Intersection Homology and Free Group Actions on Witt Spaces

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1. Introduction

The study of free G actions of a finite group on manifolds has been of great interest to topologists for many years. The space form problem is just one of many problems involving free G actions on manifolds. Poincaré duality has played an important role in studying free G actions on manifolds.

It is natural to try to study free G actions on spaces with singular sets. Witt spaces are a class of PL spaces with singularities that satisfy a Poincaré duality theorem using the intersection homology of Goresky and MacPherson ([7], [8], [24]). Irreducible complex projective varieties are an important class of Witt spaces. Thus Witt spaces are a natural candidate to choose to study free G actions on spaces with singularities. We obtain several intersection homological restrictions on Witt spaces admitting a free G action. All group actions in this paper are assumed to be groups acting through PL homeomorphisms.

The first result is a restriction on the Euler characteristic. If a compact Witt space X^n of dimension n has a free G action that acts trivially on the intersection homology of X, then the Euler characteristic $\sum_{i=0}^{n}(-1)^i \dim(IH_i^{\bar{m}}(X;\mathbf{Q}))$ is zero.

Another restriction is given on the semicharacteristic. Suppose X^{4n+1} is a compact Witt space of dimension $4n+1$, G acts freely on X, and G acts trivially on the intersection homology of X; then either the semicharacteristic $\sum_{i=0}^{2n}(-1)^i \dim(IH_i^{\bar{m}}(X;\mathbf{Q}))$ is even or G is the direct product of a cyclic 2-group and an odd order group.

Another result is a restriction on the higher signature of a compact Witt space X. Let $L(X)$ denote the L class of X as defined by Goresky and MacPherson, and let $f: X \to B\pi$ classify the universal cover of X. Then $f_*(L(X)) \in H_*(B\pi;\mathbf{Q})$ is the higher signature of X. Suppose X has a free G action such that G acts trivially on the fundamental group and trivially on the intersection homology of X with any local coefficient system. Consider a representation $\rho: \pi \to \mathrm{Sp}(2l,\mathbf{R})$ of the fundamental group of X into the real

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symplectic group in $2l$ variables, that is, the group of automorphisms of $\mathbf{R}^{2l}$ preserving the alternate form $\sum_{i=1}^{l}(x_i y_{l+i} - x_{l+i} y_i)$. We will abbreviate $\mathrm{Sp}(2l, \mathbf{R})$ by Sp. Let $B\rho: B\pi \to B\mathrm{Sp}$ be the induced map on classifying spaces. Then $\langle (B\rho)^*(u), f_*(L(X)) \rangle = 0$ for all $u \in H^*(B\mathrm{Sp}; \mathbf{Q})$.

Davis, Rosenberg, and Weinberger ([4], [22], [25], [26], [27]) have studied homological restrictions on manifolds admitting a free G action.

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2. Preliminaries

In this section we give a few preliminary definitions and show how to extend some well-known results about closed Witt spaces to noncompact Witt spaces and Witt spaces with boundary. We assume the reader is familiar with the basic results of intersection homology as given in [7]. The vanishing of the Euler characteristic of a pseudomanifold admitting a free intersection homologically trivial G action is then given as an application of the intersection homological Lefschetz fixed point theorem.

DEFINITION 2.1. A PL stratified pseudomanifold X of dimension n is a PL space X with a filtration

$$X = X_n \supset X_{n-1} = X_{n-2} \supset X_{n-3} \supset \cdots \supset X_1 \supset X_0 \supset X_{-1} = \emptyset$$

by closed PL subspaces such that

(1) $S_{n-k} = X_{n-k} \setminus X_{n-k-1}$ is a PL manifold of dimension $n-k$ (if S_{n-k} is not empty);
(2) $X \setminus X_{n-2}$ is an oriented PL manifold of dimension n that is dense in X; and
(3) for each $x \in S_{n-k}$, there is a compact PL stratified pseudomanifold of dimension $k-1$ with stratification

$$L = L_{k-1} \supset L_{k-3} \supset \cdots \supset L_0 \supset L_{-1} = \emptyset$$

and a PL homeomorphism h of an open neighborhood U of x (called a *distinguished neighborhood* of x) on the product $B \times \mathring{c}L$, where B is an open ball about x in S_{n-k} and $\mathring{c}L$ is the open cone $L \times [0, \infty)/(x, 0) \sim (x', 0)$ over L. Moreover, h preserves the stratification; namely, h maps $U \cap X_{n-l}$ PL homeomorphically onto $B \times \mathring{c}L_{k-l-1}$ (by definition, the cone over the empty set is just a point).

The subset X_{n-2} is usually called the *singular set* Σ of the PL stratified pseudomanifold X. In the relative situation, where we have a boundary, we place a collar condition on a neighborhood of the boundary.

DEFINITION 2.2. The pair (X, A) is an n-dimensional stratified pseudomanifold with boundary if

(1) A is an $(n-1)$-dimensional pseudomanifold with singular set $\Sigma(A)$;
(2) there exists a closed subspace $\Sigma(X)$ such that $X\backslash(A\cup\Sigma(X))$ is an oriented n-dimensional manifold that is dense in X; and
(3) A has a collared neighborhood: for a stratification

$$X=X_n\supset\Sigma(X)=X_{n-2}\supset\cdots\supset X_0\supset X_{-1}=\emptyset$$

of X and a stratification

$$A=A_{n-1}\supset\Sigma(A)=A_{n-3}\supset\cdots\supset A_0\supset A_{-1}=\emptyset$$

of A, each satisfying the conditions of Definition 2.1, there exists a closed neighborhood N of A in X and a PL homeomorphism h: $N\to[0,1]\times A$ such that $h(X_j\cap N)=[0,1]\times A_{j-1}$.

We let $IH_i^{\bar{p},lf}(X;\mathbf{Q})$ denote the Borel–Moore intersection homology groups of X. The main interest in studying intersection homology is the following Poincaré duality theorem.

THEOREM 2.3 ([7]; [10] for the noncompact case). *Let X be an n-dimensional pseudomanifold, $i+j=n$, and $\bar{p}+\bar{q}=\bar{t}$. Then the augmented pairing*

$$IH_i^{\bar{p}}(X;\mathbf{Q})\times IH_j^{\bar{q},lf}(X;\mathbf{Q})\xrightarrow{\cap} IH_0^{\bar{t}}(X;\mathbf{Q})\xrightarrow{\epsilon}\mathbf{Q}$$

is a nondegenerate bilinear pairing.

DEFINITION 2.4. A Witt space X^n is a pseudomanifold satisfying the following link condition: Let $x\in S_{n-l(x)-1}=X_{n-l(x)-1}\backslash X_{n-l(x)-2}$, for $l(x)$ even. Suppose $B^{n-l(x)-1}\times\mathring{c}L^{l(x)}$ is a distinguished neighborhood of x. Then $IH_{l(x)/2}^{\bar{m}}(L^{l(x)};\mathbf{Q})=0$, for each x in an odd codimensional stratum. A Witt space X^n with boundary is a pseudomanifold with boundary such that both $X\backslash\partial X$ and ∂X are Witt spaces.

There is an isomorphism between the intersection homology groups with middle perversities for Witt spaces.

