Two Problems in Multidimensional Prediction Theory

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Introduction

This paper deals with multidimensional generalizations of the following famous theorem of Szegö in prediction theory: If μ is a positive Borel measure on the circle T identified with the interval $[0,1)$ and μ has Lebesgue decomposition $\mu = w + \mu_s$, where $w \in L^1(T)$ and μ_s is a singular measure, then

$$\inf_P \int_T |1+P(\theta)|^2 \, d\mu(\theta) = \exp\left[\int_T \log w(\theta)\, d\theta\right],$$

where we use the convention that the expression on the right-hand side of the preceding equality is zero if $\log w$ is not integrable, and the infimum is taken over all trigonometric polynomials of the form

$$P(\theta) = \sum_{k=1}^{m} a_k \exp(2\pi i \theta k), \quad a_k \in \mathbf{C}, \ m \ge 1.$$

This result was first obtained by Szegö (see [2]) in the case where $\mu_s \equiv 0$; the general case is due to Kolmogorov and Krein (see [1, p. 256]). Helson and Lowdenslager [3] considered a similar multidimensional prediction theory problem. Given a positive Borel measure μ on T^n, they showed that

$$\inf_P \int_{T^n} |1+P(\theta)|^2 \, d\mu(\theta) = \exp\left[\int_{T^n} \log w(\theta)\, d\theta\right],$$

with the same convention as before and where P ranges over all trigonometric polynomials with Fourier coefficients supported in a half-space of lattice points S^+ contained in $\mathbf{Z}^n$. The definition of a half-space of lattice points is that of a set S^+ having the properties that $0 \notin S^+$; that, for $k \neq 0$, $k \in S^+$ is equivalent to $-k \notin S^+$; and that $k_1 + k_2 \in S^+$ whenever $k_1, k_2 \in S^+$. An example of such a set can be constructed in the following way. Fix a vector $x_0 \in \mathbf{R}^n$, with $x_0 \neq 0$, such that $\{k \in \mathbf{Z}^n, \langle k, x_0 \rangle = 0\} = \{0\}$. Then the set $S^+ = \{k \in \mathbf{Z}^n, \langle x_0, k \rangle > 0\}$ satisfies all the properties mentioned above. Of course, not every "usual" half-space of lattice points has those properties, and we will refer to those which do as half-spaces of lattice points in the sense of

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Helson and Lowdenslager. Let us mention that prediction problems in several dimensions corresponding to sets other than half-spaces of lattice points have been studied (see e.g. [4], where the case of lattice points in the third quadrant is considered when $n=2$).

The main objective of this paper is to investigate the following two multidimensional prediction theory problems corresponding to half-spaces of lattice points which do not satisfy the conditions of the Helson–Lowdenslager theorem. Given a vector $x_0 \in \mathbf{R}^n$, $x_0 \neq 0$, such that $\{k \in \mathbf{Z}^n, \langle x_0, k\rangle = 0\} \neq \{0\}$, we will be interested in finding an explicit expression in terms of μ for

$$\inf_P \int_{T^n} |1+P(\theta)|^2\, d\mu(\theta),$$

where P ranges over all trigonometric polynomials on T^n whose Fourier coefficients are supported either in the set $\{k \in \mathbf{Z}^n, \langle x_0, k\rangle \geq 0, k \neq 0\}$ or the set $\{k \in \mathbf{Z}^n, \langle x_0, k\rangle > 0\}$. The main technique used in obtaining the results is an appropriate change of variable on T^n which allows us to reduce the problem to a lower-dimensional one where the results of Helson and Lowdenslager, or simply Szegö's theorem, can be applied.

In the first section of this paper we establish some general facts about the mapping Φ_K which sends a positive measure μ on T^n to the number $\Phi_K(\mu)$ defined by

$$\Phi_K(\mu) = \inf_{P \in \mathcal{P}(K\setminus\{0\})} \int_{T^n} |1+P(\theta)|^2\, d\mu(\theta).$$

Here K is an arbitrary subset of $\mathbf{Z}^n$ and $\mathcal{P}(K\setminus\{0\})$ denotes the set of trigonometric polynomials with Fourier coefficients supported in $K\setminus\{0\}$. In particular, it is shown that this mapping behaves well with respect to "approximate identities" (Lemma 1.1). In Section 2, the prediction problem for $K = S_1^+ = \{k \in \mathbf{Z}^n, \langle x_0, k\rangle \geq 0\}$ is considered and the main result is Theorem 2.5. It is interesting to notice that, as in the case of half-spaces in the sense of Helson and Lowdenslager, the answer to the prediction problem for S_1^+ is independent of the singular part of the measure μ considered. This is not the case, however, for the half-space of lattice points

$$K = S_2^+ = \{k \in \mathbf{Z}^n, \langle x_0, k\rangle > 0\} \cup \{0\},$$

as shown by simple examples. We consider the prediction problem for S_2^+ in Section 3. The answer to this problem in the case of a measure absolutely continuous with respect to the Lebesgue measure is contained in Corollary 3.5. The general case is dealt with in Theorem 3.6.

0. Notation

We will identify the n-dimensional torus T^n with the product of n copies of the interval $[0,1)$. Thus, for us, $T^n = [0,1)^{(n)}$ and the Haar measure on T^n is identified with the usual n-dimensional Lebesgue measure $d\theta$ on $[0,1)^{(n)}$. We will denote by $M(T^n)$ the space of complex-valued Borel measures on T^n

which, as is well known, can be identified with the dual space of $C(T^n)$, the space of complex-valued continuous functions on T^n endowed with the sup-norm topology. $M^+(T^n)$ is the subset of $M(T^n)$ consisting of all positive measures in $M(T^n)$. If $1 \le p < \infty$ and $\mu \in M^+(T^n)$, we can define the Banach space L^p_μ of all complex-valued, μ-measurable functions f, defined μ-almost everywhere such that

$$\|f\|^p_{p,\mu} = \int_{T^n} |f(\theta)|^p \, d\mu(\theta) < \infty.$$

If $\mu \in M(T^n)$ and $k \in \mathbf{Z}^n$, we define the kth Fourier coefficients of μ, $\hat{\mu}(k)$, by

$$\hat{\mu}(k) = \int_{T^n} \exp(-2\pi i \langle k, \theta \rangle) \, d\mu(\theta);$$

if $K \subset \mathbf{Z}^n$, we will denote by $\mathcal{P}(K)$ the set of all trigonometric polynomials P on T^n satisfying $\hat{P}(k) = 0 \ \forall k \notin K$. Finally, if f is a function on T^n, we define the function $\check{f}$ by $\check{f}(\theta) = f(-\theta)$ for $\theta \in T^n$.

