Boundary Behavior of Derivatives of Analytic Functions

SHELDON AXLER & KEHE ZHU

Introduction

In this paper we study the boundary behavior of functions of the form $(1-|z|^2)^n f^{(n)}(z)$; here f is an analytic function defined on the open unit disk D in the complex plane and n is a positive integer. Many well-studied classes of analytic functions arise from requiring functions of the above form to have a certain growth rate. For example, the Bloch space $\mathcal{B}$ is defined to be the set of analytic functions f on D such that $(1-|z|^2)f'(z)$ is bounded on D, and the little Bloch space $\mathcal{B}_0$ is defined to be the set of analytic functions f on D such that $(1-|z|^2)f'(z) \to 0$ as $|z| \to 1$ (of course, z is restricted to the values in D).

Let dA denote the usual Lebesgue area measure on the complex plane. For $p \in [1, \infty)$, the Bergman space L^p_a is defined to be the set of analytic functions f on D such that

$$\int_D |f|^p \, dA < \infty.$$

As is well known (e.g., see [2, Prop. 1.7]), if $f \in L^1_a$ then

$$f(z) = \frac{1}{\pi} \int_D \frac{f(w)}{(1-z\bar{w})^2} \, dA(w) \quad \text{for every } z \in D. \tag{1}$$

Equation (1) suggests that for $f \in L^1(D, dA)$ (not necessarily analytic), we define an analytic function $P(f)$ on D by

$$P(f)(z) = \frac{1}{\pi} \int_D \frac{f(w)}{(1-z\bar{w})^2} \, dA(w).$$

It is useful to know the image under P of certain natural spaces. We begin by noting that P restricted to $L^2(D, dA)$ is the orthogonal projection of $L^2(D, dA)$ onto L^2_a; furthermore, if $p \in (1, \infty)$ then P restricted to $L^p(D, dA)$ is a bounded projection of $L^p(D, dA)$ onto L^p_a (see Theorem 1.10 of [2]; throughout this paper, we choose references most suited to our approach, so the references are not necessarily to the original source).

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Table 2

	Y	$P(Y)$
1.	$L^p(D, dA)$ $(1<p<\infty)$	L^p_a
2.	$L^\infty(D, dA)$	$\mathcal{B}$
3.	$C(\bar{D})$	$\mathcal{B}_0$
4.	$h^\infty(D)$	BMOA
5.	$C(\bar{D}) \cap h^\infty(D)$	VMOA
6.	$\mathcal{U}$	?

The images under P of some other spaces are listed in Table 2. We have already mentioned the result in line 1 of Table 2. Lines 2 and 3 of Table 2 state that P maps $L^\infty(D, dA)$ onto the Bloch space $\mathcal{B}$ and that $C(\bar{D})$ is mapped onto the little Bloch space $\mathcal{B}_0$ (see Theorem 2.7, Theorem 2.11, and page 22 of [2]); here and throughout the paper, if Ω is a topological space then $C(\Omega)$ denotes the set of continuous, complex-valued functions on Ω.

In lines 4 and 5 of Table 2, $h^\infty(D)$ denotes the set of bounded, complex-valued, harmonic functions on D. Also, BMOA and VMOA denote the usual spaces of analytic functions in the Hardy space H^2 whose boundary values have bounded mean oscillation and vanishing mean oscillation, respectively.

Let $\mathcal{U}$ denote the closed subalgebra of $L^\infty(D, dA)$ generated by the bounded, complex-valued, harmonic functions on D (the product of two harmonic functions is not necessarily harmonic, so $h^\infty(D)$ is not an algebra). In several contexts, $\mathcal{U}$ is much easier to work with than $L^\infty(D, dA)$. For example, McDonald and Sundberg [10] used $\mathcal{U}$ to produce some striking results in the theory of Toeplitz operators on the Bergman space.

Because the functions z and $\bar{z}$ are in $\mathcal{U}$, the Stone–Weierstrass theorem implies that $C(\bar{D}) \subset \mathcal{U}$; thus (by line 3 of Table 2) $P(\mathcal{U})$ contains the little Bloch space. In [2, p. 40], Axler raised the question of whether $P(\mathcal{U})$ equals the Bloch space. In the next section of this paper, we will prove (Theorem 6) that an analytic function f lies in $P(\mathcal{U})$ if and only if $(1-|z|^2)^n f^{(n)}(z) \in \mathcal{U}$ for every positive integer n. This description of $P(\mathcal{U})$ will then be used (Corollary 11) to show that $P(\mathcal{U})$ is a proper subspace of the Bloch space.

Let $H^\infty(D)$ denote the set of bounded analytic functions on D. The algebra $\mathcal{U}$ is precisely the set of continuous functions on D that extend continuously to the maximal ideal space of $H^\infty(D)$. The results in the next section, when considered in light of this identification, lead to various questions that are answered in the final section of the paper.

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The Algebra Generated by Bounded Harmonic Functions

In this section, we will attempt to fill in the last line of Table 2. Before we can prove the main result of this section (Theorem 6), we need three lemmas.

After proving Theorem 6, we will show (Corollary 11) that $P(\mathcal{U})$ is a proper subspace of the Bloch space.

The following lemma is well known; for example, see [6, Thm. 5.5].

LEMMA 3. *Let f be an analytic function on D. Then the following are equivalent:*

(a) $f \in \mathcal{B}$;
(b) $\sup\{(1-|z|^2)^n |f^{(n)}(z)|: z \in D\} < \infty$ *for every positive integer n;*
(c) $\sup\{(1-|z|^2)^n |f^{(n)}(z)|: z \in D\} < \infty$ *for some positive integer n.*

The following lemma gives a formula for representing an analytic function as an area integral of its derivatives.