THEOREM 2.5. *Suppose $(X,\partial X)$ is an n-dimensional Witt space with boundary. Then, for all i, we have the isomorphisms:*

(1) $IH_i^{\bar{m}}(X;\mathbf{Q})\cong IH_i^{\bar{n}}(X;\mathbf{Q})$,
(2) $IH_i^{\bar{m},lf}(X;\mathbf{Q})\cong IH_i^{\bar{n},lf}(X;\mathbf{Q})$,
(3) $IH_i^{\bar{m}}(X,\partial X;\mathbf{Q})\cong IH_i^{\bar{n}}(X,\partial X;\mathbf{Q})$, *and*
(4) $IH_i^{\bar{m},lf}(X,\partial X;\mathbf{Q})\cong IH_i^{\bar{n},lf}(X,\partial X;\mathbf{Q})$.

This theorem was proved by Siegel [23, Prop. 3.7] in the closed case. One need only replace $IH_i^{\bar{m}}(X;\mathbf{Q})$ with either the Borel–Moore analogue $IH_i^{\bar{m},lf}(X;\mathbf{Q})$ or the relative analogues $IH_i^{\bar{m}}(X,\partial X;\mathbf{Q})$ or $IH_i^{\bar{m},lf}(X,\partial X;\mathbf{Q})$ in Section 3 of [23] to extend the result to the noncompact and relative cases.

Lefschetz duality takes the following form.

THEOREM 2.6 (Lefschetz Duality). *Suppose X is an n-dimensional pseudomanifold with boundary, $i+j=n$, and $\bar{p}+\bar{q}=\bar{t}$. The augmented intersection homology products on intersection homology with rational coefficients*

$$IH_i^{\bar{p},lf}(X;\mathbf{Q})\times IH_j^{\bar{q}}(X,\partial X;\mathbf{Q}) \xrightarrow{\cap} IH_0^{\bar{t}}(X;\mathbf{Q}) \xrightarrow{\epsilon} \mathbf{Q}$$

and

$$IH_i^{\bar{p}}(X;\mathbf{Q})\times IH_j^{\bar{q},lf}(X,\partial X;\mathbf{Q}) \xrightarrow{\cap} IH_0^{\bar{t}}(X;\mathbf{Q}) \xrightarrow{\epsilon} \mathbf{Q}$$

are nondegenerate pairings.

The proof of this theorem is similar to the proof of Poincaré duality given by Goresky and MacPherson [7]. The main difference is a fuller use of McCrory's general position theorem [14]. See [3] for details.

Let G be a finite group. We consider free actions of G on a compact Witt space X that induces a trivial action on $IH_*^{\bar{m}}(X;\mathbf{Q})$. There is a restriction on the Euler characteristic.

THEOREM 2.7. *Suppose X is a compact n-dimensional Witt space. Suppose G is a finite group which acts freely through PL homeomorphisms on X and which induces a trivial action on the intersection homology groups $IH_i^{\bar{m}}(X;\mathbf{Q})$ for all i. Then $I\chi(X;\mathbf{Q})=\Sigma_{i=0}^{n}(-1)^i\dim(IH_i^{\bar{m}}(X;\mathbf{Q}))=0$.*

Proof. Let $g\neq 1\in G$. Since G acts freely, g has no fixed points. By the intersection homological fixed point theorem [9, Thm. I], $IL(g)=0$. Because G acts trivially on the intersection homology of X, we have $I\chi(X)=IL(g)=0$. □

3. The Intersection Semicharacteristic

For an odd-dimensional Witt space X^{2k+1} without boundary, the Euler characteristic $I\chi(X;\mathbf{Q})=\Sigma_{i=0}^{2k+1}(-1)^i\dim(IH_i^{\bar{m}}(X;\mathbf{Q}))$ is zero. However, one is naturally led to consider the intersection semicharacteristic $\widehat{I\chi}_{1/2}(X;\mathbf{Q})=\Sigma_{i=0}^{k}(-1)^i\dim(IH_i^{\bar{m}}(X;\mathbf{Q}))$. Weinberger and Davis ([4], [26], [27]) obtain a restriction on the semicharacteristic for manifolds which admit a free homologically trivial action. This was subsequently generalized by Davis and Milgram [5]. In order to extend this restriction to the semicharacteristic of a Witt space using intersection homology, we need to consider an L-theoretic semicharacteristic. Let $\Omega_n^{\text{Witt}}(BG)$ be the bordism group of compact n-dimensional Witt spaces that admit a free G action. We define a bordism invariant semicharacteristic that takes values in the symmetric L-theory group $L_h^n(\mathbf{Q}G)$:

$$I\chi_{1/2}\colon \Omega_n^{\text{Witt}}(BG)\to L_h^n(\mathbf{Q}G).$$

This allows us to use the fact that Ω_n^{Witt} is a generalized homology theory. We play off the difference of transfer and induction in bordism theory and L-theory to extract a homological restriction on the ordinary intersection semicharacteristic $\widehat{I\chi}_{1/2}(X;\mathbf{Q})$.

Suppose G acts on a Witt space X. This action makes $IH_i^{\bar{m}}(X;\mathbf{Q})$ into a $\mathbf{Q}G$ module. Because $IH_i^{\bar{m}}(X;\mathbf{Q})$ is a finitely generated $\mathbf{Q}G$ module, the semicharacteristic

$$I\chi_{1/2}(X;\mathbf{Q}) = \sum_{i=0}^{k}(-1)^i[IH_i^{\bar{m}}(X;\mathbf{Q})] \in K_0(\mathbf{Q}G)$$

is a well-defined element in the projective class group of $\mathbf{Q}G$. We push this further into the reduced projective class group $\tilde{K}_0(\mathbf{Q}G)$ of $\mathbf{Q}G$. Consider the Rothenberg exact sequence [19]

$$(*)\qquad \cdots \to L_p^{2k+2}(\mathbf{Q}G) \to H^1(\mathbf{Z}_2;\tilde{K}_0(\mathbf{Q}G)) \to L_h^{2k+1}(\mathbf{Q}G) \to L_p^{2k+1}(\mathbf{Q}G) \to \cdots$$

for the projective and stably-free L-theory groups of $\mathbf{Q}G$. A calculation of $H^1(\mathbf{Z}_2;\tilde{K}_0(\mathbf{Q}G))$ shows that

$$H^1(\mathbf{Z}_2;\tilde{K}_0(\mathbf{Q}G)) = \frac{\{[P]\in\tilde{K}_0(\mathbf{Q}G) \mid [P]=[P^*]\}}{\{[P\oplus P^*] \mid [P]\in\tilde{K}_0(\mathbf{Q}G)\}}.$$

We use Theorems 2.3 and 2.5 to observe that

$$\begin{aligned} I\chi_{1/2}(X;\mathbf{Q}) - I\chi_{1/2}(X;\mathbf{Q})^* &= \sum_{i=0}^{k}(-1)^i[IH_i^{\bar{m}}(X;\mathbf{Q})] - \sum_{i=0}^{k}(-1)^i[IH_i^{\bar{m}}(X;\mathbf{Q})^*] \\ &= \sum_{i=0}^{2k+1}[IH_i^{\bar{m}}(X;\mathbf{Q})] = 0 \in \tilde{K}_0(\mathbf{Q}G). \end{aligned}$$

This calculation shows that $I\chi_{1/2}(X;\mathbf{Q})$ is an element of $H^1(\mathbf{Z}_2;\tilde{K}_0(\mathbf{Q}G))$. By composing with the map in $(*)$ we further push $I\chi_{1/2}(X;\mathbf{Q})$ into $L_h^{2k+1}(\mathbf{Q}G)$.