1. A General Prediction Problem

If K is any subset of $\mathbf{Z}^n$ with $0 \in K$ and $\mu \in M^+(T^n)$, we can define the expression

$$\Phi_K(\mu) = \inf_{P \in \mathcal{P}(K \setminus \{0\})} \int_{T^n} |1 + P(\theta)|^2 \, d\mu(\theta).$$

In this section we will be interested in establishing some general facts about the mapping $\Phi_K : M^+(T^n) \to \mathbf{R}^+$. It is easily seen that $\Phi_K(\mu) > 0$ is equivalent to the continuity of the functional that sends a polynomial in $\mathcal{P}(K)$ to its mean value; that is, equivalent to the existence of a constant $C > 0$ such that

$$\forall P \in \mathcal{P}(K), \quad \left| \int_{T^n} P(\theta) \, d\theta \right| \le C \left[\int_{T^n} |P(\theta)|^2 \, d\mu(\theta) \right]^{1/2}.$$

In that case, $C = 1/\sqrt{\Phi_K(\mu)}$. Simple examples show that the mapping $\mu \to \Phi_K(\mu)$ is not continuous in general, for arbitrary K, even in the norm topology of $M(T^n)$. However, we will see that this mapping behaves well as far as approximate identities are concerned. Let us recall that an approximate identity in $C(T^n)$ is a family of functions $\{\varphi_\epsilon\}_{\epsilon > 0}$ in $C(T^n)$, indexed by $\epsilon > 0$, having the property that $\int_{T^n} \varphi_\epsilon(\theta) \, d\theta = 1$ for every $\epsilon > 0$, $\varphi_\epsilon \ge 0$, and such that

(1.1) $$\forall f \in C(T^n), \quad \lim_{\epsilon \to 0^+} \int_{T^n} f(\theta) \varphi_\epsilon(\theta) \, d\theta = f(0).$$

We have the following lemma.

LEMMA 1.1. *Suppose that $K \subset \mathbf{Z}^n$ with $0 \in K$, and let $\{\varphi_\epsilon\}_{\epsilon > 0}$ be any approximate identity. Then, if $\mu \in M^+(T^n)$, we have that $\Phi_K(\mu * \varphi_\epsilon) \ge \Phi_K(\mu)$ for every $\epsilon > 0$, and furthermore that $\Phi_K(\mu * \varphi_\epsilon)$ converges to $\Phi_K(\mu)$ as $\epsilon \to 0^+$.*

Proof. If $P \in \mathcal{P}(K\setminus\{0\})$, we compute

$$\int_{T^n} |1+P(\theta)|^2 (\mu * \varphi_\epsilon)(\theta)\, d\theta = \int_{T^n} \left[\int_{T^n} |1+P(\theta)|^2\, d\mu(\theta-\tau) \right] \varphi_\epsilon(\tau)\, d\tau$$
$$= \int_{T^n} \left[\int_{T^n} |1+P(\theta+\tau)|^2\, d\mu(\theta) \right] \varphi_\epsilon(\tau)\, d\tau$$
$$\geq \int_{T^n} \Phi_K(\mu) \varphi_\epsilon(\tau)\, d\tau = \Phi_K(\mu),$$

since, for every fixed τ, $P(\cdot+\tau) \in \mathcal{P}(K\setminus\{0\})$. We thus obtain the first statement of the lemma by taking the infimum over $P \in \mathcal{P}(K\setminus\{0\})$ of the left-hand side of the previous equality. Let us now prove the second statement. If $\eta > 0$ is fixed, choose $P_0 \in \mathcal{P}(K\setminus\{0\})$ such that

$$\int_{T^n} |1+P_0(\theta)|^2\, d\mu(\theta) \leq \Phi_K(\mu) + \eta.$$

Now, since $\{\check{\varphi}_\epsilon\}_{\epsilon>0}$ is also an approximate identity, we have

$$\int_{T^n} |1+P_0(\theta)|^2 (\mu * \varphi_\epsilon)(\theta)\, d\theta = \int_{T^n} [|1+P_0|^2 * \check{\varphi}_\epsilon](\theta)\, d\mu(\theta)$$
$$\to \int_{T^n} |1+P_0(\theta)|^2\, d\mu(\theta), \quad \epsilon \to 0^+,$$

and thus we can find $\epsilon_0 > 0$ such that, if $0 < \epsilon < \epsilon_0$,

$$\int_{T^n} |1+P_0(\theta)|^2 (\mu * \varphi_\epsilon)(\theta)\, d\theta \leq \int_{T^n} |1+P_0(\theta)|^2\, d\mu(\theta) + \eta.$$

Hence we obtain that $\Phi_K(\mu) \leq \Phi_K(\mu * \varphi_\epsilon) \leq \Phi_K(\mu) + 2\eta$ if $0 < \epsilon < \epsilon_0$, and, since η is arbitrary, the second assertion follows. □

COROLLARY 1.2. *If $\mu \in M^+(T^n)$ has Lebesgue decomposition $\mu = w + \mu_s$, where $w \in L^1(T^n)$ and μ_s is a singular measure, define $\Phi(\mu) = 0$ if $\log w \notin L^1(T^n)$ and $\Phi(\mu) = \exp(\int \log w(\theta)\, d\theta)$ if $\log w \in L^1(T^n)$. Then*

$$\lim_{\epsilon \to 0^+} \Phi(\mu * \varphi_\epsilon) = \Phi(\mu)$$

for every approximate identity $\{\varphi_\epsilon\}_{\epsilon>0}$.

Proof. This follows immediately, from the generalization of Szegö's theorem due to Helson and Lowdenslager [3, p. 171] and Lemma 1.1, where $K\setminus\{0\}$ is chosen to be a half-space in the sense of Helson and Lowdenslager. □

The following lemma will be useful in Section 3. We will write μ_ϵ for $\mu * \varphi_\epsilon$.

LEMMA 1.3. *Let $K \subset \mathbf{Z}^n$ with $0 \in K$ and let $\mu \in M^+(T^n)$ with $\Phi_K(\mu) > 0$. If $\{\varphi_\epsilon\}_{\epsilon>0}$ is any approximate identity, let us denote by M (resp. M_ϵ) the closure of $\mathcal{P}(K)$ in L^2_μ (resp. $L^2_{\mu_\epsilon}$), and by u (resp. u_ϵ) the unique element of M (resp. M_ϵ) satisfying*

$$\forall P \in \mathcal{P}(K), \quad \int_{T^n} \overline{P(\theta)}\, d\theta = \int_{T^n} u(\theta)\overline{P(\theta)}\, d\mu(\theta) = \int_{T^n} u_\epsilon(\theta)\overline{P(\theta)}\mu_\epsilon(\theta)\, d\theta.$$

Then $u_\epsilon \mu_\epsilon$ converges to $u\mu$, in the weak topology of $M(T^n)$, as $\epsilon \to 0^+$.*

Proof. We first remark that if $v \in L^2_{\mu_\epsilon}$, we can define $v * \check{\varphi}_\epsilon$ as an element of L^2_μ. Indeed, if $P \in \mathcal{P}(T^n)$, using Cauchy–Schwarz inequality we have that

$$\begin{aligned}\int_{T^n} |P * \check{\varphi}_\epsilon|^2(\theta)\, d\mu(\theta) &= \int_{T^n} \left| \int_{T^n} P(\tau)\varphi_\epsilon(\tau-\theta)\, d\tau \right|^2 d\mu(\theta) \\ &\leq \int_{T^n} \left[\int_{T^n} |P(\tau)|^2 \varphi_\epsilon(\tau-\theta)\, d\tau \right] d\mu(\theta) \qquad (1.2)\\ &= \int_{T^n} |P(\tau)|^2 \mu_\epsilon(\tau)\, d\tau.\end{aligned}$$

