Weakly and Strongly Outer Functions on the Bidisc

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0. Introduction

In previous work [1], certain classes of analytic functions on the bidisc were shown to arise naturally in connection with a problem in prediction theory. These were dubbed "weakly outer" and "strongly outer" functions, for it was demonstrated that a regular stationary field has the so-called weak (strong) commutation property if and only if its spectral density is the squared modulus of a weakly (strongly) outer function of the Hardy space H^2. Further applications to prediction were given.

In the present article, the related function theory is explored. We shall see that weakly and strongly outer functions exhibit some properties of the usual outer functions. Indeed, it turns out that, within this context, many of the classical one-variable results have multivariate analogues. Among these are Beurling's theorem, the Riesz factorization, and Szegö's infimum.

1. Preliminaries

Let $\mathbf{D}$ be the unit disc in the complex plane $\mathbf{C}$, and let $\mathbf{T}$ be the unit circle. For $d=1$ or 2, σ_d denotes normalized Lebesgue measure on $\mathbf{T}^d$. We are concerned with the Nevanlinna class $N_*(\mathbf{D}^d)$ as well as the Hardy classes $H^p(\mathbf{D}^d)$ of analytic functions on $\mathbf{D}^d$ (see [3] and [7]). Such a function is associated with its radial limit function on $\mathbf{T}^d$. For convenience, the same letter will be used for both, and corresponding spaces will be identified. The symbol ^ indicates a Fourier coefficient, and C_z represents the Cauchy kernel. Thus, for $d=2$ and $f \in L^1(\mathbf{T}^2)$,

$$\hat{f}(m,n) = \int f(e^{is}, e^{it}) e^{-ims-int} \, d\sigma_2(e^{is}, e^{it});$$

$$C_z = \frac{1}{1-z_1 e^{-is}} \cdot \frac{1}{1-z_2 e^{-it}}.$$

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A function f in $N_*(\mathbf{T}^d)$ is said to be outer, we recall, if $\log|f(0)| = \int \log|f|\,d\sigma_d$; in this case, $f(z) = \exp\int(2C_z - 1)\log|f|\,d\sigma_d$. When $d=1$, a key element in understanding the $H^p(\mathbf{D})$ spaces is the following.

BEURLING'S THEOREM. *Let $1 \le p < \infty$ and $f \in H^p(\mathbf{D})$. Then the (closed) span in $H^p(\mathbf{D})$ of $\{z^n f(z) : z = 0, 1, 2, \ldots\}$ is all of $H^p(\mathbf{D})$ if and only if f is outer.*

The analogous statement for $d > 1$ is false, however. In the interests of restoring symmetry, let us consider the following definition.

1.1. DEFINITION. Let $1 \le p < \infty$, and $f \in H^p(\mathbf{D}^2)$. The function f is said to be *strongly outer in* $H^p(\mathbf{D}^2)$ if the span in $H^p(\mathbf{D}^2)$ of $\{z_1^m z_2^n f(z_1, z_2): m \ge 0,\ n \ge 0\}$ is all of $H^p(\mathbf{D}^2)$.

The dependence of the defining condition on p is a major weakness; in particular, it is not clear how to extend the notion of strong outerness to all of $N_*(\mathbf{T}^2)$. In this respect, one finds an improvement in the next definition.

1.2. DEFINITION. A function $f \in N_*(\mathbf{T}^2)$ is said to be *weakly outer* if

(i) $f(\cdot, e^{it})$ is outer in $N_*(\mathbf{T})$ for almost every fixed e^{it};
(ii) $f(e^{is}, \cdot)$ is outer in $N_*(\mathbf{T})$ for almost every fixed e^{is}.

The remaining sections are devoted to studying the behavior of these classes of functions, and exhibiting some applications.

2. Elementary Properties

First, let us note that the words "weakly" and "strongly" are not ill-chosen in Definitions 1.1 and 1.2; that is, we have the following proposition.