DEFINITION 3.1. Suppose X is a Witt space of dimension $2k+1$. The intersection semicharacteristic of X is the element

$$I\chi_{1/2}(X;\mathbf{Q}) = \sum_{i=0}^{k}(-1)^i[IH_i^{\bar{m}}(X;\mathbf{Q})] \in L_h^{2k+1}(\mathbf{Q}G).$$

Davis has shown that the utility of this semicharacteristic is limited to $n=1$ (mod 4) because $L_h^{4k+3}(\mathbf{Q}G)=0$ [4, Cor. 2.5(a)]. Cobordism invariance is a useful property of the intersection semicharacteristic. The proof is essentially the same as that of Proposition 3.2 of [4].

PROPOSITION 3.2. *Let $(X,\partial X)$ be a compact Witt space with boundary of dimension $2k+2$ which admits a free orientation preserving G action. Then $I\chi_{1/2}(\partial X;\mathbf{Q}) = 0 \in L_h^{2k+1}(\mathbf{Q}G)$.*

We consider the bordism theory defined by compact Witt spaces and pairs. Denote the Witt space bordism group of X by $\Omega_n^{\mathrm{Witt}}(X)$. We will show that Witt bordism theory is a generalized homology theory. Thus Brown Representation will give us a spectrum that represents reduced Witt cobordism $\tilde{\Omega}^*_{\mathrm{Witt}}$. We then show that the Witt spectrum $M\mathrm{Witt}_*$ is a module spectrum over the ring spectrum $M\mathrm{SO}_*$.

PROPOSITION 3.3. *Witt space bordism is a generalized homology on the category of topological pairs.*

The verification of functoriality, naturality, homotopy invariance, and exactness are straightforward. We need the next lemma to verify the excision axiom.

LEMMA 3.4. *Suppose P and Q are closed disjoint subsets of the compact n-dimensional Witt space X with boundary. There exists a compact Witt space $Y \subset X$ such that $P \subset Y \backslash \partial Y$, $Q \cap Y = \emptyset$, and ∂Y is bicollared in the interior of X.*

Proof. Since X is normal, there exist open subsets U and V of X such that $P \subset U \subset V \subset X \backslash Q$, $\bar{U} \subset V$, and $\bar{V} \subset X \backslash Q$. For each $x \in \bar{U}$, let $\mathring{c}L_x$ be a distinguished neighborhood of x such that $\mathring{c}L_x \subset V$. Let $\mathring{c}L_x^{1/2} \subset \mathring{c}L_x$ be the distinguished neighborhood of x whose join lines emanating from x are half as long as those of $\mathring{c}L_x$. Similarly, let $\mathring{c}L_x^{3/4} \subset \mathring{c}L_x$ be the distinguished neighborhood of x whose join lines emanating from x are three fourths as long as those of $\mathring{c}L_x$. Since $\bar{U}$ is compact, there is a finite subcover of $\bar{U}$ by open cones $\mathring{c}L_{x_1}^{1/2}, \mathring{c}L_{x_2}^{1/2}, \dots, \mathring{c}L_{x_r}^{1/2}$. Choose a triangulation T of X that contains $\mathring{c}L_{x_i}^{1/2}$, $\mathring{c}L_{x_i}^{3/4}$, and $\mathring{c}L_{x_i}$, for $i = 1, 2, \dots, r$, as subcomplexes. Each closed cone $cL_{x_i}^{3/4}$ is a Witt subspace of X such that $\partial(cL_{x_i}^{3/4}) = L_{x_i}^{3/4}$ is bicollared in the interior of X.

We inductively replace $cL_{x_i}^{3/4}$ with a distinguished neighborhood $c\tilde{L}_{x_i}$ of x_i that is PL homeomorphic to $cL_{x_i}^{3/4}$ such that $Y = \bigcup_{j=1}^{r} c\tilde{L}_{x_j}$ has the desired properties. Let $Y_1 = cL_{x_1}^{3/4}$. Then Y_1 is a compact Witt subspace of X such that $\mathring{c}L_{x_1}^{1/2} \subset Y_1 \subset \mathring{c}L_{x_1}$, and ∂Y_1 is bicollared in the interior of X. Suppose Y_{i-1} is a compact Witt subspace of X such that $\bigcup_{j=1}^{i-1} \mathring{c}L_{x_j}^{1/2} \subset Y_{i-1} \subset \bigcup_{j=1}^{i-1} \mathring{c}L_{x_j}$ and ∂Y_{i-1} is bicollared in the interior of X. Consider the collection C of simplices Δ in $\partial Y_{i-1} \cup cL_{x_i}^{3/4}$ with respect to the triangulation T such that $\partial(Y_{i-1} \cup cL_{x_i}^{3/4})$ is not bicollared in the interior of star(Δ). Choose a subtriangulation T' of T such that $\partial(\text{star}(\Delta))$ is bicollared in X for every $\Delta \in C$. Let $c\tilde{L}_{x_i} = cL_{x_i}^{3/4} \backslash \bigcup_{\Delta \in C} \text{int}(\text{star}(\Delta))$. Then $Y_i = Y_{i-1} \cup c\tilde{L}_{x_i}$ is a compact Witt subspace of X such that $\bigcup_{j=1}^{i} \mathring{c}L_{x_j}^{1/2} \subset Y_i \subset \bigcup_{j=1}^{i} \mathring{c}L_{x_j}$ and ∂Y_i is bicollared in the interior of X. Let $Y = Y_r$. Then Y is a compact Witt subspace of X such that $P \subset U \subset Y \subset V \subset X \backslash Q$ and ∂Y is bicollared in the interior of X. □

Proof of Proposition 3.3. Let $i: (B \backslash U, A \backslash U) \to (B, A)$ be the inclusion map for an open set U with $\bar{U} \subset \text{int}(A)$. We will show that $i: \Omega_n^{\text{Witt}}(B \backslash U, A \backslash U) \to \Omega_n^{\text{Witt}}(B, A)$ is surjective. The proof that i_* is injective is similar. Let

$$[(X, \partial X), f] \in \Omega_n^{\text{Witt}}(B, A).$$

Consider the closed subsets $P = f^{-1}(B \backslash \text{int}(A))$ and $Q = f^{-1}(\bar{U})$ in X. By Lemma 3.4 there exists a Witt subspace $Y \subset X$ such that $P \subset Y \backslash \partial Y$, $Y \cap Q = \emptyset$, and ∂Y is bicollared in the interior of X. Hence, $\partial(X \times I) \cup (X \backslash \text{int}(Y)) \times 1$ is a Witt subspace of $\partial(X \times I)$ because ∂Y is in the interior of X. Let

$$\pi_1 : X \times I \to X$$

be the projection onto the first factor. Then

$$[(X \times I, \partial X \times I \cup (X \backslash \mathrm{int}(Y)) \times 1), f \circ \pi_1]$$

is a cobordism in (B, A) which yields $i_*([(Y, \partial Y), f|_Y]) = [(X, \partial X), f]$. □

We will show that there is a spectrum corresponding to this generalized homology theory which has the structure of a MSO module spectrum. The importance of this structure is a result of Browder, Luilevicius, and Petersen [2].

THEOREM 3.5. *Any module spectrum over MSO becomes a generalized Eilenberg-MacLane spectrum after localizing at* 2.

