'' is obtained by pulling both collections of turns taut, it is possible that index$(h'', K_0'') = 0$ while a nonzero index occurs as in the preceding paragraph. In this case, by the maximality of α and λ, K_0 must be an exceptional critical region. Thus, $\partial\alpha \cup \partial\lambda \subset \mathrm{Fix}(\tilde{h}'')$, α and λ are disjoint merging arcs, and $h'' \xrightarrow{(\alpha,\lambda)} h$. By using all maximal subgraphs of $G(h)$ as above, there is a map $\hat{g}: M \to M$ and a simple merging sequence Λ, so that $\hat{g} \xrightarrow{\Lambda} h$ and $\hat{g}^{-1}(A)$ does not contain any turns.

Let Γ be a taut collection of curves isotopic (rel $\partial M \cup \Lambda$) to $\hat{g}^{-1}(A)$. Let $g \equiv g_\Gamma$ and let f be obtained from g by merging along Λ. Then $f^{-1}(A)$ and $h^{-1}(A)$ differ only in the order and labelings of turns and crossings and so, by definition, we can choose $f_{\min} = h$. This proves the theorem. □

With regard to the actual computation of the minimal number of fixed points, suppose that a taut map g homotopic to f_ϕ is given. First, because $\Gamma \cap W$ was not specified in the above proof, one is free to arrange $\Gamma \cap W$ so as to optimize the number of critical regions for g having index 0. Second, note that each merging arc $\lambda \in \Lambda$ is not only simple, but must have the additional property that $g(\lambda)$ meets the same components of A in the same order as λ. As a result, given the fixed point data of g, it is straightforward to determine if two fixed points may be joined by such an arc.

Finally, after obtaining Λ and $g_\Gamma \xrightarrow{\Lambda} f$ so that each turn and crossing of $f^{-1}(A)$ is simple, $f_{\min}$ is obtained as follows. Let $G(f)$ be as in the proof of Theorem 3.1. Suppose C_1, C_2 are critical regions, each with nonzero index, which can be joined by an arc γ in $G(f)$ such that each critical region meeting the interior of γ has index zero. If the orders and labels of the turns and crossings meeting these critical regions can be altered so that at most one of the regions has nonzero index, then do so. This reduces the fixed point count. Repeat for all such pairs of critical regions to obtain $f_{\min}$.

Of course, the problem with Theorem 3.1 is that all possible taut maps g must be checked. In order to reduce the number of taut maps at the expense of introducing nonsimple merging arcs, the following theorem is given.

THEOREM 3.2. *Fix $j \in \{1, \dots, k\}$. Suppose ϕ has the property that, given any taut simple closed curve l with $l \in \ker\phi$, $l \cap H_j$ has at most one component. Then there exist $h: M \to M$ homotopic to f_ϕ* rel ∂M *and a disjoint merging sequence*

$$h \xrightarrow{\Lambda_1} h' \xrightarrow{\Lambda_2} h'' \xrightarrow{\Lambda_3} f$$

such that

(i) $h^{-1}(A)\cap(A_{2j-1}\cup A_{2j})=\emptyset$;
(ii) Λ_1, Λ_3 *are simple with* $(\Lambda_1\cup\Lambda_3)\cap H_j=\emptyset$;
(iii) *for each* $\lambda\in\Lambda_2$, $\lambda\cap H_j\neq\emptyset$; *and*
(iv) $\#\mathrm{Fix}(f_{\min})=MF[f_\phi]$.

Proof. If two curves τ_1, τ_2 in $f^{-1}(A)$ meet A_k at points p_1, p_2 so that the image under f of the arc δ of A_k joining p_1 to p_2 is homotopic rel endpoints to a subarc of A, we say that τ_1, τ_2 *are paired at* A_k. Furthermore, if no other curves of $h^{-1}(A)$ intersect δ then we can alter the collection by *surgering the curves* τ_1, τ_2 *along* δ. Replacing τ_1, τ_2 with τ_1', τ_2' which no longer intersect δ, $\tau_i'=\tau_i$ outside a prescribed neighborhood of δ.