2.1. PROPOSITION. *Let $1 \le p < \infty$ and $f \in N_*(\mathbf{T}^2)$.*

(i) *If f is strongly outer in $H^p(\mathbf{T}^2)$, then f is outer.*
(ii) *If f is outer, then f is weakly outer.*

Proof. (i) This is the assertion of Rudin [7, Thm. 4.4.6]. (ii) See Soltani [8, Lemma 1.3]. □

Moreover, as the examples that follow illustrate, both of the inclusions implied by Proposition 2.1 are proper.

2.2. EXAMPLES. Let $f(z_1, z_2) = z_1^m + \alpha z_2^n$, where m and n are positive integers and $|\alpha| = 1$. Then f is weakly outer, as can be verified by inspection. Moreover, f has zeros in the bidisc, and hence fails to be outer.

More generally, let j_1 and j_2 be inner functions on $\mathbf{D}$, and put $g(z_1, z_2) = j_1(z_1) + j_2(z_2)$. For almost every fixed e^{it}, $g(\cdot, e^{it})$ takes values in a half-plane and hence is outer; similarly, $g(e^{is}, \cdot)$ is outer for almost every fixed e^{is}. Therefore g is weakly outer. Such g typically have zeros and fail to be outer.

2.3. EXAMPLE. The bounded analytic function $e^{(z_1+z_2+2)/(z_1+z_2-2)}$ is outer, but for all p, $1\leq p<\infty$, it fails to be strongly outer in $H^p(\mathbf{T}^2)$. (For $p=2$ this is [7, 4.4.8(b)]; a straightforward modification of this argument yields the general claim.)

2.4. EXAMPLE [7, 4.4.9]. Let $1\leq p<\infty$ and $f\in H^p(\mathbf{D}^2)$. If $f=g^k$, where k is a positive integer and g is an analytic function with positive real part, then f is strongly outer in $H^p(\mathbf{D}^2)$.

Although the condition for strong outerness depends on the parameter p, it respects the inclusion relationship among the $H^p(\mathbf{T}^2)$ spaces, as the following proposition shows.

2.5. PROPOSITION. *If $1\leq p<\infty$ and f is strongly outer in $H^p(\mathbf{T}^2)$, then for all r, $1\leq r\leq p$, f is strongly outer in $H^r(\mathbf{T}^2)$.*

Proof. As ϕ varies over analytic polynomials,

$$\inf\|1+\phi f\|_r\leq\inf\|1+\phi f\|_p=0.$$

Let us turn to algebraic considerations. Recall that if f and g are outer, then fg is outer and $1/f$ is outer. These easily carry over to the weakly outer case.

2.6. PROPOSITION.

(i) *If f and g are weakly outer, then fg is weakly outer.*
(ii) *If f is weakly outer and $1/f\in N_*(\mathbf{T}^2)$, then $1/f$ is weakly outer.*

Strong versions hold as well, each necessarily dependent on the parameters p of the $H^p(\mathbf{T}^2)$ spaces involved.

2.7. PROPOSITION. *Suppose that $1\leq p<\infty$, $1\leq q<\infty$, and $1/p+1/q\leq 1$. If f is strongly outer in $H^p(\mathbf{T}^2)$ and g is strongly outer in $H^q(\mathbf{T}^2)$, then fg is strongly outer in $H^r(\mathbf{T}^2)$, where $1/r=1/p+1/q$.*

Proof. Let $\epsilon>0$. There exists an analytic polynomial ϕ_1 such that $\|1+\phi_1 f\|_p<\epsilon/2$. There exists an analytic polynomial ϕ_2 such that

$$\|1-\phi_2 g\|_q<\epsilon\cdot 2^{-1}(1+\|\phi_1 f\|_p)^{-1}.$$

So now

$$\begin{aligned}\inf\|1+\phi fg\|_r&\leq\|1+\phi_1 f\phi_2 g\|_r\\&\leq\|1+\phi_1 f\|_r+\|\phi_1 f(\phi_2 g-1)\|_r\\&\leq\|1+\phi_1 f\|_p+\|\phi_1 f\|_p\|\phi_2 g-1\|_q\\&<\epsilon.\end{aligned}$$