THEOREM 3.6. *There is a spectrum M*Witt *associated to the Witt bordism theory which is a MSO module spectrum.*

Proof. Let D denote the Spanier-Whitehead duality functor on the category of spectra. Then $\tilde{\Omega}^n_{\mathrm{Witt}}(-) \equiv \tilde{\Omega}^{\mathrm{Witt}}_{-n}(D(-))$ is a generalized cohomology theory. Let $M\mathrm{Witt}_n$ be the associated spectrum that represents this theory. There is a natural pairing $\Omega^{\mathrm{SO}}_m(A) \times \Omega^{\mathrm{Witt}}_n(B) \to \Omega^{\mathrm{Witt}}_{m+n}(A \times B)$ due to the fact that $M \times X$ is a Witt space whenever M is a manifold and X is a Witt space. Consider the diagram below:

$$\begin{array}{ccc}
\tilde{\Omega}^{\mathrm{SO}}_{-*}(D(M\mathrm{SO})) \times \tilde{\Omega}^{\mathrm{Witt}}_{-*}(D(M\mathrm{Witt})) & \longrightarrow & \tilde{\Omega}^{\mathrm{Witt}}_{-*}(D(M\mathrm{SO}) \wedge D(M\mathrm{Witt})) \\
\| & & \| \\
\tilde{\Omega}^*_{\mathrm{SO}}(M\mathrm{SO}) \times \tilde{\Omega}^*_{\mathrm{Witt}}(M\mathrm{Witt}) & & \tilde{\Omega}^*_{\mathrm{Witt}}(M\mathrm{SO} \wedge M\mathrm{Witt}) \\
\| & & \| \\
[M\mathrm{SO}, M\mathrm{SO}] \times [M\mathrm{Witt}, M\mathrm{Witt}] & & [M\mathrm{SO} \wedge M\mathrm{Witt}, M\mathrm{Witt}].
\end{array}$$

Consider the homotopy class of maps between spectra induced from the identity map on MWitt and the identity map on MSO. Following the diagram around yields a pairing on spectra $\rho: M\mathrm{SO} \wedge M\mathrm{Witt} \to M\mathrm{Witt}$. Notice that the MSO ring spectrum pairing $\mu: M\mathrm{SO} \wedge M\mathrm{SO} \to M\mathrm{SO}$ is obtained from the diagram below by starting with the identity map on MSO and following the diagram:

$$\begin{array}{ccc}
\tilde{\Omega}^{\mathrm{SO}}_{-*}(D(M\mathrm{SO})) \times \tilde{\Omega}^{\mathrm{SO}}_{-*}(D(M\mathrm{SO})) & \longrightarrow & \tilde{\Omega}^{\mathrm{SO}}_{-*}(D(M\mathrm{SO}) \wedge D(M\mathrm{SO})) \\
\| & & \| \\
\tilde{\Omega}^*_{\mathrm{SO}}(M\mathrm{SO}) \times \tilde{\Omega}^*_{\mathrm{SO}}(M\mathrm{SO}) & & \tilde{\Omega}^*_{\mathrm{SO}}(M\mathrm{SO} \wedge M\mathrm{SO}) \\
\| & & \| \\
[M\mathrm{SO}, M\mathrm{SO}] \times [M\mathrm{SO}, M\mathrm{SO}] & & [M\mathrm{SO} \wedge M\mathrm{SO}, M\mathrm{SO}].
\end{array}$$

There is a diagram corresponding to $(M \times N) \times X = M \times (N \times X)$:

$$\begin{array}{ccc}\tilde{\Omega}_*^{SO}(X)\times\tilde{\Omega}_*^{SO}(Y)\times\tilde{\Omega}_*^{Witt}(Z) & \longrightarrow & \tilde{\Omega}_*^{SO}(X\wedge Y)\times\tilde{\Omega}_*^{Witt}(Z)\\ \downarrow & & \downarrow\\ \tilde{\Omega}_*^{SO}(X)\times\tilde{\Omega}_*^{Witt}(Y\wedge Z) & \longrightarrow & \tilde{\Omega}_*^{Witt}(X\wedge Y\wedge Z).\end{array}$$

The corresponding diagram on the spectra level and an application of the Yoneda lemma yields the following homotopy commutative diagram:

$$\begin{array}{ccc}M\mathrm{SO}\wedge M\mathrm{SO}\wedge M\mathrm{Witt} & \xrightarrow{1\wedge\rho} & M\mathrm{SO}\wedge M\mathrm{Witt}\\ {\scriptstyle\mu\wedge 1}\downarrow & & \downarrow{\scriptstyle\rho}\\ M\mathrm{SO}\wedge M\mathrm{Witt} & \xrightarrow{\rho} & M\mathrm{Witt}.\end{array}$$

The ring spectrum unit $\eta: S\to M\mathrm{SO}$ is obtained on the $M\mathrm{SO}$ bordism theory level by the fact that $pt\times M=M=M\times pt$. An argument similar to the previous argument shows that

$$\begin{array}{ccc}S\wedge M\mathrm{Witt} & \xrightarrow{\eta\wedge 1} & M\mathrm{SO}\wedge M\mathrm{Witt}\\ {\scriptstyle\cong}\downarrow & & \downarrow{\scriptstyle\rho}\\ M\mathrm{Witt} & \xrightarrow{1} & M\mathrm{Witt}\end{array}$$

is homotopy commutative. □

COROLLARY 3.7. *$M\mathrm{Witt}_{(2)}$ is a generalized Eilenberg-MacLane spectrum.*

We apply Corollary 3.7 to group actions on Witt spaces. Let $i: G_2\hookrightarrow G$ be the inclusion of a 2-Sylow subgroup. There is a commutative diagram

$$\begin{array}{ccc}\Omega_n^{Witt}(BG)_{(2)} & \xrightarrow{I\chi_{1/2}} & L_h^n(\mathbf{Q}G)\\ {\scriptstyle i^*}\downarrow & & {\scriptstyle i^*}\downarrow\\ \Omega_n^{Witt}(BG_2)_{(2)} & \xrightarrow{I\chi_{1/2}} & L_h^n(\mathbf{Q}G_2)\\ {\scriptstyle i_*}\downarrow & & {\scriptstyle i_*}\downarrow\\ \Omega_n^{Witt}(BG)_{(2)} & \xrightarrow{I\chi_{1/2}} & L_h^n(\mathbf{Q}G),\end{array}$$

where i^* is the map defined by restricting the G action to a G_2 action, $i_*(X)=G\times_{G_2}X=G\times X/(gh,x)\sim(g,hx)$, and $i_*(V)=\mathbf{Q}G\otimes_{\mathbf{Q}G_2}V$. The groups on the right have exponent 2, so there is no loss of semicharacteristic information by localizing the bordism groups at 2. Because $M\mathrm{Witt}_{(2)}$ is a generalized Eilenberg-MacLane spectrum, the bordism groups $\Omega_*^{Witt}(-)_{(2)}$ behave like ordinary homology with respect to restriction and induction. Thus a transfer argument shows that

$$i_*\circ i^*: \Omega_n^{Witt}(BG)\to\Omega_n^{Witt}(BG)$$

is multiplication by $|G: G_2|$. Because $L_h^n(\mathbf{Q}G)$ has exponent 2 when n is odd, we obtain the following proposition.

PROPOSITION 3.8. *Let G act freely and orientation-preservingly on a compact Witt space X^n of odd dimension n. Then*

$$I\chi_{1/2}(X;\mathbf{Q})=i_*(I\chi_{1/2}(i^*(X);\mathbf{Q})).$$

We obtain restrictions on the intersection semicharacteristic for free intersection homologically trivial actions. In our first proposition we need no assumption about the action being trivial on intersection homology.