LEMMA 4. *Let $f \in \mathcal{B}$ and let n be a nonnegative integer such that f has a zero of order at least $2n$ at 0. Then*

$$f(z) = \frac{1}{n!\,\pi} \int_D \frac{(1-|w|^2)^n f^{(n)}(w)}{(1-z\bar{w})^2 (\bar{w})^n}\, dA(w)$$

for every $z \in D$.

Proof. First we must show that the integral in the statement of the lemma is well defined. Because f is in the Bloch space, Lemma 3 implies that the numerator in the integral is bounded. For fixed $z \in D$, the denominator in the integral is bounded away from 0 except near $w=0$. Because f has a zero of order at least $2n$ at 0, $f^{(n)}$ has a zero of order at least n at 0. Thus $|f^{(n)}(w)/w^n|$ is bounded near $w=0$, and so the integral is well defined.

Let g be the analytic function on D defined by

$$g(z) = \frac{1}{n!\,\pi} \int_D \frac{(1-|w|^2)^n f^{(n)}(w)}{(1-z\bar{w})^2 (\bar{w})^n}\, dA(w).$$

Let m be a nonnegative integer. By differentiating under the integral sign, we see that the formula for the mth derivative of g is given by

$$g^{(m)}(z) = \frac{(m+1)!}{n!\,\pi} \int_D \frac{(1-|w|^2)^n f^{(n)}(w)}{(1-z\bar{w})^{m+2} (\bar{w})^{n-m}}\, dA(w)$$

for $z \in D$. In particular,

$$g^{(m)}(0) = \frac{(m+1)!}{n!\,\pi} \int_D \frac{(1-|w|^2)^n f^{(n)}(w)}{(\bar{w})^{n-m}}\, dA(w).$$

Assume temporarily that f is analytic on a neighborhood of $\bar{D}$. In the above integral, replace $f^{(n)}(w)$ with its power series expansion

$$\sum_{j=n}^{\infty} f^{(n+j)}(0) \frac{w^j}{j!},$$

interchange the summation and the integral, and change to polar coordinates, yielding

$$g^{(m)}(0)=\begin{cases}0 & \text{if } m<n,\\ \dfrac{(m+1)!\,2f^{(m)}(0)}{n!\,(m-n)!}\displaystyle\int_0^1(1-r^2)^n r^{2m-2n+1}dr & \text{if } m\geq n.\end{cases}$$

The last integral can now be computed by making the change of variables $t=r^2$ and then integrating by parts $m-n$ times; this yields

$$g^{(m)}(0)=f^{(m)}(0)$$

for every nonnegative integer m. Thus $f=g$, as desired.

Now remove the assumption that f is analytic on a neighborhood of $\bar{D}$. Let $r\in(0,1)$. Applying the result from the previous paragraph to the function whose value at z is $f(rz)$, we have

$$f(rz)=\frac{1}{n!\,\pi}\int_D\frac{(1-|w|^2)^n f^{(n)}(rw)r^n}{(1-z\bar{w})^2(\bar{w})^n}\,dA(w)$$

for every $z\in D$. Now $(1-|w|^2)^n|f^{(n)}(rw)|\leq(1-|rw|^2)^n|f^{(n)}(rw)|$, and the latter quantity is a bounded function of w and r (by Lemma 3), so we can let r increase to 1 and apply the Lebesgue dominated convergence theorem to conclude that

$$f(z)=\frac{1}{n!\,\pi}\int_D\frac{(1-|w|^2)^n f^{(n)}(w)}{(1-z\bar{w})^2(\bar{w})^n}\,dA(w)$$

for every $z\in D$, completing the proof. □

For $w,z\in D$, the pseudohyperbolic distance from w to z, denoted $d(w,z)$, is the metric on D defined by $d(w,z)=|b_w(z)|$, where b_w is the Möbius transform from D onto D defined by

$$b_w(z)=\frac{w-z}{1-\bar{w}z}.$$

LEMMA 5. *Let $h\in\mathcal{U}$ and let $\epsilon>0$. Then there exists $\delta>0$ such that*

$$|h(b_z(\lambda))-h(b_z(\lambda'))|<\epsilon \quad \textit{for all } z\in D$$

whenever $\lambda,\lambda'\in D$ and $d(\lambda,\lambda')<\delta$.

Proof. By Lemma 4.4 of [8], $\mathcal{U}$ is the closed subalgebra of $L^\infty(D,dA)$ generated by $H^\infty(D)$ and the complex conjugates of $H^\infty(D)$ functions. By Lemma 4.6 of [2], each $H^\infty(D)$ function is uniformly continuous as a map from D (with the pseudohyperbolic metric) to the complex plane $\mathbf{C}$ (with the usual metric). Thus each function in $\mathcal{U}$ also has this property. Finally,

$$d(b_z(\lambda),b_z(\lambda'))=d(\lambda,\lambda')$$

for all $z,\lambda,\lambda'\in D$ (see [7, Chap. I, Lemma 1.2]), and so the uniform continuity of $h\in\mathcal{U}$ (as discussed above) gives the equicontinuity of the family $\{h\circ b_z : z\in D\}$, as desired. □

The following theorem is the main result of this section. Because $\mathcal{U}$ contains all bounded harmonic functions on D, the following theorem, together with line 4 of Table 2, shows that if $f \in \text{BMOA}$ then

$$(1-|z|^2)^n f^{(n)}(z) \in \mathcal{U}$$

for every positive integer n. For functions f in $H^\infty(D)$, this result was proved by Hoffman [8, Thm. 4.5 and Lemma 4.4]).