Hence, using the density of $\mathcal{P}(T^n)$ in $L^2_{\mu_\epsilon}$, the mapping $P \to P * \check{\varphi}_\epsilon$ can be extended by continuity as a mapping from $L^2_{\mu_\epsilon}$ to L^2_μ. It is also clear that if $v \in M_\epsilon$ then $v * \check{\varphi}_\epsilon \in M$, since $P * \check{\varphi}_\epsilon \in \mathcal{P}(K)$ whenever $P \in \mathcal{P}(K)$. The existence and uniqueness of u and u_ϵ, for $\epsilon > 0$, follow immediately from the Riesz representation theorem. We will first show that $u_\epsilon * \check{\varphi}_\epsilon$ converges to u in L^2_μ as $\epsilon \to 0^+$. Indeed, we have the inequalities

$$\|u_\epsilon * \check{\varphi}_\epsilon\|_{2,\mu} \leq \|u_\epsilon\|_{2,\mu_\epsilon} \leq \|u\|_{2,\mu}, \qquad (1.3)$$

the first one coming from (1.2) and the second one from Lemma 1.1. Let $\{P_j\}$ be a sequence in $\mathcal{P}(K)$ converging to u_ϵ in $L^2_{\mu_\epsilon}$. We compute

$$\begin{aligned}\int_{T^n} u(\theta)\overline{(u_\epsilon * \check{\varphi}_\epsilon)(\theta)}\, d\mu(\theta) &= \lim_{j\to\infty} \int_{T^n} u(\theta)\overline{(P_j * \check{\varphi}_\epsilon)(\theta)}\, d\mu(\theta) \\ &= \lim_{j\to\infty} \int_{T^n} \overline{(P_j * \check{\varphi}_\epsilon)(\theta)}\, d\theta = \lim_{j\to\infty} \int_{T^n} \overline{P_j(\theta)}\, d\theta \qquad (1.4)\\ &= \lim_{j\to\infty} \int_{T^n} u_\epsilon(\theta)\overline{P_j(\theta)}\mu_\epsilon(\theta)\, d\theta = \|u_\epsilon\|^2_{2,\mu_\epsilon},\end{aligned}$$

and thus it follows from Lemma 1.1 that

$$\lim_{\epsilon\to 0^+} \int_{T^n} u(\theta)\overline{(u_\epsilon * \check{\varphi}_\epsilon)(\theta)}\, d\mu(\theta) = \|u\|^2_{2,\mu}. \qquad (1.5)$$

On the other hand, applying Cauchy–Schwarz inequality to (1.4) and using (1.3), we obtain $\|u\|_{2,\mu} \geq \|u_\epsilon * \check{\varphi}_\epsilon\|_{2,\mu} \geq \|u_\epsilon\|^2_{2,\mu_\epsilon}/\|u\|_{2,\mu}$, which yields, by Lemma 1.1 again, that

$$\lim_{\epsilon\to 0^+} \|u_\epsilon * \check{\varphi}_\epsilon\|_{2,\mu} = \|u\|_{2,\mu}. \qquad (1.6)$$

It is now clear from (1.5) and (1.6) that $\|u - u_\epsilon * \check{\varphi}_\epsilon\|_{2,\mu}$ goes to zero as $\epsilon \to 0^+$, which implies in particular that $(u_\epsilon * \check{\varphi}_\epsilon)\mu$ converges to $u\mu$ in the weak* topology of $M(T^n)$ as $\epsilon \to 0^+$. Hence, we only need to show that $u_\epsilon(\mu * \varphi_\epsilon) - (u_\epsilon * \check{\varphi}_\epsilon)\mu$ converges to zero in the weak* topology of $M(T^n)$ as $\epsilon \to 0^+$. Now, using (1.3), it is easy to see that the family $\{u_\epsilon(\mu * \varphi_\epsilon) - (u_\epsilon * \check{\varphi}_\epsilon)\mu\}_{\epsilon>0}$ is bounded in $M(T^n)$. It is therefore sufficient to show that, for every $k \in \mathbf{Z}^n$,

$$\lim_{\epsilon\to 0^+}\int_{T^n} u_\epsilon(\theta)e^{2\pi i\langle k,\theta\rangle}\mu_\epsilon(\theta)\,d\theta-\int_{T^n}(u_\epsilon*\check{\varphi}_\epsilon)(\theta)e^{2\pi i\langle k,\theta\rangle}\,d\mu(\theta)=0.$$

If $\{P_j\}_{j\geq 0}$ is a sequence in $\mathcal{P}(K)$ converging to u_ϵ in $L^2_{\mu_\epsilon}$, we have

$$\left|\int_{T^n} u_\epsilon(\theta)e^{2\pi i\langle k,\theta\rangle}\mu_\epsilon(\theta)\,d\theta-\int_{T^n}(u_\epsilon*\check{\varphi}_\epsilon)(\theta)e^{2\pi i\langle k,\theta\rangle}\,d\mu(\theta)\right|$$

$$=\lim_{j\to\infty}\left|\int_{T^n}P_j(\theta)\left[\int_{T^n}\varphi_\epsilon(\theta-\tau)[e^{2\pi i\langle k,\theta\rangle}-e^{2\pi i\langle k,\tau\rangle}]\,d\mu(\tau)\right]d\theta\right|$$

$$\leq\lim_{j\to\infty}\left[\int_{T^n}|P_j(\theta)|^2\mu_\epsilon(\theta)\,d\theta\right]^{1/2}\left[\int_{T^n}d\mu(\tau)\right]^{1/2}\left[\int_{T^n}|1-e^{2\pi i\langle k,\tau\rangle}|^2\varphi_\epsilon(\tau)\,d\tau\right]^{1/2}$$

$$\leq\|u\|_{2,\mu}\left[\int_{T^n}d\mu(\tau)\right]^{1/2}\left[\int_{T^n}|1-e^{2\pi i\langle k,\tau\rangle}|^2\varphi_\epsilon(\tau)\,d\tau\right]^{1/2},$$

and this last expression converges to zero, by (1.1), as $\epsilon\to 0^+$, which proves the lemma. □

2. The Prediction Problem for S_1^+

Let $x_0\in\mathbf{R}^n$ with $x_0\neq 0$, and define $S_1^+=\{v\in\mathbf{Z}^n,\ \langle x_0,v\rangle\geq 0\}$ and $S=\{v\in\mathbf{Z}^n,\ \langle x_0,v\rangle=0\}$. In this section we will be interested in finding an explicit expression in terms of $\mu\in M^+(T^n)$ for

$$\Phi_1(\mu)=\Phi_{S_1^+}(\mu)=\inf_{P\in\mathcal{P}(S_1^+\setminus\{0\})}\int_{T^n}|1+P(\theta)|^2\,d\mu(\theta).\tag{2.1}$$

If $S=\{0\}$, this problem falls into the theory of half-spaces of lattice points considered by Helson and Lowdenslager, and so we will assume that $S\neq\{0\}$.

DEFINITION 2.1. M_1 denotes the closure of $\mathcal{P}(S_1^+)$ in L^2_μ.