PROPOSITION 3.9. *Suppose $\mathbf{Z}_2 \times \mathbf{Z}_2$ acts freely through PL homeomorphisms and orientation-preservingly on a compact Witt space X of dimension $4n+1$. Then $\widehat{I\chi_{1/2}}(X;\mathbf{Q})$ is even.*

Proof. Consider the free $\mathbf{Z}_2$ action on X by restricting to a factor of $\mathbf{Z}_2 \times \mathbf{Z}_2$. The diagram below commutes

$$\begin{array}{ccccccc} X/\mathbf{Z}_2 & \xrightarrow{\tilde{\phi}} & B\mathbf{Z}_2 \times E\mathbf{Z}_2 & \xrightarrow{\tilde{p}_1} & B\mathbf{Z}_2 & \xrightarrow{\tilde{j}} & B\mathbf{Z}_2 \times E\mathbf{Z}_2 \\ \downarrow & & \downarrow \pi & & \| & & \downarrow \pi \\ X/\mathbf{Z}_2 \times \mathbf{Z}_2 & \xrightarrow{\phi} & B\mathbf{Z}_2 \times B\mathbf{Z}_2 & \xrightarrow{p_1} & B\mathbf{Z}_2 & \xrightarrow{j} & B\mathbf{Z}_2 \times B\mathbf{Z}_2, \end{array}$$

where $\tilde{p}_1$ and p_1 are projections onto the first factors, $\pi: B\mathbf{Z}_2 \times E\mathbf{Z}_2 \to B\mathbf{Z}_2 \times B\mathbf{Z}_2$ is the 2-fold covering projection, and $\tilde{j}$ and j are the inclusions into the first factors. Let $h_t: B\mathbf{Z}_2 \times E\mathbf{Z}_2 \to B\mathbf{Z}_2 \times E\mathbf{Z}_2$ be a deformation retract of $B\mathbf{Z}_2 \times E\mathbf{Z}_2$ to $B\mathbf{Z}_2 \times pt$ ($E\mathbf{Z}_2$ is contractible). Let $k_t = \pi \circ h_t$. Then $k: \pi \simeq j \circ \tilde{p}_1$. Let $\mathrm{tr}_*: \Omega^{\mathrm{Witt}}_{4n+1}(B\mathbf{Z}_2 \times B\mathbf{Z}_2) \to \Omega^{\mathrm{Witt}}_{4n+1}(B\mathbf{Z}_2 \times E\mathbf{Z}_2)$ denote the transfer map associated to the covering projection π. Then $\pi_* \circ \mathrm{tr}_*$ is multiplication by 2.

The image of tr_* is divisible by 2. Let $\alpha \in \Omega^{\mathrm{Witt}}_{4n+1}(B\mathbf{Z}_2 \times B\mathbf{Z}_2)$. Then

$$\begin{aligned} \mathrm{tr}_*(\alpha) &= (\tilde{j}_* \circ \tilde{p}_{1*}) \circ \mathrm{tr}_*(\alpha) \\ &= (\tilde{j}_* \circ p_{1*}) \circ (j_* \circ \tilde{p}_{1*}) \circ \mathrm{tr}_*(\alpha) \\ &= (\tilde{j}_* \circ p_{1*}) \circ (\pi_* \circ \mathrm{tr}_*)(\alpha) \\ &= 2 \cdot (\tilde{j}_* \circ p_{1*})(\alpha). \end{aligned}$$

Notice that the image of the intersection semicharacteristic

$$I\chi_{1/2}: \Omega^{\mathrm{Witt}}_{4n+1}(B\mathbf{Z}_2) \to L_h^{4n+1}(\mathbf{Q}\mathbf{Z}_2) = \mathbf{Z}_2$$

is detected by the ordinary intersection semicharacteristic $\widehat{I\chi_{1/2}}(X;\mathbf{Q}) = \sum_{i=0}^{2n} (-1)^i \dim(IH_i^{\bar{m}}(X;\mathbf{Q})) \pmod 2$. Since $[X/\mathbf{Z}_2, \tilde{\phi}] \in \Omega^{\mathrm{Witt}}_{4n+1}(B\mathbf{Z}_2)$ comes from the image of tr_*, the ordinary intersection semicharacteristic $\widehat{I\chi_{1/2}}(X;\mathbf{Q})$ is even. □

A similar argument to that given in Example 1.5 in [5] yields the following proposition.

PROPOSITION 3.10. *Suppose D_{2p} acts freely through PL homeomorphisms and intersection homologically trivially on a compact $(4n+1)$-dimensional Witt space X. Then $\widehat{I\chi_{1/2}}(X;\mathbf{Q})$ is even.*

An argument similar to that of Theorem 8.1 in [4] yields the following theorem.

THEOREM 3.11. *Suppose G acts freely through PL homeomorphisms and intersection homologically trivially on a compact $(4n+1)$-dimensional Witt space X. Then either the intersection semicharacteristic $\widehat{I\chi_{1/2}}(X;\mathbf{Q})$ is even or G is a direct product of a cyclic 2-group and an odd order group.*

We note that there is a minor error in the proof of Theorem 8.1 in [4]. It is not shown that G is a semidirect product of its Sylow 2-subgroup G_2 and an odd order group H. One needs to apply the Schur–Zassenhaus theorem [6, Thm. 6.2.1] to the short exact sequence

$$1 \to G_2 \to G \to H \to 1.$$

The Schur–Zassenhaus theorem applies because $(2, |H|) = 1$ and G_2 is solvable.

4. Restrictions on Higher Signature

We consider the restrictions on the higher signature of a Witt space that admits a free G action which acts trivially on the intersection homology of X with any local coefficient system. Let $f: X \to B\pi$ classify the universal cover of X. Consider a representation $\rho: \pi \to \mathrm{Sp}(2l, \mathbf{R})$ of the fundamental group of X into the real symplectic group in $2l$ variables, that is, the group of automorphisms of $\mathbf{R}^{2l}$ preserving the alternate form $\sum_{i=1}^{l}(x_i y_{l+i} - x_{l+i} y_i)$. We will abbreviate $\mathrm{Sp}(2l, \mathbf{R})$ by Sp when no confusion will arise from doing so. Let $L(X) \in H_*(X;\mathbf{Q})$ be the L-class of a Witt space X. Then

$$\langle (B\rho)^*(u), f_*(L(X))\rangle = 0 \quad \text{for all } u \in H^*(B\mathrm{Sp};\mathbf{Q}),$$

where $B\rho: B\pi \to B\mathrm{Sp}$ is the induced map on classifying spaces. Our methods follow those of Lusztig [12] and Miščenko [16].

Let V be a finite-dimensional complex vector space with a nondegenerate hermitian form h. A homomorphism $\psi: \pi \to \mathrm{GL}(V)$ such that the hermitian form h is invariant allows us to define a local coefficient system for the intersection homology of X. Let $\tilde{X}$ be the universal cover of X. Then $IC_i^{\bar{m}}(\tilde{X};\mathbf{Q})$ is a free $\mathbf{Q}\pi$ module. We use the homomorphism ψ to make V into a left $\mathbf{Q}\pi$ module. Hence, $IC_i^{\bar{m}}(\tilde{X};\mathbf{Q}) \otimes_{\mathbf{Q}\pi} V$ is the i-chains of X with the local coefficient system induced by the homomorphism ψ.