The corollary in [11, p. 254] states that $\mathcal{U}$ may be replaced by $L^p(D, dA)$ (for $1<p<\infty$) in statements (a), (b), and (c) of Theorem 6. Thus the following theorem demonstrates one sense in which $\mathcal{U}$ is a good replacement for $L^\infty(D, dA)$.

THEOREM 6. *Let f be an analytic function on D. Then the following are equivalent:*

(a) $f \in P(\mathcal{U})$;
(b) $(1-|z|^2)^n f^{(n)}(z) \in \mathcal{U}$ *for every positive integer n;*
(c) $(1-|z|^2)^n f^{(n)}(z) \in \mathcal{U}$ *for some positive integer n.*

Proof. First suppose that (a) holds, so there exists $g \in \mathcal{U}$ such that $f = Pg$. Thus

$$f(z) = \frac{1}{\pi}\int_D \frac{g(w)}{(1-z\bar{w})^2}\, dA(w)$$

for every $z \in D$. To prove that (b) holds, let n be a positive integer. Differentiating both sides of the above equation n times, we have

$$f^{(n)}(z) = \frac{(n+1)!}{\pi}\int_D \frac{g(w)(\bar{w})^n}{(1-z\bar{w})^{n+2}}\, dA(w)$$

for every $z \in D$. For convenience, let $h(w) = g(w)(\bar{w})^n$, so $h \in \mathcal{U}$. Make the change of variables $w = b_z(\lambda)$ in the above integral (here λ is the new variable, because z is fixed in the integral), so that $dA(w) = |b_z'(\lambda)|^2\, dA(\lambda)$, and the above integral formula becomes

$$(1-|z|^2)^n f^{(n)}(z) = \frac{(n+1)!}{\pi}\int_D \frac{(1-z\bar{\lambda})^{n+2}}{|1-z\bar{\lambda}|^4} h(b_z(\lambda))\, dA(\lambda) \tag{7}$$

for every $z \in D$.

For $z, \lambda \in D$, let

$$H(z,\lambda) = \frac{(n+1)!}{\pi}\frac{(1-z\bar{\lambda})^{n+2}}{|1-z\bar{\lambda}|^4} h(b_z(\lambda)),$$

so that

$$(1-|z|^2)^n f^{(n)}(z) = \int_D H(z,\lambda)\, dA(\lambda) \tag{8}$$

for every $z \in D$.

Let $\epsilon > 0$. We claim that there exists $t \in (0,1)$ such that

$$\left|\int_{D\setminus tD} H(z,\lambda)\, dA(\lambda)\right| < \epsilon \quad \text{for every } z \in D. \tag{9}$$

If $n \geq 2$ then H is bounded on $D \times D$, and the claim is clearly true. To verify the claim when $n=1$, note that the function sending w to $(1-w)^{-1/2}$ is analytic on D, so we have a power series expansion

$$\frac{1}{(1-w)^{1/2}} = \sum_{j=0}^{\infty} a_j w^j$$

for $w \in D$. Thus, if $z \in D$ we have

$$\frac{1}{(1-\bar{z}re^{i\theta})^{1/2}} = \sum_{j=0}^{\infty} a_j (\bar{z})^j r^j e^{ji\theta}$$

for all $r \in (0,1)$ and all real θ. Thus, if $z \in D$ and $r \in (0,1)$, then

$$\begin{aligned}\int_0^{2\pi} \frac{1}{|1-\bar{z}re^{i\theta}|}\, d\theta &= 2\pi \sum_{j=0}^{\infty} |a_j|^2 |z|^{2j} r^{2j} \leq 2\pi \sum_{j=0}^{\infty} |a_j|^2 r^{2j} \\ &= \int_0^{2\pi} \frac{1}{|1-re^{i\theta}|}\, d\theta.\end{aligned}$$

Thus, if $t \in (0,1)$ and $z \in D$, then

$$\begin{aligned}\int_{D\setminus tD} \frac{1}{|1-\bar{z}\lambda|}\, dA(\lambda) &= \int_t^1 r \int_0^{2\pi} \frac{1}{|1-\bar{z}re^{i\theta}|}\, d\theta\, dr \leq \int_t^1 r \int_0^{2\pi} \frac{1}{|1-re^{i\theta}|}\, d\theta\, dr \\ &= \int_{D\setminus tD} \frac{1}{|1-\lambda|}\, dA(\lambda).\end{aligned} \tag{10}$$

The function that sends λ to $1/|1-\lambda|$ is integrable on D (as is easily verified by changing to polar coordinates based at the point 1), so the above inequality shows that there exists $t \in (0,1)$ such that

$$\int_{D\setminus tD} \frac{1}{|1-\bar{z}\lambda|}\, dA(\lambda) < \epsilon \quad \text{for every } z \in D.$$

If $n=1$ then $|H(z,\lambda)|$ is bounded by a constant (independent of z and λ) times $1/|1-\bar{z}\lambda|$, so the above inequality establishes the claim made at the beginning of this paragraph.