In order to solve this prediction problem, we will need to introduce a change of variables on T^n depending upon the set S introduced at the beginning of this section. We need the following algebraic fact, not too difficult to verify: It is always possible to find vectors $v_1,\dots,v_n\in\mathbf{Z}^n$ and a unique integer k with $0<k<n$ such that $S=\{\sum_{j=1}^k m_jv_j,\ m_j\in\mathbf{Z}\}$ and $\mathbf{Z}^n=\{\sum_{j=1}^n m_jv_j,\ m_j\in\mathbf{Z}\}$. Consider now the mapping $B\colon[0,1)^{(n)}\to[0,1)^{(n)}\colon\theta\mapsto B(\theta)$, where the jth component of $B(\theta)$ is defined to be $\langle v_j,\theta\rangle$ (mod 1) for $j=1,\dots,n$. To simplify the notation we will write $B(\theta)=(B_1(\theta),B_2(\theta))$, where the mappings $B_1\colon T^n\to T^k$ and $B_2\colon T^n\to T^{n-k}$ are defined in the obvious way. We note that, since B is by construction the dual automorphism of an automorphism of $\mathbf{Z}^n$, it preserves the Haar measure on T^n. We will also denote by A the inverse mapping of B and consider A as a mapping from $T^k\times T^{n-k}$, again defined in the obvious way.

LEMMA 2.2. *Let $\mu\in M^+(T^n)$ and consider the measure $\mu_S\in M^+(T^n)$ defined by the identity*

(2.2) $\quad \forall \phi \in C(T^n), \quad \int_{T^n} \phi(\theta)\, d\mu_S(\theta) = \int_{T^n} \left[\int_{T^{n-k}} \phi(A(B_1(\theta), \eta))\, d\eta \right] d\mu(\theta).$

Then $(\mu_S)\hat{}(s) = \hat{\mu}(s)$ *if* $s \in S$, *and* $(\mu_S)\hat{}(s) = 0$ *if* $s \notin S$.

Proof. If $s \in \mathbf{Z}^n$, we can write $s = \sum_{j=1}^n m_j v_j$, where $m = (m_1, \ldots, m_n) \in \mathbf{Z}^n$. Now, if $(\tau, \eta) \in T^k \times T^{n-k}$, we have $\langle s, A(\tau, \eta)\rangle = \sum_{j=1}^n m_j \langle v_j, A(\tau, \eta)\rangle = \langle m, (\tau, \eta)\rangle$. Hence, writing $m = (m', m'')$, with $m' \in \mathbf{Z}^k$ and $m'' \in \mathbf{Z}^{n-k}$, it follows that

$$(\mu_S)\hat{}(s) = \int_{T^n} \left[\int_{T^{n-k}} \exp(-2\pi i \langle s, A(B_1(\theta), \eta)\rangle)\, d\eta \right] d\mu(\theta)$$
$$= \int_{T^n} \left[\int_{T^{n-k}} \exp(-2\pi i \langle m, (B_1(\theta), \eta)\rangle)\, d\eta \right] d\mu(\theta)$$
$$= \int_{T^n} \exp(-2\pi i \langle m', B_1(\theta)\rangle) \left[\int_{T^{n-k}} \exp(-2\pi i \langle m'', \eta\rangle)\, d\eta \right] d\mu(\theta).$$

Therefore, if $s \notin S$, we have $m'' \neq 0$ and $(\mu_S)\hat{}(s) = 0$. On the other hand, if $s \in S$ then we have $m'' = 0$ and thus

$$(\mu_S)\hat{}(s) = \int_{T^n} \exp(2\pi i \langle m', B_1(\theta)\rangle)\, d\mu(\theta) = \int_{T^n} \exp\left(-2\pi i \sum_{j=1}^k m_j \langle v_j, \theta\rangle\right) d\mu(\theta)$$
$$= \int_{T^n} \exp(-2\pi i \langle s, \theta\rangle)\, d\mu(\theta) = \hat{\mu}(s),$$

which proves the lemma. □

The following lemma shows that the answer of our prediction problem (2.1) for S_1^+ depends only on the absolutely continuous part of the measure μ considered.

LEMMA 2.3. *Let* $\mu \in M^+(T^n)$ *have the Lebesgue decomposition* $\mu = w + \mu_s$, *where* $\mu \in L^1(T^n)$ *and* μ_s *is singular. Then* $\Phi_1(\mu) = \Phi_1(w)$.

Proof. This is clear if $\Phi_1(\mu) = 0$. If $\Phi_1(\mu) > 0$ we obtain, using the Riesz representation theorem, the existence of a unique element $u \in M_1$ satisfying

(2.3) $\quad \forall P \in \mathcal{P}(S_1^+), \quad \int_{T^n} \overline{P(\theta)}\, d\theta = \int_{T^n} u(\theta) \overline{P(\theta)}\, d\mu(\theta).$

To prove the lemma, it is sufficient to show that $u = 0$ a.e. $(d\mu_s)$. In order to do so, let us consider a sequence $\{P_j\}_{j \geq 0}$ in $\mathcal{P}(S_1^+)$ converging to u in L^2_μ. It follows from (2.3) that $(u\overline{P_j}\mu)\hat{}(s) = 0$ if $s \in S_1^+ \setminus S$, and thus, letting $j \to \infty$, we obtain that $(|u|^2\mu)\hat{}(s) = 0$ whenever $s \in S_1^+ \setminus S$. Since $|u|^2\mu$ is a positive measure, this implies that $(|u|^2\mu)\hat{}(s) = 0$ whenever $s \notin S$. On the other hand, if $s \in S$ then $(u\overline{P_j}\mu)\hat{}(s) = 0$ except for, possibly, finitely many $s \in S$. Indeed, if $P_j(\theta) = \sum_{l \in F} a_l \exp(2\pi i \langle l, \theta\rangle)$, where $F \subset S_1^+$ is a finite set, we have, using (2.3), that

$$\int_{T^n} u(\theta)\overline{P_j(\theta)}e^{-2\pi i\langle s,\theta\rangle}\,d\mu(\theta)=\sum_{l\in F}\overline{a_l}\int_{T^n} u(\theta)e^{-2\pi i\langle l+s,\theta\rangle}\,d\mu(\theta)$$
$$=\sum_{l\in F}\overline{a_l}\int_{T^n} e^{-2\pi i\langle l+s,\theta\rangle}\,d\theta=0$$

whenever $-s\notin F$. It thus follows from Lemma 2.2 that $(|u|^2\mu)_S=|u|^2\mu$ and that $(u\overline{P_j}\mu)_S\in\mathcal{P}(T^n)$. Furthermore, if we denote by $\langle\cdot,\cdot\rangle$ the duality between measures in $M(T^n)$ and functions in $C(T^n)$, from (2.2) we easily obtain the inequality

$$|\langle[|u|^2\mu-u\overline{P_j}\mu]_S,\phi\rangle|\le\|\phi\|_\infty\|u\|_{2,\mu}\|u-P_j\|_{2,\mu},$$

valid for all $\phi\in C(T^n)$, which shows that $(u\overline{P_j}\mu)_S$ converges to $|u|^2\mu$ in the norm topology of $M(T^n)$. Hence, we conclude that $|u|^2\mu\in L^1(T^n)$ and thus that $u=0$ a.e. $(d\mu_s)$. □

In the following proposition, an explicit form is given for the expression $|u|^2\mu$ $(=|u|^2w)$ appearing in the proof of the previous lemma.