□

2.8. PROPOSITION. *Suppose that $1\leq p<\infty$, $1\leq q\leq\infty$, and $1/p+1/q\leq 1$. If $f\in H^p(\mathbf{T}^2)$ and $f^{-1}\in H^q(\mathbf{T}^2)$, then f is strongly outer in $H^r(\mathbf{T}^2)$, where $1/r=1/p+1/q$.*

Proof. $\inf\|1+\phi f\|_r \le \inf(\|f\|_p\|f^{-1}-\phi\|_q)$, where each infimum is taken over analytic polynomials ϕ. For finite q, such ϕ are dense in $H^q(\mathbf{T}^2)$, hence the infimum is zero. For $q=\infty$, we can choose ϕ_n converging to f^{-1} boundedly almost everywhere, so that $(f^{-1}-\phi_n)f$ converges to zero in $H^r(\mathbf{T}^2)$. In either case, we find that f is strongly outer in $H^r(\mathbf{T}^2)$. □

It is known that outer functions enjoy the following integral representation: A function $f\in N_*(\mathbf{D}^2)$ is outer and only if

$$f(z)=\exp\int(2C_z-1)\Phi\, d\sigma_2,$$

where Φ is a real integrable function on $\mathbf{T}^2$, and $\hat{\Phi}(m,n)=0$ whenever $mn<0$. In this case, $\Phi=\log|f|$. Weakly outer functions, on the other hand, admit the following structural characterization.

2.9. THEOREM. *Let $f\in N_*(\mathbf{T}^2)$. Then* (A) *f is weakly outer if and only if* (B) $f(e^{is},e^{it})=a(e^{is})g(e^{is},e^{it})\overline{h(e^{is},e^{-it})}$, *where*

(i) *$g(e^{is},e^{it})$ and $h(e^{is},e^{it})$ are outer;*
(ii) $(\log|h|)\hat{}(m,n)=0$ *whenever* $mn\le 0$;
(iii) *$a(e^{is})$ is unimodular;*
(iv) $h(e^{is},e^{it})/a(e^{is})\overline{h(e^{is},e^{it})}=b(e^{it})$ *for some unimodular function* $b(e^{it})$.

Moreover, the representation in (B) *is unique.*

Proof. In any case, $\log|f|\in L^1(\mathbf{T}^2,\sigma_2)$, and so we can define

$$g(z_1,z_2)=\exp\int(2C_z-1)\log|f|\, d\sigma_2.$$

Note that g is outer, $\log|g|\in L^1(\mathbf{T}^2,\sigma_2)$, and $(\log|g|)\hat{}(m,n)=0$ if $mn<0$. Moreover, $(\log|f|)\hat{}(m,n)=(\log|g|)\hat{}(m,n)$ whenever $mn\ge 0$.

If f is weakly outer, then so is f/g. By construction, $(\log|f/g|)\hat{}(m,n)=0$ whenever $mn\ge 0$. Hence $c_{mn}=(\log|f/g|)\hat{}(m,-n)$ are the Fourier coefficients of $\log|h|$, for some outer function h. This produces g and h satisfying (i), (ii), and the condition $|f(e^{is},e^{it})|=|g(e^{is},e^{it})||h(e^{is},e^{-it})|$.

Since f/g is weakly outer, the univariate outer functions $f(\cdot,e^{it})/g(\cdot,e^{it})$ and $h(\cdot,e^{it})$ differ by a constant unimodular factor for almost every fixed e^{it}. Similarly, $f(e^{is},\cdot)/g(e^{is},\cdot)$ and $\overline{h(\cdot,e^{-it})}$ are so related. That is,

$$\frac{f(e^{is},e^{it})}{g(e^{is},e^{it})}=\overline{b(e^{-it})}h(e^{is},e^{-it}) \quad\text{and}\quad \frac{f(e^{is},e^{it})}{g(e^{is},e^{it})}=a(e^{is})\overline{h(e^{is},e^{-it})}$$

for some univariate unimodular functions a and b. Thus (iii) and (iv) hold, and f has the stated representation. Uniqueness is evident.