Consider the factorization $g\xrightarrow{\Lambda}f$ given in Theorem 3.1. Choose $\lambda\in\Lambda$ such that $\lambda\cap H_j\neq\emptyset$. First, by repeatedly surgering curves of $f^{-1}(A)$ that are on opposite sides of λ in H_j and are paired at A_{2j-1} (or A_{2j}), by pulling curves taut we can define a new map $\hat{f}$. We say $\hat{f}$ *is obtained from* f *by surgering* λ *along* H_j. Notice that this is the opposite of $\hat{f}\xrightarrow{\lambda}f$ except that either of the following may happen when pulling a curve taut: (I) it passes over another merging arc $\lambda'\in\Lambda$; (II) it passes over an endpoint of another $\lambda'\in\Lambda$; or (III) it passes over an endpoint of λ.

In the case of (II) or (III), when pulling a curve taut we simply leave an inessential closed curve that encloses the endpoint and a segment of λ. In (III) this is the opposite of a nonsimple merge along λ with $\theta=3$. In (II), λ' will be a nonsimple merging arc with $\theta=1$.

Consider a pair λ, λ' as in (I) for which both λ and λ' intersect H_j. Let x denote the endpoint of λ' that is passed over first when pulling curves taut. Then there exists a disk contained in D which meets λ' in an arc containing x and meets λ in an interior arc but meets no other arc in Λ. Let $I(\lambda, \lambda')$ be an arc in this disk so that its interior misses $\lambda\cup\lambda'$ and has endpoints x and a point on λ. Let Γ be the graph in M consisting of all possible λ, λ', with $I(\lambda, \lambda')$ as above. By simultaneously surgering all the merging arcs in Γ along H_j, without worrying about endpoints, the resulting taut curves will only intersect Γ along a free edge. Thus, for any loop l in Γ, $[l]\in\ker\phi$. But, by construction, each edge of Γ must traverse H_j, so by hypothesis Γ is simply connected.

We are now ready to give the factorization for Theorem 3.2. First, $\Lambda_3=\{\lambda' \mid \lambda, \lambda'$ satisfy (I) or (II) and $\lambda'\cap H_j=\emptyset\}$. For these turns we only remove the turns corresponding to arcs in Λ_3 as in Theorem 3.1. Let h'' be the corresponding map so that $h''\xrightarrow{\Lambda_3}f$. The arcs in Λ_2 are those from Λ that meet H_j. The order of the surgeries is determined by Γ, which being simply connected means there exists an arc $\tau\in\Lambda$ which appears as a λ' in (I) but not as a λ. Surger τ along H_j first, then continue using the graph $\Gamma-\tau$. For those in Λ_2 that are not contained in Γ, any order may be used. As a result, we obtain a map h' and a merging sequence $h'\xrightarrow{\Lambda_2}h''$, where the order in Λ_2 is the opposite of the order of surgering. Finally, Λ_1 consists of all other arcs in Λ, with only the turns being removed and h defined accordingly. □

The additional hypothesis imposed on the endomorphism ϕ in Theorem 3.2 is quite restrictive. All examples indicate that surgering along H_j should be possible, independent of any conditions on ker ϕ. In fact, by using a careful choice of surgeries, a factorization of the form $f_\phi \xrightarrow{\Lambda} f$, where $\#\text{Fix}(f) = MF[f]$, seems reasonable to expect. This is still not ideal, as nonsimple merges are much more difficult to work with than simple ones—especially, it seems, those having one of $\theta_i = 1$. This is because, in the proof of Theorem 3.2, when (II) occurs the order of merging is (λ', λ) where λ' is nonsimple, $\theta_1 = 1$. The corresponding inessential curves $\mathcal{L}$ created for λ' must meet λ in a subarc. The problem is that, when merging λ', the arc λ has not yet been determined.

The ideal situation would be a merging sequence $f_\phi \xrightarrow{\Lambda} f$ where Λ is simple. This would be a step in the direction of showing that an algorithm exists for computing $MF[f]$. But the following example shows that this is not always possible.