By Theorem 4.5 and Lemma 4.4 of Hoffman's paper [8], if $F \in H^\infty(D)$ then $F(b_z(\lambda))$ (as a function of z) is in $\mathcal{U}$ for every $\lambda \in D$. Because h is in the norm closed algebra generated by $H^\infty(D)$ and the complex conjugates of $H^\infty(D)$ functions, we conclude that $h(b_z(\lambda))$ (as a function of z) is in $\mathcal{U}$ for every $\lambda \in D$.

Lemma 5 implies that the map from D to $\mathcal{U}$ that sends λ to $h(b_z(\lambda))$ is continuous as a map from D (with the pseudohyperbolic metric) to $\mathcal{U}$ (with its usual sup-norm metric). Because the topology induced on D by the pseudohyperbolic metric is the same as the usual topology on D, the map from D to $\mathcal{U}$ that sends λ to $h(b_z(\lambda))$ is continuous as a map from D (with the Euclidean metric) to $\mathcal{U}$ (with its sup-norm metric).

Thus the map from the compact set $t\bar{D}$ to $\mathcal{U}$ that sends λ to $H(z,\lambda)$ is uniformly continuous, so there exists $\delta > 0$ such that, for every $z \in D$, we have

$$|H(z,\lambda)-H(z,\lambda')|<\epsilon \quad \text{whenever } \lambda,\lambda'\in t\bar{D} \text{ and } |\lambda-\lambda'|<\delta.$$

Write $t\bar{D}$ as a disjoint union of non-empty measurable sets $E_1, \ldots, E_m$, where each E_j has diameter less than δ. For each j, choose a point $\lambda_j \in E_j$. Then

$$\left|\int_{tD} H(z,\lambda)\,dA(\lambda)-\sum_{j=1}^{m} H(z,\lambda_j)A(E_j)\right|<\pi\epsilon \quad \text{for every } z\in D.$$

Thus the above inequality, along with equality (8) and inequality (9), shows that

$$\left|(1-|z|^2)^n f^{(n)}(z)-\sum_{j=1}^{m} H(z,\lambda_j)A(E_j)\right|<(1+\pi)\epsilon \quad \text{for every } z\in D.$$

The function

$$\sum_{j=1}^{m} H(z,\lambda_j)A(E_j)$$

is in $\mathfrak{U}$; thus $(1-|z|^2)^n f^{(n)}(z)\in\mathfrak{U}$, and so (b) holds, as desired.

Clearly (b) implies (c).

Now suppose that (c) holds, and let n be a positive integer such that

$$(1-|z|^2)^n f^{(n)}(z)\in\mathfrak{U}.$$

To prove that (a) holds, define functions g and h on D by

$$g(z)=\sum_{j=0}^{2n}\frac{f^{(j)}(0)}{j!}z^j, \qquad h(z)=\sum_{j=2n+1}^{\infty}\frac{f^{(j)}(0)}{j!}z^j,$$

so $f=g+h$. Clearly $(1-|z|^2)^n g^{(n)}(z)\in\mathfrak{U}$, so $(1-|z|^2)^n h^{(n)}(z)\in\mathfrak{U}$. Let

$$s(z)=\frac{(1-|z|^2)^n h^{(n)}(z)}{n!\,(\bar{z})^n}.$$

Clearly s is the uniform limit of functions of the form

$$\frac{(1-|z|^2)^n h^{(n)}(z)}{n!\,(\bar{z})^n}Q(z),$$

where $Q\in C(\bar{D})$ is identically 0 on a small neighborhood of the origin and is identically 1 outside another small neighborhood of the origin. Because $(1-|z|^2)^n h^{(n)}(z)\in\mathfrak{U}$ and $Q(z)/(\bar{z})^n\in C(\bar{D})\subset\mathfrak{U}$, we see that $s\in\mathfrak{U}$. By Lemma 4, we have $h=P(s)$. Thus $f=g+P(s)=P(g+s)$, and so $f\in P(\mathfrak{U})$. Thus (c) implies (a), and the proof of the theorem is complete. □

We can now answer the question mentioned in the introduction.

COROLLARY 11. *$P(\mathfrak{U})$ is properly contained in the Bloch space $\mathcal{B}$.*

Proof. Let f be the analytic function on D defined by

$$f(z)=\sum_{n=1}^{\infty} z^{n!}.$$

By Lemma 2.1(b) of [1], $f\in\mathcal{B}$. We will show that $f\notin P(\mathfrak{U})$.

Suppose that $f \in P(\mathcal{U})$. Then by Theorem 6, the function $(1-|z|^2)f'(z)$ is in $\mathcal{U}$. The function $z/(1+|z|)$ is continuous on the closed disk and thus is in $\mathcal{U}$; multiplying $(1-|z|^2)f'(z)$ by $z/(1+|z|)$, we see that $(1-|z|)zf'(z) \in \mathcal{U}$. Because $\mathcal{U}$ is the algebra generated by the bounded harmonic functions, and because every bounded harmonic function on D has a nontangential limit at almost every point of ∂D, we conclude that $(1-|z|)zf'(z)$ has a nontangential limit at almost every point of ∂D. To complete the proof, we will show that this last statement leads to a contradiction.