Conversely, let f have the form in (B). Then

$$f(e^{is},e^{it})=a(e^{is})g(e^{is},e^{it})\overline{h(e^{is},e^{-it})}$$

is outer in the variable e^{it} for almost every e^{is}; and

$$f(e^{is}, e^{it}) = \overline{b(e^{-it})}g(e^{is}, e^{it})h(e^{is}, e^{-it})$$

is outer in e^{is} for almost every e^{it}. Therefore, f is weakly outer. □

2.10. EXAMPLE. Consider the weakly outer function $F(e^{is}, e^{it}) = e^{is} + e^{it}$. In this case, $f(e^{is}, e^{it}) = a(e^{is})g(e^{is}, e^{it})\overline{h(e^{is}, e^{-it})}$, where $a(e^{is}) = e^{is}$, $g(e^{is}, e^{it}) = 2$, and $h(e^{is}, e^{-it}) = \frac{1}{2}(1 + e^{is}e^{it})$. Note that h is outer and that $h(e^{is}, e^{it})/a(e^{is})\overline{h(e^{is}, e^{it})} = e^{it}$, a unimodular function of e^{it}.

Such a description of strongly outer functions has not been found. A necessary condition, however, is provided by the following, a straightforward extension of [7, 4.4.8(a)].

2.11. PROPOSITION. *If f is strongly outer in $H^p(\mathbf{T}^2)$ for some p, $1 \leq p < \infty$, then $f(\frac{1}{2} + \frac{1}{2}\zeta, \frac{1}{2} + \frac{1}{2}\zeta)$ is outer (as a function of ζ).*

3. Other Developments

The original appearance of weakly and strongly outer functions was in a prediction problem [1]. The main result is quoted below as an introduction to further ramifications.

We begin by considering the space $L^p(\mu)$, where $1 \leq p < \infty$ and μ is a finite Borel measure on $\mathbf{T}^2$. Every subset S of $\mathbf{Z}^2$ generates a natural subspace of $L^p(\mu)$: we define $\mathcal{L}^p_\mu(S)$ to be the span of $\{e^{ims+int}: (m, n) \in S\}$. (If μ is not specified, it is understood to be σ_2.) Of particular interest are the "right half-plane" $R = \{(m, n) \in \mathbf{Z}^2: m \geq 0\}$ and the "top half-plane" $T = \{(m, n) \in \mathbf{Z}^2: n \geq 0\}$. If $p = 2$ then $L^p(\mu)$ is a Hilbert space, and each $\mathcal{L}^2_\mu(S)$ is the range of a projection operator $P(S)$.

3.1. DEFINITION. The space $L^2(\mu)$ is *regular* if $\bigcap_{m=0}^{\infty} e^{ims} \cdot \mathcal{L}^2_\mu(R) = (0)$ and $\bigcap_{n=0}^{\infty} e^{int} \cdot \mathcal{L}^2_\mu(T) = (0)$. The space $L^2(\mu)$ has the *weak commutation property* if $P(R)$ commutes with $P(T)$. The space $L^2(\mu)$ has the *strong commutation property* if it has the weak commutation property and, in addition, it satisfies $\mathcal{L}^2_\mu(R) \cap \mathcal{L}^2_\mu(T) = \mathcal{L}^2_\mu(R \cap T)$. (See [6] for further exploration of these notions.)

3.2. THEOREM [1]. *$L^2(\mu)$ is regular and has the weak (strong) commutation property if and only if $d\mu = |f|^2\, d\sigma_2$, where f is a weakly (strongly) outer function in $H^2(\mathbf{T}^2)$.*

An immediate consequence of Theorem 3.2 is a sort of Riesz factorization for $H^2(\mathbf{T}^2)$.

3.3. PROPOSITION. *Let $f \in H^2(\mathbf{T}^2)$. Then f has a factorization $f = gh$, where g is strongly outer in $H^2(\mathbf{T}^2)$ and h is inner, if and only if the space $L^2(|f|^2\, d\sigma_2)$ has the strong commutation property.*

Proof. If the factorization exists, then $|f|^2=|g|^2$. Since g is strongly outer in $H^2(\mathbf{T}^2)$, $L^2(|g|^2\,d\sigma_2)$ has the strong commutation property, and hence so does $L^2(|f|^2\,d\sigma_2)$.