EXAMPLE 3.3. Let M be a disk with three holes defined, using the notation of this paper, so that w_1, w_2, w_3, and $w_1 w_2 w_3$ are homotopic to boundary curves. Let ϕ be the endomorphism defined by

$$w_1 \mapsto w_2^{-1} w_1 w_3 w_2^{-1},$$
$$w_2 \mapsto w_3 w_2^{-1},$$
$$w_3 \mapsto 1.$$

Then $\text{Fix}(f_\phi) = \{x_0, x_1, x_2\}$ with $x_1 \in H_1$ and $x_2 \in H_2$. It is easy to check that there are no simple merging arcs.

A nonsimple merging arc can be constructed in the following manner. Choose a point $y_0 \in D$ and let α be an arc from x_1 to y_0 which traverses A in the order A_2, A_5, then A_6, thus looping around w_3. Add two inessential simple closed curves around y_0, each meeting α in three points, with μ-values 3 and 4 so that $\tilde{f}_\phi(\alpha) \sim \alpha$. Then, by merging along α, we obtain a map g. It can be shown that $\text{Fix}(g) = \{x_0, y_1, x_2\}$ with $y_1 \in D$. Also, there exists a merging arc β with $\partial\beta = \{y_1, x_2\}$, $\beta \cap \alpha = \emptyset$, and $\beta \cap A \subset A_4$. Then $g \xrightarrow{\beta} f$ yields $\text{Fix}(f) = \{x_0\}$.

REMARK 3.4. If ϕ is replaced by ϕ', where $\phi'(w_i) = w_2^{-1}\phi(w_i)w_2$, then $f_{\phi'} \sim f_\phi$ and there exists a simple merge to a map having one fixed point. But by taking two copies of the example above and a boundary connected sum (to obtain a disk with six holes), there is then no way to replace $\phi \# \phi$ by conjugating so as to avoid nonsimple merging arcs.

4. Applications of the Factorization Theorems

EXAMPLE 4.1 (Weier's example). Let $M = D \cup H_1 \cup H_2 \cup H_3 \cup H_4$, where the handles are joined so that $A_1, \ldots, A_8$ are in order on ∂D. Thus M is homeomorphic to a disk with four holes removed. Let ϕ denote the endomorphism of $\pi_1(M, x_0)$ defined by

$$\begin{aligned} w_1 &\mapsto w_1, \\ w_2 &\mapsto w_4^{-1} w_2 w_4, \\ w_3 &\mapsto w_3, \\ w_4 &\mapsto 1. \end{aligned}$$

The taut map

$$f_\phi : M \to M$$

has $\mathrm{Fix}(f_\phi) = \{x_0, x_1, x_2, x_3\}$ with indices $\{1, -1, -1, -1\}$, respectively. An easy calculation shows that there exists, for each j, taut arcs α_j such that $f_\phi(\alpha_j) \sim \alpha_j$ rel $\partial\alpha_j$ and $\partial\alpha_j = \{x_0, x_j\}$. Hence $N(f_\phi) = 1$. Weier [We] claimed without proof that $MF[f_\phi] = 2$. Recently, Tong [T] was able to establish this claim. We present here an alternative proof based on the factorization Theorem 3.2.