PROPOSITION 2.4. *Let $\mu\in M^+(T^n)$ with $\Phi_1(\mu)>0$, and let u be the unique element in M_1 satisfying* (2.3). *Then $|u|^2w=G\circ B_1$ a.e. on T^n, where w is the absolutely continuous part of μ and $G\in L^1(T^k)$ is defined for a.e. $\tau\in T^k$ by*

$$G(\tau)=\exp\left[-\int_{T^{n-k}}\log w(A(\tau,\zeta))\,d\zeta\right].$$

Proof. If $P_1\in\mathcal{P}(T^k)$ and $P_2\in\mathcal{P}(T^{n-k})$, we can define $P\in\mathcal{P}(T^n)$ by $P=(P_1\circ B_1)(P_2\circ B_2)$. Let $y_0\in\mathbf{R}^{n-k}$ have its jth component equal to $\langle x_0,v_{j+k}\rangle$ for $j=1,\dots,n-k$. Consider the sets $T_1^+=\{h\in\mathbf{Z}^{n-k},\langle y_0,h\rangle\ge0\}$ and $T=\{h\in\mathbf{Z}^{n-k},\langle y_0,h\rangle=0\}$. Our construction of the v_i's shows that $T=\{0\}$ and that $T_1^+\backslash\{0\}$ is a half-space of lattice points in the sense of Helson and Lowdenslager in $\mathbf{Z}^{n-k}$. Furthermore, $P\in\mathcal{P}(S_1^+)$ is equivalent to $P_2\in\mathcal{P}(T_1^+)$. For every $P_1\in\mathcal{P}(T^k)$ and every $P_2\in\mathcal{P}(T_1^+)$, using the invariance of the Haar measure on T^n under B we compute

$$\begin{aligned}\int_{T^n}\overline{P(\theta)}\,d\theta&=\int_{T^n}\overline{P_1(B_1(\theta))P_2(B_2(\theta))}\,d\theta\\&=\int_{T^k}\overline{P_1(\tau)}\,d\tau\int_{T^{n-k}}\overline{P_2(\zeta)}\,d\zeta.\end{aligned}\tag{2.4}$$

On the other hand, using (2.3), we have also

$$\begin{aligned}\int_{T^n}\overline{P(\theta)}\,d\theta&=\int_{T^n}u(\theta)\overline{P(\theta)}\,d\mu(\theta)=\int_{T^n}u(\theta)\overline{P(\theta)}w(\theta)\,d\theta\quad\text{(by Lemma 2.3)}\\&=\int_{T^k\times T^{n-k}}u(A(\tau,\zeta))\overline{P_1(\tau)}\overline{P_2(\zeta)}w(A(\tau,\zeta))\,d\zeta\,d\tau\\&=\int_{T^k}\left[\int_{T^{n-k}}u(A(\tau,\zeta))\overline{P_2(\zeta)}w(A(\tau,\zeta))\,d\zeta\right]\overline{P_1(\tau)}\,d\tau.\end{aligned}\tag{2.5}$$

Since we have an equality between (2.5) and (2.6) for every $P_1 \in \mathcal{P}(T^k)$, it follows that, except for a set of zero measure depending on $P_2 \in \mathcal{P}(T_1^+)$, every $\tau \in T^k$ satisfies

$$\int_{T^{n-k}} u(A(\tau,\zeta))\overline{P_2(\zeta)}w(A(\tau,\zeta))\,d\zeta = \int_{T^{n-k}} \overline{P_2(\zeta)}\,d\zeta. \tag{2.6}$$

We can remove the dependence on P_2 of this set of zero measure by noticing the existence of a countable dense set in $\mathcal{P}(T_1^+)$ for the topology of uniform convergence on T^{n-k}. For example, one can take the set of trigonometric polynomials in $\mathcal{P}(T_1^+)$ of which the real and imaginary parts of the Fourier coefficients are rational numbers. Thus, for a.e. $\tau \in T^k$, (2.6) holds for every $P_2 \in \mathcal{P}(T_1^+)$. If $\tau \in T^k$, we define the weight w_τ on T^{n-k} by $w_\tau(\zeta) = w(A(\tau,\zeta))$, and u_τ by $u_\tau(\zeta) = u(A(\tau,\zeta))$, for $\zeta \in T^{n-k}$. It is easy to see, using Fubini's theorem, that for a.e. $\tau \in T^k$,

$$w_\tau \in L^1(T^{n-k}) \quad \text{and} \quad \int_{T^{n-k}} |u_\tau(\zeta)|^2 w_\tau(\zeta)\,d\zeta < \infty.$$

Furthermore, if $\{P_j\}_{j\geq 0}$ is a sequence in $\mathcal{P}(S_1^+)$ converging to u in L^2_μ, we have

$$\begin{aligned} 0 &= \lim_{j\to\infty} \int_{T^n} |u(\theta) - P_j(\theta)|^2\,d\mu(\theta) \\ &= \lim_{j\to\infty} \int_{T^k} \left[\int_{T^{n-k}} |u(A(\tau,\zeta)) - P_j(A(\tau,\zeta))|^2 w(A(\tau,\zeta))\,d\zeta\right] d\tau. \end{aligned}$$

After extraction of a subsequence converging pointwise a.e. to 0 on T^k and taking into account the fact that, for a fixed τ, the function $\zeta \to P_j(A(\tau,\zeta))$ belongs to $\mathcal{P}(T_1^+)$, it follows that, for a.e. $\tau \in T^k$, u_τ belongs to the closure of $\mathcal{P}(T_1^+)$ in $L^2_{w_\tau}$. Since, by (2.6), it satisfies

$$\forall P_2 \in \mathcal{P}(T_1^+), \quad \int_{T^{n-k}} u_\tau(\zeta)\overline{P_2(\zeta)}w_\tau(\zeta)\,d\zeta = \int_{T^{n-k}} \overline{P_2(\zeta)}\,d\zeta,$$

we conclude from the result of Helson and Lowdenslager [3, p. 171] that, for a.e. $\tau \in T^k$,

$$\int_{T^{n-k}} |u_\tau(\zeta)|^2 w_\tau(\zeta)\,d\zeta = \exp\left[-\int_{T^{n-k}} \log w_\tau(\zeta)\,d\zeta\right]. \tag{2.7}$$

On the other hand, it follows from the proof of Lemma 2.3 that $|u|^2 w = G \circ B_1$ a.e. on T^n, where $G \in L^1(T^k)$. This implies that, for a.e. $\tau \in T^k$, $|u_\tau|^2 w_\tau = G(\tau)$ a.e. on T^{n-k}, and thus we deduce from (2.7) that, for a.e. $\tau \in T^k$,

$$G(\tau) = \exp\left[-\int_{T^{n-k}} \log w_\tau(\zeta)\,d\zeta\right] = \exp\left[-\int_{T^{n-k}} \log w(A(\tau,\zeta)\,d\zeta\right],$$

which proves our assertion. □

We can now state our extension of the Helson–Lowdenslager theorem in the case of the half-space of lattice points S_1^+.

THEOREM 2.5. *Let $\mu \in M^+(T^n)$ have Lebesgue decomposition $\mu = w + \mu_s$, where $w \in L^1(T^n)$ and μ_s is singular. Then*

$$\Phi_1(\mu) = \left[\int_{T^k} \exp\left[-\int_{T^{n-k}} \log w(A(\tau, \zeta))\, d\zeta\right] d\tau\right]^{-1}, \tag{2.8}$$

where we use the convention that the right-hand side of (2.8) *is zero if*

$$G\colon \tau \mapsto \exp\left[-\int_{T^{n-k}} \log w(A(\tau, \zeta))\, d\zeta\right]$$

does not belong to $L^1(T^k)$.