Since $f\in N_*(\mathbf{T}^2)$, the logarithm of $|f|$ is integrable, which implies that $L^2(|f|^2\,d\sigma_2)$ is regular. If, in addition, $L^2(|f|^2\,d\sigma_2)$ has the strong commutation property, then $|f|^2=|g|^2$ for some strongly outer g in $H^2(\mathbf{T}^2)$. Since g is outer, the function $h=f/g$ belongs to $N_*(\mathbf{T}^2)$. Since h is unimodular as well, it is inner. □

This proposition does not generalize to $L^p(\mu)$ with $p\neq 2$, for then the Hilbert space methods do not apply. But other general descriptions of weak and strong outerness can be obtained, however. The next lemma brings in the spaces $\mathcal{L}^2(\cdot)$.

3.4. LEMMA. *Let $1\le p<\infty$ and $f\in\mathcal{L}^p(R)$. Then $f(\cdot,e^{it})$ is outer in $H^p(\mathbf{T})$ for almost every fixed e^{it} if and only if the span M in $\mathcal{L}^p(R)$ of $\{e^{ims+int}f(e^{is},e^{it}): m\ge 0,\ n\in Z\}$ is all of $\mathcal{L}^p(R)$.*

Proof. Necessity can be established as a simple modification of Cheng [1, Lemma 2.2], which treats the case $p=2$.

For sufficiency, suppose that $f(\cdot,e^{it})$ fails to be outer in $H^p(\mathbf{T}^2)$ for each e^{it} in a set A of positive measure. Consider

$$\epsilon=\inf\int|1+\phi f|^p\,d\sigma_2.$$

where the infimum is taken over trigonometric polynomials ϕ of $\mathcal{L}^p(R)$. For e^{it} fixed, put

$$\epsilon(e^{it})=\inf\int|1+\psi(e^{i\theta})f(e^{i\theta},e^{it})|^p\,d\sigma_1(e^{i\theta}),$$

where $\psi(e^{i\theta})$ varies over analytic (univariate) polynomials. By Beurling's theorem, $\epsilon(e^{it})>0$ for all $e^{it}\in A$.

Now $\epsilon(e^{it})$ is obviously a bounded function. It is also measurable: The set $\{e^{it}:\epsilon(e^{it})>\alpha\}$ can be written $\bigcap_{n=1}^{\infty}\{e^{it}:\int|1+\psi_n(e^{i\theta})f(e^{i\theta},e^{it})|^p\,d\sigma_1(e^{i\theta})>\alpha\}$, where $\{\psi_n\}_{n=1}^{\infty}$ is the collection of analytic polynomials with complex rational coefficients.

Finally, observe that

$$\epsilon\ge\int_A\epsilon(e^{it})\,d\sigma_1>0,$$

and hence M fails to be all of $\mathcal{L}^p(R)$. □

It follows that weakly outerness has the following Beurling-type characterization.

3.5. THEOREM. *Let $1\le p<\infty$ and $f\in H^p(\mathbf{T}^2)$. Then f is weakly outer if and only if*

(i) *the span of* $\{e^{ims+int}f(e^{is},e^{it}) : (m,n)\in R\}$ *in* $\mathcal{L}^p(R)$ *is all of* $\mathcal{L}^p(R)$, *and*
(ii) *the span of* $\{e^{ims+int}f(e^{is},e^{it}) : (m,n)\in T\}$ *in* $\mathcal{L}^p(T)$ *is all of* $\mathcal{L}^p(T)$.

This, in turn, can be restated in terms of spaces of the form $L^p_\mu(\cdot)$. A strong version exists as well.