Let $h \xrightarrow{\Lambda_1} h' \xrightarrow{\Lambda_2} \tilde{h} \xrightarrow{\Lambda_3} f$ be a factorization as given in Theorem 3.2. In order to calculate #Fix(f) we need some additional information about Λ_2. First consider the factorization $g \xrightarrow{\Lambda} f$ of Theorem 3.1. The number of merging arcs in Λ that traverse a given H_i is the same as the number of components of $M \backslash g^{-1}(A)$ that are bounded by paired curves in $g^{-1}(A)$ and intersect H_i. In the present example there is a nice way to describe, up to isotopy, the collection of curves in $g^{-1}(A)$ as being obtained from the collection $f_\phi^{-1}(A)$. Outside of joining together the paired curves in $H_2 \cap f_\phi^{-1}(A)$, the only way to change $f_\phi^{-1}(A)$ is to add a collection of simple closed curves each isotopic to w_4 and successively join them to other curves, each time joining curves along some fixed taut arc. If this arc traverses H_2 then Λ has one merging arc meeting H_2; otherwise there are none. Thus, by the proof of Theorem 3.2, we can assume that either Λ_2 is empty or consists of a single arc λ. If λ is not simple, then one end loops around w_4, $\theta = 3$, and the other end is contained in Fix(h'). Also, since there are only three choices (up to isotopy) for $h^{-1}(A)$ and in each case $\mathrm{Fix}(h) = \mathrm{Fix}(f_\phi)$, we can assume that $h = f_\phi$.

Let σ_i denote either of the two arcs in w_i going from x_i to x_0. Let α_{ij} denote a merging arc for f_ϕ with $\partial\alpha_{ij} = \{x_i, x_j\}$. For $\alpha_{ij} \in \Lambda_1$ the only possibilities are $\alpha_{01} = \sigma_1$, $\alpha_{01} = \sigma_3$, and $\alpha_{13} = \sigma_1\sigma_3^{-1}$, and Λ_1 is one of $\{\alpha_{01}\}$, $\{\alpha_{03}\}$, $\{\alpha_{13}\}$, or $\{\alpha_{01}, \alpha_{03}\}$. If $\Lambda_2 = \emptyset$ then without loss $\Lambda_3 = \emptyset$ as well, and it is easy to check that in each case #Fix(f) ≥ 2.

If $\Lambda_2 \neq \emptyset$ then $\Lambda_2 = \{\lambda\}$, as mentioned earlier. Since $\Lambda_1 \cap H_2 = \emptyset$, any merging arc starting at x_2 must pass through H_4 and so λ is the only possible choice. If $x_2 \notin \partial\lambda$, then $x_2 \in \mathrm{Fix}(f)$ and by an index calculation #Fix(f) ≥ 2. For the following assume that $x_2 \in \partial\lambda$. Then, up to isotopy, λ must be one of the following: (i) $\sigma_2 w_4 \sigma_1^{-1} \equiv \alpha_{12}$, $\partial\lambda = \{x_1, x_2\}$; (ii) $\sigma_2 w_4 \equiv \alpha_{02}$, $\partial\lambda = \{x_0, x_2\}$; or

(iii) λ is not simple, isotopic to $\sigma_2 w_4$ but $x_0 \notin \partial\lambda$. In this last case we denote $\partial\lambda = \{x_2, x_4\}$ and denote λ as α_{24}. Also in (iii), since $h'(\lambda) \sim \lambda$, a straightforward calculation shows that no inessential simple closed curves are added around x_4.

There are three possibilities, depending on Λ_1.

Case 1: $\Lambda_1 = \{\alpha_{13}\}$ or $\{\alpha_{01}, \alpha_{03}\}$. Here $\mathrm{Fix}(h') = \{y_0, x_2\}$, where y_0 is the result of merging along Λ_1. But $\lambda \cap \Lambda_1 \neq \emptyset$, so $\Lambda_2 = \emptyset$, $\Lambda_3 = \emptyset$, and $\#\mathrm{Fix}(f) = 2$.

Case 2: $\Lambda_1 = \{\alpha_{03}\}$ with $\mathrm{Fix}(h') = \{x_1, x_2\}$. In this case λ is either α_{12} or α_{24}, where $\lambda \cap \alpha_{03} = \emptyset$. As x_4 and x_1 can be joined by a merging arc in $D \cup H$, we can replace α_{24} by α_{12} by doing this additional merge. So we can assume that $\lambda = \alpha_{12}$. See Figure 5a for an illustration of α_{12} and α_{03}. Let L, R be arcs