DEFINITION 4.1. The intersection homology of X with local coefficients ψ is the homology of the complex

$$IC_i^{\bar{m}}(\tilde{X};\mathbf{Q}) \otimes_{\mathbf{Q}\pi} V, \ \partial \otimes \mathrm{id}.$$

We denote these homology groups by $IH_i^{\bar{m}}(X;\psi)$.

REMARK 4.2. We induce a Hermitian form on $IH_*^{\bar{m}}(X;\psi)$ by the induced Hermitian form $\langle \sigma \otimes v, \tau \otimes w\rangle = \epsilon'(\sigma \cap \tau) \cdot h(v, w)$ on the chain level for all $\sigma \in IC_i^{\bar{m}}(\tilde{X},\mathbf{Q})$, $\tau \in IC_{n-i}^{\bar{n}}(\tilde{X};\mathbf{Q})$, and $v, w \in V$, where $\cap$ is the intersection product and ϵ' is the augmentation homomorphism $\epsilon': IC_0^{\bar{t}}(\tilde{X};\mathbf{Q}) \to \mathbf{Q}$. From

Theorems 2.3 and 2.6 we obtain Poincaré and Lefschetz duality for Witt spaces on the intersection homology with local coefficients. In particular, the intersection products

$$IH_i^{\bar m}(X;\psi)\times IH_{n-i}^{\bar m}(X;\psi)\xrightarrow{\cap} IH_0^{\bar t}(X;\psi)\xrightarrow{\epsilon}\mathbf{C}$$

and

$$IH_i^{\bar m}(X,\partial X;\psi)\times IH_{n-i}^{\bar m}(X;\psi)\xrightarrow{\cap} IH_0^{\bar t}(X;\psi)\xrightarrow{\epsilon}\mathbf{C}$$

are nondegenerate Hermitian pairings when X has no boundary and X has boundary, respectively.

DEFINITION 4.3. Suppose X is a Witt space of dimension $4n$. The signature with local coefficients ψ will be the signature of the intersection product

$$IH_{2n}^{\bar m}(X^{4n};\psi)\times IH_{2n}^{\bar m}(X^{4n};\psi)\to\mathbf{C}.$$

We will denote this local coefficients signature by $\mathrm{sign}_\psi(X)$.

LEMMA 4.4. *The local coefficients signature* $\mathrm{sign}_\psi(X)$ *is a bordism invariant in* $\Omega_{4n}^{\mathrm{Witt}}(B\pi)$.

Proof. Suppose $X=\partial Y$ is a boundary of a Witt space Y with a map $g: Y\to B\pi$ that restricts to f on X. By Remark 4.2, the long exact sequences

$$\begin{array}{ccccc} IH_{2n+1}^{\bar m}(Y,X;\psi) & \xrightarrow{\partial} & IH_{2n}^{\bar m}(X,\psi) & \xrightarrow{i} & IH_{2n}^{\bar m}(Y;\psi) \\ \downarrow\cong & & \downarrow\cong & & \downarrow\cong \\ IH_{2n}^{\bar m}(Y;\psi)^* & \xrightarrow{i^*} & IH_{2n}^{\bar m}(X,\psi)^* & \xrightarrow{\partial^*} & IH_{2n+1}^{\bar m}(Y,X;\psi)^* \end{array}$$

are paired by the nondegenerate intersection products. As in the usual proof of the bordism invariance of the signature for manifolds, we see that im i^* is a submodule of $IH_{2n}^{\bar m}(X;\psi)$ of half the dimension for which the intersection product is zero. Hence, $\mathrm{sign}_\psi(X)=0$. □

We define the L-class for a Witt space X. Our method follows that of Goresky and MacPherson [7, §5.3] who define the L-class of a Whitney stratified pseudomanifold with even codimensional strata. This element lies in the rational homology of X, $L(X)\in H_*(X;\mathbf{Q})$. When X is a manifold, $L(X)$ is the dual of $\tilde{L}(X)$, the Hirzebruch L-class of X. The signature $\sigma(Y)$ of a $4n$-dimensional Witt space Y is a well-defined integer. Siegel [23] has shown that it is a Witt bordism invariant. If $\dim(X)\not\equiv 0 \pmod 4$ then we define $\sigma(Y)=0$.

Let $p\in S^k$ be a point in the interior of a k-simplex. We say that a simplicial map $f: X\to S^k$ is *transverse* if there exists a regular neighborhood N of p such that $f^{-1}(N)$ is PL homeomorphic to $f^{-1}(p)\times N$. For each transverse simplicial map $f: X\to S^k$ we get a stratification of $f^{-1}(p)$ that makes $f^{-1}(p)$ into a Witt space. Then the signature $\sigma(f^{-1}(p))$ is defined. This signature depends only on the homotopy class of $f: X\to S^k$, by the following lemma.

LEMMA 4.5. *There is a unique map* $\theta: [X, S^k] \to \mathbf{Z}$ *such that* $\theta([f]) = \sigma(f^{-1}(p))$ *for each transverse map* $f: X \to S^k$.

The proof is similar to the proof of Lemma 5.3 of [7].

DEFINITION 4.6. The L-class $L_k(X) \in H_k(X; \mathbf{Q})$ is the homomorphism $\theta \otimes \mathbf{Q}: H^k(X; \mathbf{Q}) \to \mathbf{Q}$, where we apply the isomorphism

$$[X, S^k] \otimes \mathbf{Q} \cong H^k(X; \mathbf{Q}) \quad \text{when } 2k > n+1.$$

As noted in [7], the restriction $2k > n+1$ can be removed by suspending X with a sphere, as in [15].

DEFINITION 4.7. Suppose $f: X \to B\pi$ classifies the universal cover of X. We define $f_*(L(X)) \in H_*(B\pi; \mathbf{Q})$ to be the higher signature of X.

LEMMA 4.8. *The higher signature* $f_*(L(X))$ *is a bordism invariant in* $\Omega_n^{\text{Witt}}(B\pi)$.

Proof. Suppose that $X = \partial Y$ and that $g: Y \to B\pi$ restricts to f on X. We need to show that $f_*(L(X)) = 0$. Consider a transverse map $h: Y \to S^k$ and let $h_0: X \to S^k$ be the restriction of h to X. Then $h^{-1}(p)$ is a Witt space of Y that bounds $h_0^{-1}(p)$ in X. Thus $\sigma(h_0^{-1}(p)) = 0$. Hence $L(X) = \theta \otimes \mathbf{Q} = 0$, and therefore $f_*(L(X)) = 0 \in H_*(B\pi; \mathbf{Q})$. □

COROLLARY 4.9. *Let* $v \in H^*(B\pi; \mathbf{Q})$. *Then* $\langle v, f_*(L(X)) \rangle$ *is a bordism invariant in* $\Omega_n^{\text{Witt}}(B\pi)$.

We apply this higher signature to group actions on a Witt space. Consider a smooth connected manifold M. We state a result of Lusztig [12, §5.2].