3.6. THEOREM. *Let* $1\le p<\infty$, $f\in H^p(\mathbf{T}^2)$, *and* $d\mu=|f|^p\,d\sigma_2$. *Then*

(i) *f is weakly outer if and only if* $\mathcal{L}^p(R)=f\cdot\mathcal{L}^p_\mu(R)$ *and* $\mathcal{L}^p(T)=f\cdot\mathcal{L}^p_\mu(T)$;
(ii) *f is strongly outer if and only if* $H^p(\mathbf{T}^2)=f\cdot\mathcal{L}^p_\mu(R\cap T)$.

3.7. EXAMPLE. Consider again the weakly outer function $f(e^{is},e^{it})=e^{is}+e^{it}$, and put $d\mu=|f|^2\,d\sigma_2$. For $0<r<1$, the function $(re^{is}+e^{it})^{-1}=e^{-it}\sum_{k=0}^{\infty}(-re^{is}e^{-it})^k$ lies in $\mathcal{L}^p_\mu(R)$. As r increases to 1,

$$\left\|1-\frac{e^{is}+e^{it}}{re^{is}+e^{it}}\right\|^2=\frac{1-r}{1+r}\to 0.$$

Hence $\mathcal{L}^p(R)=f\cdot\mathcal{L}^p_\mu(R)$. Similarly, $\mathcal{L}^p(T)=f\cdot\mathcal{L}^p_\mu(T)$.

The next assertion is, in a sense, a partial extension of Theorem 3.2 to the case of arbitrary p, with "weak commutation" assumed. It also gives the extent to which strong outerness is more restrictive than weak outerness, namely, through condition (ii).

3.8. THEOREM. *Let* $1\le p<\infty$, $f\in H^p(\mathbf{T}^2)$, *and* $d\mu=|f|^p\,d\sigma_2$. *Then f is strongly outer in* $H^p(\mathbf{T}^2)$ *if and only if*

(i) *f is weakly outer, and*
(ii) $\mathcal{L}^p_\mu(R)\cap\mathcal{L}^p_\mu(T)=\mathcal{L}^p_\mu(R\cap T)$.

Proof. If f is strongly outer in $H^p(\mathbf{T}^2)$, then f is weakly outer, that is, (i) holds. Thus

$$\begin{aligned} f(\mathcal{L}^p_\mu(R)\cap\mathcal{L}^p_\mu(T)) &= f\cdot\mathcal{L}^p_\mu(R)\cap f\cdot\mathcal{L}^p_\mu(T)\\ &=\mathcal{L}^p(R)\cap\mathcal{L}^p(T)\\ &=\mathcal{L}^p(R\cap T)\\ &=H^p(\mathbf{T}^2).\end{aligned}$$

Now by strong outerness $H^p(\mathbf{T}^2)=f\cdot\mathcal{L}^p_\mu(R\cap T)$, yielding (ii).

If (i) and (ii) hold, then

$$\begin{aligned} f\cdot\mathcal{L}^p_\mu(R\cap T) &= f\cdot\mathcal{L}^p_\mu(R)\cap f\cdot\mathcal{L}^p_\mu(T)\\ &=\mathcal{L}^p(R)\cap\mathcal{L}^p(T)\\ &=H^p(\mathbf{T}^2).\end{aligned}$$

Therefore, f is strongly outer. □

We now turn to a number of extremal problems. The first is to find the least-squares error in estimating 1 by a vector in $e^{iNs}\cdot \mathfrak{L}_\mu^2(R)$. This N-step prediction error formula resembles its univariate analogue (see [4, §10.9]) and is a consequence of Lemma 3.4.

3.9. PROPOSITION. *Let $w(e^{is}, e^{it})$ be a nonnegative integrable function on T^2, and define $d\mu = w\, d\sigma_2$. If $\int \log w(e^{i\theta}, e^{it})\, d\sigma_1(e^{i\theta}) > -\infty$ a.e. $[\sigma, (e^{it})]$, then*

$$\inf\{\|1+\phi\|_{L^2(\mu)}^2 : \phi \in z_1^N \cdot \mathfrak{L}^2(R)\} = \sum_{m=0}^{N-1} \int |h_m(e^{it})|^2\, d\sigma_1(e^{it}), \quad N = 1, 2, 3, \ldots,$$

where, with

$$h(e^{is}, e^{it}) = \lim_{r\to 1} \exp \frac{1}{2} \int \frac{e^{i\theta}+re^{is}}{e^{i\theta}-re^{is}} \log w(e^{i\theta}, e^{it})\, d\sigma_1(e^{i\theta}),$$

we define

$$h_m(e^{it}) = \int h(e^{i\theta}, e^{it}) e^{-im\theta}\, d\sigma_1(e^{i\theta}).$$