Note that

$$(1-|z|)zf'(z) = (1-|z|)\sum_{n=1}^{\infty} n!\, z^{n!}.$$

For each positive integer N, let

$$r_N = 1 - \frac{1}{N!}.$$

If $N \geq 2$, then

$$\begin{aligned}(1-r_N)\sum_{n=1}^{N-1} n!\,(r_N)^{n!} &= \frac{1}{N!}\sum_{n=1}^{N-1} n!\left(1-\frac{1}{N!}\right)^{n!} \leq \frac{1}{N!}\sum_{n=1}^{N-1} n! \\ &= \frac{1}{N!}\left(\sum_{n=1}^{N-2} n! + (N-1)!\right) \\ &\leq \frac{1}{N!}((N-2)(N-2)! + (N-1)!) \leq \frac{2}{N},\end{aligned}$$

and so

$$(1-r_N)\sum_{n=1}^{N-1} n!\,(r_N)^{n!} \to 0 \quad \text{as } N \to \infty. \tag{12}$$

Also, if $N \geq 1$ then

$$\begin{aligned}(1-r_N)\sum_{n=N+1}^{\infty} n!\,(r_N)^{n!} &= \sum_{n=N+1}^{\infty} \frac{n!}{N!}\left[\left(1-\frac{1}{N!}\right)^{N!}\right]^{n!/N!} \leq \sum_{n=N+1}^{\infty} \frac{n!}{N!}\left[\frac{1}{e}\right]^{n!/N!} \\ &\leq \sum_{k=N+1}^{\infty} k\left[\frac{1}{e}\right]^{k},\end{aligned}$$

where the last inequality holds because, for each $n \geq N+1$, $n!/N!$ is some integer greater than or equal to $N+1$. The last series above converges, and so

$$(1-r_N)\sum_{n=N+1}^{\infty} n!\,(r_N)^{n!} \to 0 \quad \text{as } N \to \infty. \tag{13}$$

Now let $\zeta \in \partial D$. If N is a positive integer greater than 1, then

$$\begin{aligned}&(1-|r_N\zeta|)r_N\zeta f'(r_N\zeta) \\ &\quad = (1-r_N)\sum_{n=1}^{\infty} n!\,(r_N\zeta)^{n!} \\ &\quad = (1-r_N)\sum_{n=1}^{N-1} n!\,(r_N\zeta)^{n!} + \left(1-\frac{1}{N!}\right)^{N!}\zeta^{N!} + (1-r_N)\sum_{n=N+1}^{\infty} n!\,(r_N\zeta)^{n!}.\end{aligned}$$

By statements (12) and (13), the first and third terms above tend to zero as $N \to \infty$. The absolute value of the middle term tends to $1/e$. Thus if $\zeta \in \partial D$ is such that $(1-|z|)zf'(z)$ has a nontangential limit at ζ, then this limit is nonzero, and so f' must have a nontangential limit of ∞ at ζ. Thus f' has a nontangential limit of ∞ at almost every point of ∂D. However, by Privalov's theorem (see [5, Cor. 1 to Thm. 8.1]; replace f by $1/f$ to obtain the version needed here), no analytic function can have a nontangential limit of ∞ at almost every point of ∂D. This contradiction completes the proof. □

The Maximal Ideal Space of $H^\infty(D)$

Let $\mathfrak{M}$ denote the maximal ideal space of $H^\infty(D)$. Thus $\mathfrak{M}$ is the set of multiplicative linear functionals from $H^\infty(D)$ onto the complex numbers **C**. With the usual topology (the weak-star topology that $\mathfrak{M}$ inherits as a subset of the dual of $H^\infty(D)$), $\mathfrak{M}$ becomes a compact Hausdorff space. We will think of the disk D as a subset of $\mathfrak{M}$ by identifying each point of the disk with the multiplicative linear functional of point evaluation. The topology that D inherits as a subset of $\mathfrak{M}$ coincides with the usual topology on D.

Lemma 4.4 of [8] allows us to identify $\mathfrak{U}$ with $C(\mathfrak{M})$ in the following manner:

A complex-valued function u defined on D is in $\mathfrak{U}$ if and only if u can be extended to a continuous, complex-valued function defined on $\mathfrak{M}$.

Carleson's corona theorem (see [7, Chap. VIII]) states that D is dense in $\mathfrak{M}$, so if a function u on D has a continuous extension to $\mathfrak{M}$ (or any subset of $\mathfrak{M}$) then the extension is unique. We will denote a function u on D and its continuous extension to any subset of $\mathfrak{M}$ by the same symbol u.

Theorem 6 and Corollary 11 from the previous section show that if n is a positive integer, then there is a function f in the Bloch space such that $(1-|z|^2)^n f^{(n)}(z)$ cannot be extended to a continuous function on $\mathfrak{M}$. But Theorem 16, our first major result in this section, states that $(1-|z|^2)^n f^{(n)}(z)$ extends to a continuous function on a large subset of $\mathfrak{M}$ for every function f in the Bloch space and every positive integer n. This large subset of $\mathfrak{M}$ is the set of nontrivial Gleason parts (defined below). Theorem 17, our second major result in this section, states that if $(1-|z|^2)^n f^{(n)}(z)$ has a continuous extension to some trivial Gleason part of $\mathfrak{M}$, then the value of the extension at this trivial Gleason part is 0.

For $\varphi \in \mathfrak{M}$, the Gleason part of φ is defined to be the set of all $\tau \in \mathfrak{M}$ such that

$$\sup\{|\tau(f)| : f \in H^\infty(D),\ f(D) \subset D, \text{ and } \varphi(f) = 0\} < 1.$$

The Gleason part of φ is called *nontrivial* if it contains more than just φ. We define G to be the union of all the nontrivial parts in $\mathfrak{M} \setminus D$. Hoffman's paper [8] contains several remarkable descriptions of the Gleason parts of $\mathfrak{M}$. In particular, G is a dense open subset of $\mathfrak{M} \setminus D$ (see [8, pp. 89, 102]; note that our $D \cup G$ is equal to Hoffman's G).