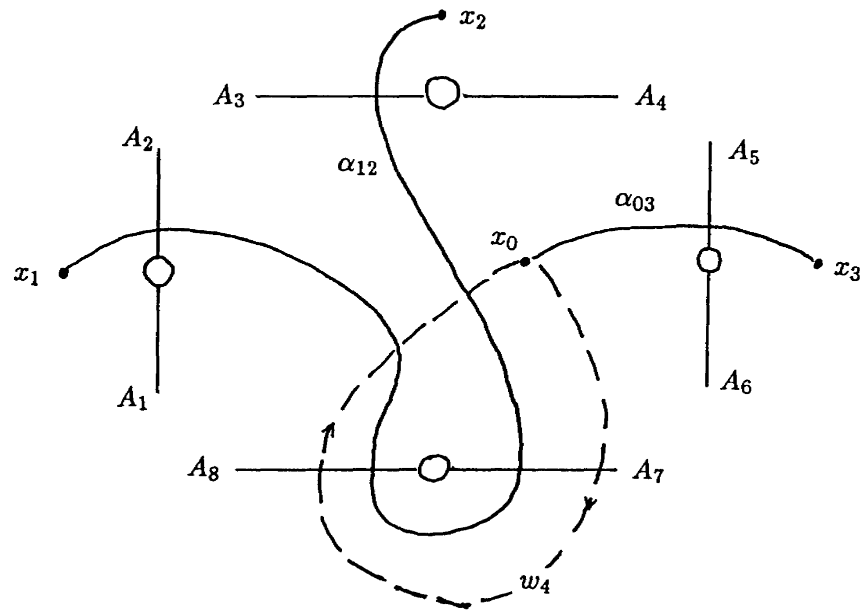

Figure 5a

in D from A_3 to A_7 which enclose one component of α_{12} and miss $\alpha_{12} \cup \alpha_{03}$ (see Figure 5b). Without loss of generality we can assume that $g(L \cup R) = x_0$. After merging along α_{12} to the map $\tilde{h}$ also satisfying $\tilde{h}(L \cup R) = x_0$, the two regions in D bounded by L, R and $\tilde{h}^{-1}(A_8)$ each contain a fixed point. Denote these by l and r, respectively. Then $\mathrm{Fix}(\tilde{h}) = \{l, r, z\}$, with respective indices $\{1, -1, -2\}$, where z is obtained by merging x_1 and x_2. By construction, l and z are in the same critical region and can be combined into a single fixed point l'. Finally, as l' and r can only be merged by an arc traversing α_{12}, $\Lambda_3 = \emptyset$ and thus $\#\mathrm{Fix}(f) = 2$.

Case 3: $\Lambda_1 = \{\alpha_{01}\}$ with $\mathrm{Fix}(h') = \{x_2, x_3\}$. Then $\lambda = \alpha_{24}$, and proceeding exactly as in case 2 we obtain $\#\mathrm{Fix}(f) \geq 2$.

This establishes that $MF[f_\phi] = 2$.

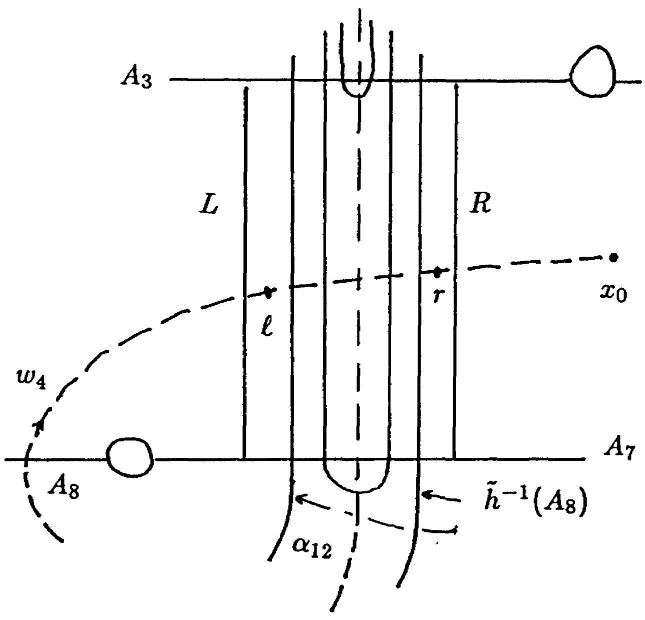

Figure 5b

EXAMPLE 4.2. In [K] an algorithm was given for computing $MF[f]$ for any self-map of the disk with two holes, P. Here we give a brief interpretation in terms of merging arcs and then a natural extension of that result.