THEOREM 4.10. *Suppose that* π *is the fundamental group of a smooth closed manifold* M *and that* $\rho: \pi \to \mathrm{Sp}$ *is a homomorphism. For any* $u \in H^*(B\mathrm{Sp}; \mathbf{Q})$, *there exists a positive integer* m *as well as two representations of* Sp, E *and* F, *with invariant nondegenerate Hermitian forms (possibly indefinite) such that*

$$mu - \mathrm{ch}(\phi(E)^+ - \phi(E)^-) + \mathrm{ch}(\phi(F)^+ - \phi(F)^-)$$

is zero in degrees $\leq \dim(M)$. *Here,* $\phi(E)$ *is the associated Hermitian bundle over the classifying space* $B\mathrm{Sp}$ *and* $\phi(E) = \phi(E)^+ \oplus \phi(E)^-$ *is the splitting of the induced Hermitian metric such that the metric is positive definite on* $\phi(E)^+$ *and negative definite on* $\phi(E)^-$. *Also,* $\mathrm{ch}(\phi(E)^+ - \phi(E)^-) \in H^*(M; \mathbf{Q})$ *is the Chern character of the bundle* $\phi(E)^+ - \phi(E)^-$ *over* $B\mathrm{Sp}$.

Suppose $f: M \to B\pi$ classifies the universal cover of M. Let $h = B\rho \circ f$. Then

$$h^*(mu) = h^* \mathrm{ch}(\phi(E)^+ - \phi(E)^-) - h^* \mathrm{ch}(\phi(F)^+ - \phi(F)^-) \in H^*(M; \mathbf{Q}).$$

Now E and F give rise to flat Hermitian bundles on M, denoted by $\tilde{E}$ and $\tilde{F}$, whose Chern characters are $\mathrm{ch}(\tilde{E}^+ - \tilde{E}^-) = h^* \mathrm{ch}(\phi(E)^+ - \phi(E)^-)$ and $\mathrm{ch}(\tilde{F}^+ - \tilde{F}^-) = h^*(\phi(F)^+ - \phi(F)^-)$. Thus

$$\begin{aligned}&\langle h^*(mu)\tilde{L}(M),[M]\rangle\\&\quad=\langle \mathrm{ch}(\tilde{E}^+-\tilde{E}^-)\tilde{L}(M),[M]\rangle-\langle \mathrm{ch}(\tilde{F}^+-\tilde{F}^-)\tilde{L}(M),[M]\rangle\\&\quad=\mathrm{sign}(M,\tilde{E})-\mathrm{sign}(M,\tilde{F}).\end{aligned}$$

This last equality is obtained from the Atiyah–Singer index theorem. Here, $\mathrm{sign}(M,\tilde{E})$ is the signature of the cohomology of the chain complex of smooth forms on M with values in $\tilde{E}$. The vector bundles $\tilde{E}$ and $\tilde{F}$ are equivalent to representations of the fundamental group π into $\mathrm{GL}(E)$ and $\mathrm{GL}(F)$, respectively. These representations of π factor through Sp. Let $\psi:\mathrm{Sp}\to\mathrm{GL}(E)$ and $\theta:\mathrm{Sp}\to\mathrm{GL}(F)$ be the representations of Sp such that $\psi\circ\rho:\pi\to\mathrm{GL}(E)$ and $\theta\circ\rho:\pi\to\mathrm{GL}(F)$ are the representations of π which classify $\tilde{E}$ and $\tilde{F}$, respectively. Let $C^*(\tilde{X};\psi\circ\rho)$ and $C^*(\tilde{X};\theta\circ\rho)$ denote the cochain complexes of the chain complexes $C_*(\tilde{X};\mathbf{Q})\otimes_{\mathbf{Q}\pi}E, \partial\otimes\mathrm{id}$ and $C_*(\tilde{X};\mathbf{Q})\otimes_{\mathbf{Q}\pi}F, \partial\otimes\mathrm{id}$, respectively, as defined in 4.1 (with ordinary chains in place of intersection chains). The proof of the usual de Rham theorem can be extended to show that the chain complex of smooth forms on M with values in $\tilde{E}$ and $\tilde{F}$ are chain homotopic to the chain complexes $C^*(\tilde{X};\psi\circ\rho)$ and $C^*(\tilde{X};\theta\circ\rho)$, respectively. Thus

$$\langle h^*(mu)\tilde{L}(M),[M]\rangle=\mathrm{sign}_{\psi\circ\rho}(M)-\mathrm{sign}_{\theta\circ\rho}(M).$$

THEOREM 4.11. *Suppose that π is the fundamental group of a smooth closed connected manifold M and that $\rho:\pi\to\mathrm{Sp}$ is a homomorphism. Then, for any $u\in H^*(B\mathrm{Sp};\mathbf{Q})$, there exists a positive integer m as well as representations $\psi:\mathrm{Sp}\to\mathrm{GL}(E)$ and $\theta:\mathrm{Sp}\to\mathrm{GL}(F)$ such that*

$$\langle B\rho^*(mu),f^*(L(M))\rangle=\mathrm{sign}_{\psi\circ\rho}(M)-\mathrm{sign}_{\theta\circ\rho}(M).$$

Proof. By dualizing the Hirzebruch L-class to the rational homology L-class we obtain

$$\begin{aligned}\mathrm{sign}_{\psi\circ\rho}(M)-\mathrm{sign}_{\theta\circ\rho}(M)&=\langle f^*\circ(B\rho)^*(mu)\tilde{L}(M),[M]\rangle\\&=\langle (B\rho)^*(mu),f_*(\tilde{L}(M)\cap[M])\rangle\\&=\langle (B\rho)^*(mu),f_*(L(M))\rangle.\end{aligned}$$ □

We extend this theorem to Witt spaces.

THEOREM 4.12. *Suppose that $f:X\to B\pi$ classifies the universal cover of a compact connected Witt space X and that $\rho:\pi\to\mathrm{Sp}$ is a homomorphism. Then, for all $u\in H^*(B\mathrm{Sp};\mathbf{Q})$, there exists a positive integer m as well as representations $\psi:\mathrm{Sp}\to\mathrm{GL}(E)$ and $\theta:\mathrm{Sp}\to\mathrm{GL}(F)$ such that*

$$\langle B\rho^*(mu),f^*(L(X))\rangle=\mathrm{sign}_{\psi\circ\rho}(X)-\mathrm{sign}_{\theta\circ\rho}(X).$$

Proof. Siegel has shown that $\Omega_n^{\mathrm{Witt}}(pt)=0$ if $n\neq 0 \pmod 4$, and that $\Omega_{4k}^{\mathrm{Witt}}(pt)\cong W(\mathbf{Q})$ if $k>0$, where $W(\mathbf{Q})$ is the Witt ring of symmetric bilinear forms over the rationals [24, Prop. IV.1.1]. Now $W(\mathbf{Q})\cong\mathbf{Z}\oplus T$, where T is an infinite torsion group and the $\mathbf{Z}$ factor is detected by the signature of a symmetric bilinear form over the rationals. On the other hand, $\Omega_{4k}^{\mathrm{SO}}(pt)\otimes\mathbf{Q}$

has rank $p(k)$, the number of partitions of k [15, Thm. 18.8]. Thus the natural map $\Omega_n^{SO}(pt)\otimes\mathbf{Q}\to\Omega_n^{\mathrm{Witt}}(pt)\otimes\mathbf{Q}$ is a surjection.