The topological space $D \cup G$ (which of course has the topology it inherits as a subset of $\mathfrak{M}$) is not compact. Nevertheless, the following proposition states that every complex-valued, continuous function on $D \cup G$ is bounded. In this connection, we note that although $D \cup G$ is not a closed subset of $\mathfrak{M}$, it is a sequentially closed subset of $\mathfrak{M}$ (see [3, Cor. 9]).

The proof of the following proposition will use the connection between G and interpolating sequences discovered by Hoffman. Recall that a sequence $(z_n)_{n=1}^{\infty}$ in D is called *interpolating* if, for every bounded sequence $(\alpha_n)_{n=1}^{\infty} \subset \mathbf{C}$, there exists $f \in H^{\infty}(D)$ such that $f(z_n) = \alpha_n$ for every n.

PROPOSITION 14. *Let $u \in C(D \cup G)$. Then u is bounded on $D \cup G$.*

Proof. Suppose that $u \in C(D \cup G)$ and that u is unbounded on $D \cup G$. By the corona theorem, D is dense in $D \cup G$. Thus u is unbounded on D, and so there is a sequence $(z_n)_{n=1}^{\infty}$ in D such that $|u(z_n)| \to \infty$ as $n \to \infty$. Clearly we must have $|z_n| \to 1$ as $n \to \infty$. Dropping to a subsequence, we can assume without loss of generality that $(z_n)_{n=1}^{\infty}$ is an interpolating sequence (see [6, Thms. 9.1 and 9.2]), so $(z_n)_{n=1}^{\infty}$ has a limit point $\varphi \in G$ [8, Thm. 5.5]. The continuity of u on $D \cup G$ now implies that $|u(\varphi)| = \infty$, which is nonsense because u is supposed to be a complex-valued function on $D \cup G$. This contradiction completes the proof. □

We need the following lemma to prove Theorem 16. Here and throughout the paper, when we write that a function on D is in $C(D \cup G)$, we mean that the function has a continuous, complex-valued extension to $D \cup G$.

LEMMA 15. *Let u be a bounded, continuous, complex-valued function on D such that*

$$\sup\left\{(1-|z|^2)\limsup_{w \to z}\left|\frac{u(w)-u(z)}{w-z}\right| : z \in D\right\} < \infty.$$

Then $u \in C(D \cup G)$.

Proof. In Theorem 4 of [4], Brown and Gauthier proved that a normal meromorphic function on D can be extended to a continuous function on $D \cup G$ with values in the Riemann sphere $\mathbf{C} \cup \{\infty\}$. Although Brown and Gauthier restrict their attention to normal meromorphic functions, the only property of such functions that they use in their proof is that each normal meromorphic function is uniformly continuous as a map from D (with the hyperbolic metric) to the Riemann sphere (with the chordal metric); see [4, Lemma 2]. We have assumed that u is bounded, so no continuous extension of u to a function on $D \cup G$ with values in $\mathbf{C} \cup \{\infty\}$ could take on the value ∞. Thus to complete the proof we need only show that u is uniformly continuous as a map from D (with the hyperbolic metric) to the Riemann sphere (with the chordal metric). This statement follows immediately from our hypothesis on u and Theorems 1 and 3 of Lappan's paper [9]. □

The following theorem should be compared to Theorem 6 in the previous section, identifying $\mathcal{U}$ (in parts (b) and (c) of Theorem 6) with $C(\mathfrak{M})$.

THEOREM 16. *Let f be an analytic function on D. Then the following are equivalent:*

(a) $f \in \mathcal{B}$;
(b) $(1-|z|^2)^n f^{(n)}(z) \in C(D \cup G)$ *for every positive integer n;*
(c) $(1-|z|^2)^n f^{(n)}(z) \in C(D \cup G)$ *for some positive integer n.*

Proof. First suppose that (a) holds, so that $f \in \mathcal{B}$. To show that (b) holds, let n be a positive integer. Define a function u on D by

$$u(z) = (1-|z|^2)^n f^{(n)}(z).$$

Thus u is a bounded continuous function on D.

We now show that u satisfies the hypothesis of Lemma 15. Let $z \in D$. Then

$$\begin{aligned}
&\limsup_{w \to z} \left| \frac{u(w)-u(z)}{w-z} \right| \\
&\quad \le \limsup_{w \to z} (1-|w|^2)^n \left| \frac{f^{(n)}(w)-f^{(n)}(z)}{w-z} \right| \\
&\qquad + |f^{(n)}(z)| \limsup_{w \to z} \left| \frac{(1-|w|^2)^n-(1-|z|^2)^n}{w-z} \right| \\
&\quad = (1-|z|^2)^n |f^{(n+1)}(z)| + |f^{(n)}(z)| \limsup_{w \to z} \left| \frac{(1-|w|^2)^n-(1-|z|^2)^n}{w-z} \right| \\
&\quad \le (1-|z|^2)^n |f^{(n+1)}(z)| + 2n(1-|z|^2)^{n-1} |f^{(n)}(z)|.
\end{aligned}$$

Thus

$$\begin{aligned}
(1-|z|^2) \limsup_{w \to z} \left| \frac{u(w)-u(z)}{w-z} \right| &\le (1-|z|^2)^{n+1} |f^{(n+1)}(z)| \\
&\quad + 2n(1-|z|^2)^n |f^{(n)}(z)|,
\end{aligned}$$

and by Lemma 3 the right-hand side of the above inequality is bounded by a constant independent of $z \in D$. Thus, by Lemma 15, we see that

$$(1-|z|^2)^n f^{(n)}(z) \in C(D \cup G),$$

completing the proof that (a) implies (b).