Write $P = H_1 \cup H_2$ with $\pi_1(P, x_0) = \langle w_1, w_2 \rangle$. By composing with a suitable inner automorphism of $\pi_1(P, x_0)$ and conjugating by a homeomorphism, arrange that for $\phi: \pi_1(P, x_0) \to \pi_1(P, x_0)$ only one of $\phi(w_i)\phi(w_i)$, $\phi(w_1^{\pm 1})\phi(w_2^{\mp 1})$ contains a reduction. Consequently, there is only one way to choose merging arcs for f_ϕ; namely, by arcs parallel to the reduction. The only exception is when $\phi(w_2) = 1$ and $\phi(w_1)$ is cyclically reduced. Then any merging arc is isotopic to $\sigma_1(w_2w_1)^k w_2(w_2w_1)^{-k}\sigma_2$ $(k \geq 0)$ and σ_i is an arc from a fixed point to x_0 along w_1, or $\tau_1(w_1w_2)^k w_2(w_1w_2)^{-k}\tau_2$ $(k \leq 0)$ and τ_i is an arc from a fixed point to x_0 along w_1^{-1}. This gives two different types of merging arcs which do intersect. Express the set of merging arcs for f_ϕ as $\{\alpha_1, \alpha_2, \alpha_3, \ldots\}$ in order of increasing length, or (equivalently) by increasing cancellation occurring in $f(\alpha_i)$. There are two such sequences in the exceptional case. Let α_0 denote the set of all merging arcs for f_ϕ that have one endpoint x_0 and that meet only one component of A. Observe that if this collection is not finite then, as $i \to \infty$, α_i spirals around one or two of the boundary components of P. Let f_i be the map obtained from f_ϕ by merging along α_i.

LEMMA 4.3. *Suppose* $i < j < k$ *and* $\partial\alpha_i = \partial\alpha_j = \partial\alpha_k$. *Then* $\#\text{Fix}(f_j) = \#\text{Fix}(f_k)$.

Proof. Since $\partial\alpha_i = \partial\alpha_j$, α_j is isotopic to $S_1^{l_1}\alpha_i S_2^{l_2}$ where S_1 and S_2 are simple closed curves isotopic to boundary components.

Similarly, $\alpha_k = S_1^{m_1}\alpha_j S_2^{m_2}$. Up to isotopy, curves in $f_j^{-1}(A)$ can be obtained from curves in $f_i^{-1}(A)$ by twisting around the boundary components of P corresponding to S_1 and S_2, $-l_1$ and l_2 times (respectively); likewise for $f_k^{-1}(A)$. Since α_j itself contains full spirals, the difference in critical regions for f_j and f_k are only those created by twisting the boundary components. By Lemmas 5.1 and 5.2 in [K], these have zero index. □

LEMMA 4.4. *There exists a merging sequence* $f_\phi \xrightarrow{(\alpha_i,\alpha_0)} f$, $i \geq 1$, *such that* $\#\text{Fix}(f) = MF[f_\phi]$.

Proof. In [K, §6] the set of curves, up to isotopy, for $f^{-1}(A)$ is described explicitly. Excepting the case when a simple turn is formed at A_i, there is only one sequence of turns in the merging sequence $g \xrightarrow{\Lambda} f$ of Theorem 3.1. Thus $\Lambda = (\beta, \alpha_0)$. By surgering all pairs of curves parallel to β, as in the proof of Theorem 3.2, we obtain a factorization $h \xrightarrow{(\beta,\alpha_0)} f$, where $h^{-1}(A) \cap A = \emptyset$. Hence, $h = f_\phi$ and $\beta = \alpha_i$ for some $i \geq 1$. □

Combining Lemmas 4.3 and 4.4 yields an algorithm for $MF[f_\phi]$.