The Atiyah-Hirzebruch spectral sequences for

$$\Omega_n^{SO}(B\pi)\otimes\mathbf{Q} \quad \text{and} \quad \Omega_n^{\mathrm{Witt}}(B\pi)\otimes\mathbf{Q}$$

both collapse, and we obtain

$$\Omega_n^{\mathrm{Cat}}(B\pi)\otimes\mathbf{Q}\cong\bigoplus_k H_{n-4k}(B\pi;\Omega_{4k}^{\mathrm{Cat}}(pt)\otimes\mathbf{Q}).$$

Hence, the map $\Omega_n^{SO}(B\pi)\otimes\mathbf{Q}\to\Omega_n^{\mathrm{Witt}}(B\pi)\otimes\mathbf{Q}$ is a surjection. Thus a positive integral multiple of X is Witt bordant to the image of a manifold M in $\Omega_n^{\mathrm{Witt}}(B\pi)\otimes\mathbf{Q}$. By Lemma 4.4, Corollary 4.9, and Theorem 4.10, for some positive integer r we have

$$\begin{aligned} r(\mathrm{sign}_{\psi\circ\rho}(X)-\mathrm{sign}_{\theta\circ\rho}(X)) &= \mathrm{sign}_{\psi\circ\rho}(rX)-\mathrm{sign}_{\theta\circ\rho}(rX)\\ &= \mathrm{sign}_{\psi\circ\rho}(M)-\mathrm{sign}_{\theta\circ\rho}(M)\\ &= \langle B\rho^*(mu), f^*(L(M))\rangle\\ &= \langle B\rho^*(mu), f^*(L(rX))\rangle\\ &= r\langle B\rho^*(mu), f^*(L(X))\rangle. \end{aligned}$$

Hence, $\mathrm{sign}_{\psi\circ\rho}(X)-\mathrm{sign}_{\theta\circ\rho}(X)=\langle B\rho^*(mu), f^*(L(X))\rangle$. □

DEFINITION 4.13. A free G action on a Witt space X is said to be *unextended* if the exact sequence for covering spaces

$$1\to\pi_1(X)\to\pi_1(X/G)\to G\to 1$$

is such that $\pi_1(X)\to\pi_1(X/G)$ is a split monomorphism.

REMARK 4.14. The map $\pi_1(X/G)\to G$ is a split surjection and $\pi_1(X/G)\cong\pi_1(X)\times G$.

THEOREM 4.15. *Suppose that G acts freely through PL homeomorphisms and unextendedly on a compact connected Witt space X, and that $\rho\colon\pi\to\mathrm{Sp}$ is a representation. Suppose $f\colon X\to B\pi$ classifies the universal cover of X and $B\rho\colon B\pi\to B\mathrm{Sp}$ is the induced map of ρ on the classifying spaces. If G acts trivially on the intersection homology of X with any local coefficient system, then*

$$\langle (B\rho)^*(u), f_*(L(X))\rangle = 0 \quad \textit{for all } u\in H^*(B\mathrm{Sp};\mathbf{Q}).$$

Proof. Consider the relation of X and X/G in the bordism of $B\pi$:

$$\begin{array}{ccc} X & \longrightarrow & B\pi\\ {\scriptstyle p}\downarrow & & \uparrow\\ X/G & = & X/G. \end{array}$$

This can be done because G is an unextended action. We show that

$$(*) \qquad [X]=|G|[X/G]\in\Omega_n^{\mathrm{Witt}}(B\pi)\otimes\mathbf{Q}.$$

The analogous result is true for manifolds in the bordism group $\Omega_n^{SO}(B\pi)\otimes\mathbf{Q}$ (see [27, Thm. 2.9]). By the proof of Theorem 4.12, the map

$$\Omega_n^{SO}(B(\pi\times G))\otimes\mathbf{Q}\to\Omega_n^{\mathrm{Witt}}(B(\pi\times G))\otimes\mathbf{Q}$$

is a surjection. So there is a manifold M with a free unextended G action such that $[M]=[X]$ and $[M/G]=[X/G]$ in $\Omega_n^{\mathrm{Witt}}(B\pi)\otimes\mathbf{Q}$. Hence, $[X]=[M]=|G|[M/G]=|G|[X/G]$. Applying Lemma 4.4 to (*) yields

$$(**)\qquad \mathrm{sign}_\beta(X)=|G|\,\mathrm{sign}_\beta(X/G),$$

where $\beta:\pi\to\mathrm{GL}(V)$ is any representation of π. However, G acts trivially on the intersection homology of X with local coefficient system β. Then the following transfer argument shows that $IH_*^{\bar m}(X;\beta)\cong IH_*^{\bar m}(X/G;\beta)$.

Let $p:X\to X/H$ be the covering projection, where H is any subgroup of G, and let

$$\mathrm{tr}=\sum_{h\in H}h_*\otimes\mathrm{id}:IC_i^{\bar m}(\tilde X/H;\mathbf{Q})\otimes_{\mathbf{Q}\pi}V\to IC_i^{\bar m}(\tilde X;\mathbf{Q})\otimes_{\mathbf{Q}\pi}V$$

be the chain map that induces the transfer map

$$\mathrm{tr}_*:IH_i^{\bar m}(X/H;\beta)\to IH_i^{\bar m}(X;\beta).$$

Then a calculation yields $p_*\circ\mathrm{tr}_*=|H|:IH_i^{\bar m}(X/H;\beta)\to IH_i^{\bar m}(X/H;\beta)$ and $\mathrm{tr}_*\circ p_*=|H|:IH_i^{\bar m}(X;\beta)\to IH_i^{\bar m}(X;\beta)$ because G acts trivially on the intersection homology of X with any local coefficient system. Since $|H|$ is invertible, tr_* is an isomorphism. Hence,

$$(***)\qquad \mathrm{sign}_\beta(X)=\mathrm{sign}_\beta(X/G).$$

Combining (**) and (***) yields $\mathrm{sign}_\beta(X)=0$. By Theorem 4.12, for all $u\in H^*(B\mathrm{Sp};\mathbf{Q})$, there exists a positive integer m and representations $\psi:\mathrm{Sp}\to\mathrm{GL}(E)$ and $\theta:\mathrm{Sp}\to\mathrm{GL}(F)$ such that

$$\langle B\rho^*(mu),f^*(L(X))\rangle=\mathrm{sign}_{\psi\circ\rho}(X)-\mathrm{sign}_{\theta\circ\rho}(X)=0.$$

Hence, $\langle(B\rho)^*(u),f_*(L(X))\rangle=0.$ □

The following result due to Matsushima [13] shows that for some fundamental groups π the cohomology classes $(B\rho)^*(u)\in H^*(B\pi;\mathbf{Q})$ are nontrivial.

THEOREM 4.16. *Let Γ be a discrete, torsion-free subgroup of* Sp *such that* Sp/Γ *is compact. Then the homomorphism $H^i(B\mathrm{Sp};\mathbf{Q})\to H^i(B\Gamma;\mathbf{Q})$ induced by the inclusion $\Gamma\subset\mathrm{Sp}$ is surjective for $i<(l+2)/4$.*

This gives us the following corollary.

COROLLARY 4.17. *Suppose that G acts freely through PL homeomorphisms and unextendedly on a compact connected Witt space X, and that $\rho:\pi\to\mathrm{Sp}$ is a representation. Suppose that $f:X\to B\pi$ classifies the universal cover of X and $B\rho:B\pi\to B\mathrm{Sp}$ is the induced map of ρ on the classifying spaces. Suppose that π is a discrete, torsion-free subgroup of* Sp *such that*

Sp/π is compact. If G acts trivially on the intersection homology of X with any local coefficient system, then $f_*(L_k(X)) = 0 \in H_k(B\pi; \mathbf{Q})$ for all $k < (l+2)/4$.

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