Proof. If $\Phi_1(\mu) > 0$ then, using Lemma 2.3, we have $\Phi_1(\mu) = \|u\|_{2,w}^{-2}$, where $u \in M_1$ satisfies (2.3). Proposition 2.4 yields

$$\|u\|_{2,w}^2 = \int_{T^n} |u(\theta)|^2 w(\theta)\, d\theta = \int_{T^n} G(B_1(\theta))\, d\theta = \int_{T^k} G(\tau)\, d\tau$$
$$= \int_{T^k} \exp\left[-\int_{T^{n-k}} \log w(A(\tau, \zeta))\, d\zeta\right] d\tau,$$

where $G \in L^1(T^k)$, and our assertion is thus proved in that case. If the infimum is zero then $G \notin L^1(T^k)$. This can be seen by considering the weights $\epsilon + w$, where $\epsilon > 0$, and using the first part of the proof. The details are left to the reader. □

EXAMPLES 2.6. If $n = 2$ and $x_0 = (1, 0)$, then we have $S_1^+ = \{(m, n) \in \mathbf{Z}^2, m \geq 0\}$, $S = \{(0, n), n \in \mathbf{Z}\}$, and $k = 1$. We can choose $v_1 = (0, 1)$ and $v_2 = (1, 0)$. Then $A\colon (\tau, \zeta) \to (\zeta, \tau)$, and Theorem 2.5 reads

$$\Phi_1(\mu) = \left[\int_T \exp\left[-\int_T \log w(\zeta, \tau)\, d\zeta\right] d\tau\right]^{-1}.$$

If $n = 2$ and $x_0 = (1, 1)$, then we have that $S_1^+ = \{(m, n) \in \mathbf{Z}^2, m + n \geq 0\}$, $S = \{(-n, n), n \in \mathbf{Z}\}$, and $k = 1$. We can choose $v_1 = (-1, 1)$ and $v_2 = (1, 0)$. Thus $A\colon (\tau, \zeta) \to (\zeta, \tau + \zeta)$ and, in that case, we have

$$\Phi_1(\mu) = \left[\int_T \exp\left[-\int_T \log w(\zeta, \tau + \zeta)\, d\zeta\right] d\tau\right]^{-1}.$$

3. The Prediction Problem for S_2^+

In this section, we will consider the prediction problem for the half-space of lattice points $S_2^+ = \{v \in \mathbf{Z}^n, \langle x_0, v\rangle > 0\} \cup \{0\}$, where $x_0 \in \mathbf{R}^n$ and $x_0 \neq 0$. We assume again that $S = \{v \in \mathbf{Z}^n, \langle x_0, v\rangle = 0\} \neq \{0\}$; our goal is to find an explicit expression for

$$\Phi_2(\mu) = \Phi_{S_2^+}(\mu) = \inf_{P \in \mathcal{P}(S_2^+ \setminus \{0\})} \int_{T^n} |1 + P(\theta)|^2\, d\mu(\theta)$$

in terms of $\mu \in M^+(T^n)$. We will use the same change of variable as defined at the beginning of Section 2 as well as the same notations.

DEFINITION 3.1. M_2 will denote the closure of $\mathcal{P}(S_2^+)$ in L^2_μ. If $\nu \in M(T^k)$, we will denote by $\nu \circ B_1$ the element of $M(T^n)$ defined by the identity

$$\forall \phi \in C(T^n), \quad \int_{T^n} \phi(\theta)\, d(\nu \circ B_1)(\theta) = \int_{T^k} \left[\int_{T^{n-k}} \phi(A(\tau, \zeta))\, d\zeta \right] d\nu(\tau).$$

We have the following lemma.

LEMMA 3.2. *If $\Phi_2(\mu) > 0$ then there exists a measure $\nu \in M^+(T^k)$ such that $|u|^2 \mu = \nu \circ B_1$, where u is the unique element in M_2 satisfying*

$$\text{(3.1)} \qquad \forall P \in \mathcal{P}(S_2^+), \quad \int_{T^n} \overline{P(\theta)}\, d\theta = \int_{T^n} u(\theta) \overline{P(\theta)}\, d\mu(\theta).$$

Proof. The proof is similar to the first part of the proof of Lemma 2.3, and is left to the reader. □

REMARK 3.3. Let us notice that, in contrast to the situation in Lemma 2.3, we can no longer state that the measure ν appearing in Lemma 3.2 is actually in $L^1(T^k)$, except of course when μ is itself in $L^1(T^k)$ (see Example 3.7).

We will first consider the prediction problem for S_2^+ in the case of a weight.

LEMMA 3.4. *Let $w \in L^1(T^k)$ with $w \ge 0$, and suppose that $\Phi_2(w) > 0$. Then the set $\{\tau \in T^k, \int_{T^{n-k}} \log w(A(\tau, \zeta))\, d\zeta > -\infty\}$ has nonzero Lebesgue measure on T^k and*

$$\Phi_2(w) = \int_{T^k} \exp\left[\int_{T^{n-k}} \log w(A(\tau, \zeta))\, d\zeta \right] d\tau.$$

Proof. Let $u \in M_2$ satisfy (3.1) with $\mu = w$. By Lemma 3.2, there exists $H \in L^1(T^k)$ with $H \ge 0$ such that $|u|^2 w = H \circ B_1$. Consider a sequence $\{P_j\}_{j \ge 0}$ in $\mathcal{P}(S_2^+)$ converging to u in L^2_w. We can write $P_j = \hat{P}_j(0) + Q_j$, where $Q_j \in \mathcal{P}(S_2^+ \setminus \{0\})$. Using (3.1), it follows that $(uw)\hat{}(s) = 0$ whenever $s \in S_2^+ \setminus \{0\}$ and therefore that $(u\overline{Q_j} w)\hat{}(s) = 0$ whenever $s \in S_1^+$. Hence, if $P \in \mathcal{P}(S_1^+)$ we have

$$\begin{aligned}
&\int_{T^n} |u|^2(\theta) \overline{P(\theta)} w(\theta)\, d\theta \\
&\quad = \lim_{j \to \infty} \int_{T^n} u(\theta) \overline{P_j(\theta)} \overline{P(\theta)} w(\theta)\, d\theta \\
&\quad = \lim_{j \to \infty} \left[\int_{T^n} \overline{P_j(\theta)}\, d\theta \int_{T^n} u(\theta) \overline{P(\theta)} w(\theta)\, d\theta + \int_{T^n} u(\theta) \overline{Q_j(\theta)} \overline{P(\theta)} w(\theta)\, d\theta \right] \\
&\quad = \lim_{j \to \infty} \left(\int_{T^n} \overline{P_j(\theta)}\, d\theta \right) \int_{T^n} u(\theta) \overline{P(\theta)} w(\theta)\, d\theta = \|u\|^2_{2,w} \int_{T^n} u(\theta) \overline{P(\theta)} w(\theta)\, d\theta.
\end{aligned}$$

Therefore, for every $P \in \mathcal{P}(S_1^+)$, we obtain the identity

$$\text{(3.2)} \qquad \int_{T^n} u(\theta) \overline{P(\theta)} w(\theta)\, d\theta = \|u\|^{-2}_{2,w} \int_{T^n} \overline{P(\theta)} (H \circ B_1)(\theta)\, d\theta.$$