REMARK 4.5. Conjugation by a homeomorphism of P was not really necessary. It only served to convert the situation $\phi'(C) = 1$, C isotopic to a boundary curve, to the exceptional case $\phi(w_2) = 1$. So, without bothering to conjugate, the above argument goes through using the corresponding pair of sequences of merging arcs.

If M is any surface obtained by attaching two 1-handles, the above analysis holds. The main observation is that an infinite sequence of merging arcs must eventually spiral around boundary curves and the proofs of the above lemmas are still valid. Consequently, we have the following theorem.

THEOREM 4.6. *If M is a compact surface with* $\text{rank}(H_1(M)) \leq 2$, *then there is an algorithm for computing* $MF[f]$.

As a final application of the factorization theorems in Section 3, we have the following proposition. This will be the main step in the proof of Theorem 1.1.

PROPOSITION 4.7. *Suppose ϕ is an endomorphism of $\pi_1(M, x_0)$ such that, for each i, $\phi(w_i)$ does not contain either w_1 or w_1^{-1}. Then there exists a map f homotopic to f_ϕ* rel ∂M *such that* $\#\text{Fix}(f) = MF[f_\phi]$ *and* $f(M) \cap H_1 = \emptyset$.

Proof. Consider the factorization $g \xrightarrow{\Lambda} f$ given by Theorem 3.1. Without loss of generality we may assume that f is minimal, in the sense that $f^{-1}(A) \cap A$ cannot be reduced using the lemmas of [K, §3]. The proof is completed by showing that $g^{-1}(A_1 \cup A_2) = \emptyset$ and that $\lambda \cap H_1 = \emptyset$ for each $\lambda \in \Lambda$. The latter follows as $g(\lambda)$ is homotopic (rel endpoints) to a curve missing H_1. Thus λ, being taut and homotopic to $g(\lambda)$, must be disjoint from H_1.

Now suppose that α is a curve in $g^{-1}(A_1)$. By hypothesis, $f_\phi^{-1}(A_1) = \emptyset$. Since $g \sim f_\phi$ rel ∂M, it must be that α is a simple closed curve. Moreover, there exists a surface R contained in M such that $g(\partial R) \subset A_1$ and $g(R)$ is null homotopic. As a result, there is a surface $Q \subset R$ such that $g(\partial Q) \subset A_k$ for some k and $g(\text{int}\, Q) \cap A = \emptyset$. After merging along Λ, there is a surface $\tilde{Q}$ in M, homeomorphic to Q, with $f(\partial \tilde{Q}) \subset A_k$ and $f(\text{int}\, \tilde{Q}) \cap A = \emptyset$. But, for $j \neq k$, if $\tilde{Q} \cap A_j \neq \emptyset$ one could reduce $f^{-1}(A) \cap A$. Thus, by the minimality of f, $\tilde{Q} \cap A \subset A_k$ which implies that $\tilde{Q}$ is a disk. But ∂Q is taut, so Q cannot be a disk. Hence no such α can exist. Similarly, $g^{-1}(A_2) = \emptyset$. □

Proof of Theorem 1.1. Without loss of generality, we assume that P is embedded in the interior of M. Let K denote the closure of a component of $M \setminus P$. Since the 1-chain $[\partial K]$ bounds in K, and since i_* is a monomorphism, it follows that K must meet both of ∂P and ∂M. Now if K is not an annulus then 1-handles can be successively added to ∂P in K, forming a surface $\hat{P}$, so that the closure of $K \setminus \hat{P}$ is homeomorphic to $(\partial M \cap K) \times I$. Consequently, M is homeomorphic to the union of P and a finite number of 1-handles (embedded in M) attached to ∂P. Thus, it suffices to consider the case $M = P \cup H_1$, with w_1 being a generator of $\pi_1(M)$ corresponding to the 1-handle H_1. Now apply Proposition 4.7 to obtain the result. □

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