Clearly (b) implies (c).

Now suppose that (c) holds, so that there is a positive integer n such that

$$(1-|z|^2)^n f^{(n)}(z) \in C(D \cup G).$$

By Proposition 14, $(1-|z|^2)^n f^{(n)}(z)$ is bounded on D, which implies by Lemma 3 that $f \in \mathcal{B}$, proving that (c) implies (a) and completing the proof of the theorem. □

We say that φ is a *trivial* Gleason part of $\mathfrak{M}$ if the Gleason part of φ contains only φ. Theorem 16 says that for every Bloch function f and every positive integer n, the function $(1-|z|^2)^n f^{(n)}(z)$ can be continuously extended to the nontrivial Gleason parts. The function $(1-|z|^2)^n f^{(n)}(z)$ may extend continuously to some other points of $\mathfrak{M}$. The following theorem says that when this happens, the extension equals 0 on every trivial Gleason part on which the extension is defined. Theorems 6 and 17 combine to say that if $f \in P(\mathcal{U})$ (in particular, if $f \in$ BMOA) and n is a positive integer, then

$$(1-|z|^2)^n f^{(n)}(z) = 0$$

on every trivial Gleason part of $\mathfrak{M}$.

For $\varphi \in \mathfrak{M}$, $c \in \mathbf{C}$, and u a function on D, the statement that $\lim_{z \to \varphi} u(z)$ exists (and equals c) means that u has an extension to a continuous, complex-valued function on $D \cup \{\varphi\}$ (and that this extension equals c at φ).

THEOREM 17. *Let $f \in \mathcal{B}$, let n be a positive integer, and let φ be a trivial Gleason part of $\mathfrak{M}$. If $\lim_{z \to \varphi}(1-|z|^2)^n f^{(n)}(z)$ exists, then*

$$\lim_{z \to \varphi} (1-|z|^2)^k f^{(k)}(z) = 0$$

for every positive integer k.

Proof. Suppose that $\lim_{z \to \varphi}(1-|z|^2)^n f^{(n)}(z)$ exists. By subtracting a polynomial from f (which affects neither the hypothesis nor the conclusion), we can assume without loss of generality that f has a zero of order at least $2n$ at 0.

Define a function g on D by

$$g(w) = \frac{1}{n!} \frac{(1-|w|^2)^n f^{(n)}(w)}{(\bar{w})^n}.$$

By Lemma 4, $f = Pg$. Let k be a positive integer. From formula (7) we obtain

$$(1-|z|^2)^k f^{(k)}(z) = \frac{(k+1)!}{\pi} \int_D \frac{(1-z\bar{\lambda})^{k+2}}{|1-z\bar{\lambda}|^4} h(b_z(\lambda))\, dA(\lambda),$$

where h is the function on D defined by $h(w) = g(w)(\bar{w})^k$. Note that h is bounded on D (by Lemma 3). Let

$$c = \lim_{z \to \varphi} h(z);$$

the above limit exists because $\lim_{z \to \varphi}(1-|z|^2)^n f^{(n)}(z)$ exists.

Let $\epsilon > 0$. Then, as shown by inequality (9) and equality (8) (with n replaced by k), there exists $t \in (1-\epsilon, 1)$ such that

$$\left| (1-|z|^2)^k f^{(k)}(z) - \frac{(k+1)!}{\pi} \int_{tD} \frac{(1-z\bar{\lambda})^{k+2}}{|1-z\bar{\lambda}|^4} h(b_z(\lambda))\, dA(\lambda) \right| < \epsilon$$

for every $z \in D$.

We claim that there is an open set E of $\mathfrak{M}$ containing φ such that

$$|z| > 1-\epsilon \quad \text{for every } z \in E \cap D$$

and

$$|h(b_z(\lambda))-c|<\epsilon \quad \text{for every } z\in E\cap D \text{ and every } \lambda\in tD.$$

To prove this claim, note that $\{\tau\in D\cup\{\varphi\}:|h(\tau)-c|<\epsilon\}$ is an open subset of $D\cup\{\varphi\}$ containing φ. This open subset of $D\cup\{\varphi\}$ is equal to an open subset of $\mathfrak{M}$ intersected with $D\cup\{\varphi\}$. Because of the manner in which the topology on $\mathfrak{M}$ is defined, this means that there are finitely many functions $g_1,\ldots,g_m\in H^\infty(D)$ such that $\varphi(g_j)=0$ for each j and

$$\{z\in D:|g_j(z)|<\epsilon \text{ for each } j\}\subset\{z\in D:|h(z)-c|<\epsilon\}.$$

Because φ is a trivial Gleason part, Theorem 5.5 of [8] shows that, for each $\lambda\in D$,

$$\lim_{z\to\varphi} b_z(\lambda)=\varphi;$$

here we are taking the limit in $\mathfrak{M}$, so $b_z(\lambda)$ is thought of as the multiplicative linear functional on $H^\infty(D)$ of point evaluation at $b_z(\lambda)$. The last sentence implies that for each $\lambda\in D$ there is an open subset E_λ of $\mathfrak{M}$ such that $\varphi\in E_\lambda$ and

$$|g_j(b_z(\lambda))|<\epsilon \quad \text{for every } j\in\{1,\ldots,m\} \text{ and every } z\in E_\lambda\cap D.$$

Thus there is an open subset E of $\mathfrak{M}$ such that $\varphi\in E$ and

$$|g_j(b_z(\lambda))|<\epsilon \quad \text{for every } j\in\{1,\ldots,m\}, \text{ every } z\in E\cap D, \text{ and every } \lambda\in tD;$$

to see this, cover $t\bar{D}$ with a finite number of pseudohyperbolic disks with small pseudohyperbolic radii, let E be the intersection of the E_λ's corresponding to the pseudohyperbolic centers of these disks, and apply Lemma 5 to each g_j. If necessary, replace E by the smaller set $E\setminus\{z\in D:|z|\le 1-\epsilon\}$ to obtain a set E satisfying the claim.