In particular, for $P=(P_1\circ B_1)(P_2\circ B_2)$, where $P_1\in\mathcal{P}(T^k)$ and $P_2\in\mathcal{P}(T^{n-k})$, after using the change of variable $\theta=A(\tau,\zeta)$ we have that

$$\int_{T^k}\left[\int_{T^{n-k}}u(A(\tau,\zeta))\overline{P_2(\zeta)}w(A(\tau,\zeta))\,d\zeta\right]\overline{P_1(\tau)}\,d\tau$$
$$=\|u\|_{2,w}^{-2}\left[\int_{T^{n-k}}\overline{P_2(\zeta)}\,d\zeta\right]\left[\int_{T^k}\overline{P_1(\tau)}H(\tau)\,d\tau\right],$$

from which we conclude, as in Proposition 2.4, that for a.e. $\tau\in T^k$ we have

$$\forall P_2\in\mathcal{P}(T_1^+),\quad \int_{T^{n-k}}u_\tau(\zeta)\overline{P_2(\zeta)}w_\tau(\zeta)\,d\zeta=\|u\|_{2,w}^{-2}H(\tau)\int_{T^{n-k}}\overline{P_2(\zeta)}\,d\zeta,$$

where we use the same notation as in the proof of Proposition 2.4. We define $C=\{\tau\in T^k,\ H(\tau)=0\}$. Let us first consider the case of a $\tau\in T^k$ for which $H(\tau)\neq 0$, that is, $\tau\notin C$. Because (as can be easily checked) for a.e. $\tau\in T^k$, u_τ belongs to the closure of $\mathcal{P}(T_1^+)$ in $L^2_{w_\tau}$, it follows from the previous computation and the result of Helson and Lowdenslager [3, p. 171] that, for a.e. $\tau\notin C$,

$$(3.3)\qquad \int_{T^{n-k}}|u_\tau|^2(\zeta)w_\tau(\zeta)\,d\zeta=\|u\|_{2,w}^{-4}H^2(\tau)\exp\left[-\int_{T^{n-k}}\log w_\tau(\zeta)\,d\zeta\right].$$

If, on the contrary, $\tau\in C$, then $u_\tau=0$ a.e. on T^{n-k}. Since $\hat{P}_j(0)$ converges to $\|u\|_{2,w}^2$ as $j\to\infty$, it follows easily that the sequence $\{\|u\|_{2,w}^2+Q_j\}_{j\geq 0}$ converges to u in L^2_w. We therefore obtain, after using the change of variable $\theta=(\tau,\zeta)$, that

$$0=\lim_{j\to\infty}\int_{T^k}\left[\int_{T^{n-k}}|u_\tau(\zeta)-\|u\|_{2,w}^2-Q_j(A(\tau,\zeta))|^2w_\tau(\zeta)\,d\zeta\right]d\tau.$$

After extracting a subsequence which converges pointwise a.e. to zero on T^k and using the fact that, for a fixed τ, the function $\zeta\to Q_j(A(\tau,\zeta))$ belongs to $\mathcal{P}(T_1^+\setminus\{0\})$, we obtain that, for a.e. $\tau\in T^k$, $u_\tau-\|u\|_{2,w}^2$ belongs to the closure of $\mathcal{P}(T_1^+\setminus\{0\})$ in $L^2_{w_\tau}$. If $\tau\in C$, this implies in particular that $\Phi_{T_1^+}(w_\tau)=0$. Since $T_1^+\setminus\{0\}$ is a half-space in the sense of Helson and Lowdenslager, it follows from their theorem [3, p. 171] that, for a.e. $\tau\in C$,

$$\int_{T^{n-k}}\log w_\tau(\zeta)\,d\zeta=-\infty.$$

Hence we deduce that, for a.e. $\tau\in T^k$,

$$H(\tau)=\|u\|_{2,w}^4\exp\left[\int_{T^{n-k}}\log w_\tau(\zeta)\,d\zeta\right],$$

this last equality following immediately from the previous computation if $\tau\in C$, and from (3.3) if $\tau\notin C$, using the fact that for a.e. $\tau\in T^k$, $|u_\tau|^2w_\tau=H(\tau)$ a.e. on T^{n-k}. We thus have that

$$\|u\|_{2,w}^2=\int_{T^n}|u(\theta)|^2 w(\theta)\,d\theta=\int_{T^n}(H\circ B_1)(\theta)\,d\theta$$

$$=\int_{T^k}H(\tau)\,d\tau=\|u\|_{2,w}^4\int_{T^k}\exp\left[\int_{T^{n-k}}\log w(A(\tau,\zeta))\,d\zeta\right]d\tau,$$

from which it follows that the set $\{\tau\in T^k, \int_{T^{n-k}}\log w(A(\tau,\zeta))\,d\zeta>-\infty\}$ has nonzero measure in T^k and that

$$\Phi_2(w)=\|u\|_{2,w}^{-2}=\int_{T^k}\exp\left[\int_{T^{n-k}}\log w(A(\tau,\zeta))\,d\zeta\right]d\tau. \qquad \square$$

COROLLARY 3.5. *Let $w\in L^1(T^k)$ with $w\geq 0$. Then, with the usual convention* $\exp(-\infty)=0$, *we have*

$$\Phi_2(w)=\int_{T^k}\exp\left[\int_{T^{n-k}}\log w(A(\tau,\zeta))\,d\zeta\right]d\tau.$$

Proof. If $\Phi_2(w)>0$, this is the statement of Lemma 3.4. If $\Phi_2(w)=0$ then the conclusion is easily obtained by considering the weights $\epsilon+w$, $\epsilon>0$, via a limiting argument. $\square$

We now have all the ingredients to solve the prediction problem for S_2^+ in the case of an arbitrary measure $\mu\in M^+(T^n)$.

THEOREM 3.6. *Let $\mu\in M^+(T^k)$ and let $\{\varphi_\epsilon\}_{\epsilon>0}$ be any approximate identity. Define, for $\epsilon>0$, the positive function $G_\epsilon\in L^1(T^k)$ by*

$$G_\epsilon(\tau)=\exp\left[\int_{T^{n-k}}(\log\mu_\epsilon)(A(\tau,\zeta))\,d\zeta\right],\quad \tau\in T^k,$$

where $\mu_\epsilon=\mu*\varphi_\epsilon$. Then, if $\Phi_2(\mu)>0$, G_ϵ converges to $\|u\|_{2,\mu}^{-4}\nu$ as $\epsilon\to 0^+$ in the weak topology of $M(T^k)$, where u and ν are defined in Lemma 3.2. On the other hand, if $\Phi_2(\mu)=0$ then G_ϵ converges to zero in $L^1(T^k)$. Furthermore, in both cases we have*

$$\Phi_2(\mu)=\int_{T^k}d\sigma(\tau)=\lim_{\epsilon\to 0^+}\int_{T^k}G_\epsilon(\tau)\,d\tau, \tag{3.4}$$

where σ denotes the weak limit of G_ϵ in $M^+(T^k)$ as $\epsilon\to 0^+$.*

Proof. If $\Phi_2(\mu)>0$ then it is easily checked, as in the proof of (3.2) in Lemma 3.4, that the following identity holds for every $P\in\mathcal{P}(S_1^+)$:

$$\int_{T^n}u(\theta)\overline{P(\theta)}\,d\mu(\theta)=\|u\|_{2,\mu}^{-2}\int_{T^n}\overline{P(\theta)}\,d(\nu\circ B_1)(\theta).$$

In particular, if $P=P_1\circ B_1$ where $P_1\in\mathcal{P}(T^k)$, we have

$$\int_{T^n}u(\theta)\overline{P_1(B_1(\theta))}\,d\mu(\theta)=\|u\|_{2,\mu}^{-2}\int_{T^n}\overline{P_1(B_1(\theta))}\,d(\nu\circ B_1)(\theta)$$

$$=\|u\|_{2,\mu}^{-2}\int_{T^k}\overline{P_1(\tau)}\,d\nu(\tau),$$

by definition of the measure $\nu \circ B_1$. Since P_1 is arbitrary in $\mathcal{P}(T^k)$, it follows that

$$\text{(3.5)} \quad \forall \phi \in C(T^k), \quad \int_{T^n} u(\theta)\phi(B_1(\theta))\, d\mu(\theta) = \|u\|_{2,\mu}^{-2} \int_{T^k} \phi(\tau)\, d\nu(\tau).$$

Since $\Phi_2(\mu_\epsilon) > 0$ by Lemma 1.1, there exists a unique element u_ϵ, belonging to the closure of $\mathcal{P}(S_2^+)$ in $L^2_{\mu_\epsilon}$, that satisfies (3.1) with μ replaced by μ_ϵ and u by u_ϵ. Since $|u_\epsilon|^2\mu_\epsilon = \|u_\epsilon\|_{2,\mu_\epsilon}^4 G_\epsilon$ by the proof of Lemma 3.4, we obtain, by the same argument as above, that

$$\text{(3.6)} \qquad \int_{T^n} u_\epsilon(\theta)\phi(B_1(\theta))\mu_\epsilon(\theta)\, d\theta = \|u_\epsilon\|_{2,\mu_\epsilon}^2 \int_{T^k} \phi(\tau) G_\epsilon(\tau)\, d\tau$$

for all $\phi \in C(T^k)$. It therefore follows from Lemma 1.1 and Lemma 1.3, using (3.5) and (3.6), that $\lim_{\epsilon\to 0} G_\epsilon = \|u\|_{2,\mu}^{-4}\nu$ in the weak* topology of $M(T^k)$. In particular, if $\phi \equiv 1$, using Lemma 1.1 and Lemma 3.4 we obtain that

$$\|u\|_{2,\mu}^2 = \lim_{\epsilon\to 0^+} \|u_\epsilon\|_{2,\mu_\epsilon}^2 = \lim_{\epsilon\to 0^+} \left[\int_{T^k} G_\epsilon(\tau)\, d\tau\right]^{-1} = \left[\int_{T^k} d\sigma(\tau)\right]^{-1},$$

where $\sigma = \|u\|_{2,\mu}^{-4}\nu$, which proves (3.5). If $\Phi_2(\mu) = 0$ then, for a fixed $\epsilon > 0$, we have either that $G_\epsilon = 0$ a.e. on T^k or that $\Phi_2(\mu_\epsilon) > 0$. In that latter case, it follows from (3.6) that, for every $\phi \in C(T^k)$,

$$\begin{aligned}\left|\int_{T^k} \phi(\tau) G_\epsilon(\tau)\, d\tau\right| &= \|u_\epsilon\|_{2,\mu_\epsilon}^{-2} \left|\int_{T^n} u_\epsilon(\theta)(\phi\circ B_1)(\theta)\mu_\epsilon(\theta)\, d\theta\right| \\ &\le \|u_\epsilon\|_{2,\mu_\epsilon}^{-2}\|u_\epsilon\|_{2,\mu_\epsilon}\|\phi\|_\infty \left[\int_{T^n} d\mu(\theta)\right]^{1/2} \\ &= [\Phi_2(\mu_\epsilon)]^{1/2}\|\phi\|_\infty \left[\int_{T^n} d\mu(\theta)\right]^{1/2}.\end{aligned}$$

We thus obtain that, for every $\epsilon > 0$,

$$\forall \phi \in C(T^k), \quad \left|\int_{T^k} \phi(\tau) G_\epsilon(\tau)\, d\tau\right| \le [\Phi_2(\mu_\epsilon)]^{1/2} \left[\int_{T^n} d\mu(\theta)\right]^{1/2} \|\phi\|_\infty,$$

this last inequality being trivial if $\Phi_2(\mu_\epsilon) = 0$. This implies, by Lemma 1.1, that G_ϵ converges to zero in $L^1(T^k)$ as $\epsilon \to 0^+$. In that case, (3.4) is clearly satisfied with $\sigma = 0$, and the theorem is proved. □

EXAMPLE 3.7. The following example shows that the solution of the prediction problem for S_2^+ can depend upon the singular part of the measure μ considered. If $n = 2$, let $S_2^+ = \{(m, n), m > 0, n \in \mathbf{Z}\} \cup \{(0, 0)\}$ and suppose that μ has the form $\mu = \mu_1 \otimes \mu_2$, where $\mu_1, \mu_2 \in M(T)$. We can choose an approximate identity $\{\varphi_\epsilon\}_{\epsilon>0}$ on T^2 of the form $\varphi_\epsilon = \varphi_{1,\epsilon} \otimes \varphi_{2,\epsilon}$, where $\{\varphi_{1,\epsilon}\}_{\epsilon>0}$ and $\{\varphi_{2,\epsilon}\}_{\epsilon>0}$ are approximate identities on T. In that case we can choose $v_1 = (0, 1)$ and $v_2 = (1, 0)$, and thus $A(\tau, \zeta) = (\zeta, \tau)$. Hence, if $\tau \in T$,

$$G_\epsilon(\tau) = \exp\left[\int_T (\log \mu_\epsilon)(\zeta, \tau)\, d\zeta\right] = \exp\left[\int_T \log(\mu_1 * \varphi_{1,\epsilon})(\zeta)\, d\zeta\right](\mu_2 * \varphi_{2,\epsilon})(\tau),$$

and, using Corollary 1.2,

$$\lim_{\epsilon \to 0^+} G_\epsilon = \exp\left[\int_T \log w_1(\zeta)\, d\zeta\right]\mu_2$$

in the weak* topology of $M(T)$, where w_1 denotes the absolutely continuous part of the measure μ_1. Hence, using Theorem 3.6, we conclude that

$$\Phi_2(\mu) = \exp\left[\int_T \log w_1(\zeta)\, d\zeta\right] \int_T d\mu_2(\tau).$$

REMARK 3.8. A result of Kolmogorov states that, if $\mu \in M^+(T^k)$ and $\mu = w + \mu_s$, then $\Phi_{Z^k}(\mu) = (\int_{T^k} 1/w(\theta)\, d\theta)^{-1}$, the harmonic mean of w, while, trivially, $\Phi_{\{0\}}(\mu) = \int_{T^k} d\mu(\theta)$, the arithmetic mean of μ. Now, if one looks at the explicit expressions for $\Phi_1(\mu)$ and $\Phi_2(\mu)$, one can see, at least when $\mu_s = 0$, that they are obtained by computing a geometric mean in one set of variables and then by calculating the harmonic mean of the result (in the case of Φ_1) or its arithmetic mean (in the case of Φ_2) with respect to the other variables. This suggests, perhaps, that the methods presented in this paper might be applicable to other multidimensional prediction problems. For example, other combinations of subsets of the "k-dimensional boundary" of the half-space might be considered in the approximation process.

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