Now

$$\left|(1-|z|^2)^k f^{(k)}(z)-\frac{(k+1)!}{\pi}c\int_{tD}\frac{(1-z\bar{\lambda})^{k+2}}{|1-z\bar{\lambda}|^4}\,dA(\lambda)\right|$$
$$<\epsilon\left(1+\frac{(k+1)!}{\pi}\int_{tD}|1-z\bar{\lambda}|^{k-2}\,dA(\lambda)\right)$$

for every $z\in E\cap D$. The integral on the right-hand side of the above inequality is bounded by $\pi 2^{k-2}$ if $k\ge 2$ and by

$$\int_D\frac{1}{|1-\lambda|}\,dA(\lambda)$$

if $k=1$ (see the derivation of inequality (10)). As noted after inequality (10), $1/|1-\lambda|$ is integrable on D, so regardless of the value of k, we see that for every $z\in E\cap D$,

$$\left|(1-|z|^2)^k f^{(k)}(z)-\frac{(k+1)!}{\pi}c\int_{tD}\frac{(1-z\bar{\lambda})^{k+2}}{|1-z\bar{\lambda}|^4}\,dA(\lambda)\right|$$

is bounded by a constant (depending only on k) times ϵ. Thus, to complete the proof we need only show that

$$\int_{tD} \frac{(1-z\bar{\lambda})^{k+2}}{|1-z\bar{\lambda}|^4}\, dA(\lambda)$$

is small for every $z \in E \cap D$.

Replacing the denominator in the last integral with $(1-z\bar{\lambda})^2(1-\bar{z}\lambda)^2$, we see that

$$\int_{tD} \frac{(1-z\bar{\lambda})^{k+2}}{|1-z\bar{\lambda}|^4}\, dA(\lambda) = \int_{tD} \frac{(1-z\bar{\lambda})^{k}}{(1-\bar{z}\lambda)^2}\, dA(\lambda).$$

Now make a change of variables $\lambda = tw$; we see that the complex conjugate of the last integral equals

$$t^2 \int_D \frac{(1-t\bar{z}w)^k}{(1-tz\bar{w})^2}\, dA(w),$$

and that by equation (1) this integral equals $\pi t^2(1-t^2|z|^2)^k$. Because $t \in (1-\epsilon, 1)$ and $|z| \in (1-\epsilon, 1)$ for every $z \in E \cap D$, we see that

$$\lim_{z \to \varphi} (1-|z|^2)^k f^{(k)}(z) = 0,$$

completing the proof. □

Let $\mathcal{J}$ denote the ideal of $C(\mathfrak{M})$ consisting of those functions in $C(\mathfrak{M})$ that vanish on every trivial Gleason part of $\mathfrak{M}$. The ideal $\mathcal{J}$ plays an important role in the theory of Toeplitz operators on the Bergman space; see [10, Thm. 6]. Because $\mathcal{U}$ can be identified with $C(\mathfrak{M})$, we can think of $\mathcal{J}$ as an ideal of $\mathcal{U}$. The following corollary states that even though $\mathcal{J}$ is strictly smaller than $\mathcal{U}$, the sets $P(\mathcal{U})$ and $P(\mathcal{J})$ are equal.

COROLLARY 18. $P(\mathcal{U}) = P(\mathcal{J})$.

Proof. The ideal $\mathcal{J}$ is contained in $\mathcal{U}$, so clearly $P(\mathcal{J}) \subset P(\mathcal{U})$.

To prove the other direction, let $f \in P(\mathcal{U})$. Let g be a polynomial of degree 2 such that $f-g$ has a zero at 0 of order at least 2, and let $h = f-g$. Theorem 6 now implies that $(1-|z|^2)h'(z) \in \mathcal{U}$, and so Theorem 17 implies that $(1-|z|^2)h'(z) \in \mathcal{J}$. Thus

$$\frac{(1-|z|^2)h'(z)}{\bar{z}} \in \mathcal{J}.$$

By Lemma 4,

$$h = P\left(\frac{(1-|z|^2)h'(z)}{\bar{z}}\right),$$

so $h \in P(\mathcal{J})$. Easy calculations, which we leave to the reader, show that $P(1-|z|)$ is the constant function $1/3$, that $P(z(1-|z|))$ is the function $z/3$, and that $P(z^2(1-|z|))$ is the function $z^2/21$. Hence every polynomial of degree 2 is in $P(\mathcal{J})$. Thus f, which is equal to $h+g$, is in $P(\mathcal{J})$, completing the proof. □

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Sheldon Axler
Department of Mathematics
Michigan State University
East Lansing, MI 48824

Kehe Zhu
Department of Mathematics
State University of New York
Albany, NY 12222