Proof. If $\log w(\cdot, e^{it})$ is integrable a.e. $[\sigma, (e^{it})]$, then h and each h_m do indeed exist as shown. Note that $h(\cdot, e^{it})$ is outer in $H^2(\mathbf{T})$ a.e. $[\sigma, (e^{it})]$, and $|h|^2 = w$ a.e. $[\sigma_2]$. It follows that $h\cdot \mathfrak{L}_\mu^2(R) = \mathfrak{L}^2(R)$, and hence

$$\begin{aligned}\inf\{\|1+\phi\|_{L^2(\mu)}^2 : \phi \in z_1^N \cdot \mathfrak{L}_\mu^2(R)\} &= \inf\{\|h+h\phi\|_2^2 : \phi \in z_1^N \cdot \mathfrak{L}_\mu^2(R)\} \\ &= \inf\{\|h+\Phi\|_2^2 : \Phi \in z_1^N \cdot \mathfrak{L}^2(R)\} \\ &= \sum_{m=0}^{N-1} \int |h_m(e^{i\theta})|^2\, d\sigma_1(e^{i\theta}). \qquad \square\end{aligned}$$

Secondly, Theorem 3.6 makes possible these weak and strong versions of Szegö's infimum for $H^p(\mathbf{T}^2)$.

3.10. THEOREM. *Let $1 \le p < \infty$, $f \in H^p(\mathbf{T}^2)$, and $d\mu = |f|^p\, d\sigma_2$.*

(i) *If f is weakly outer, then*

$$\inf\left\{\int |1+z_1\phi_1+z_2\phi_2|^p\, d\mu : \phi_1 \text{ and } \phi_2 \in \mathfrak{L}_\mu^p(R) \cap \mathfrak{L}_\mu^p(T)\right\} = |f(0,0)|^p.$$

(ii) *If f is strongly outer in $H^p(\mathbf{T}^2)$, then*

$$\inf\left\{\int |1+z_1\phi_1+z_2\phi_2|^p\, d\mu : \phi_1 \text{ and } \phi_2 \text{ are analytic polynomials}\right\}$$
$$= |f(0,0)|^p = \exp \int \log|f|^p\, d\sigma_2.$$

Proof. In either case, $\int |1+z_1\phi_1+z_2\phi_2|^p\, d\mu = \int |f+z_1\phi_1 f+z_2\phi_2 f|^p\, d\sigma_2$. If f is weakly outer, we can estimate $f(0,0)-f(z_1,z_2)$ by functions of the form $z_1\phi_1 f + z_2\phi_2 f$. Hence the infimum in (i) is not greater than $|f(0,0)|^p$. On the other hand,

$$\int |f+z_1\phi_1 f+z_2\phi_2 f|^p \, d\sigma_2 \geq \left| \int (f+z_1\phi_1 f+z_2\phi_2 f) \, d\sigma_2 \right|^p$$
$$= |f(0,0)|^p,$$

yielding (i). The case (ii) is treated similarly. □

We close with a prediction problem with $d=2$, analogous to the work of Ibragimov and Solev [5] in which $d = 1$. Accordingly, let us define P_1, P_2, Q_1, and Q_2 to be the orthogonal projection operators of $L^2(\mu)$ onto $L^2_\mu(R)\cap L^2_\mu(T)$, $L^2_\mu(R^c)\cap L^2_\mu(T^c)$, $L^2_\mu(R\cap T)$, and $L^2_\mu(R^c\cap T^c)$, respectively. We consider the traces of the products $P_1P_2P_1$ and $Q_1Q_2Q_1$ to be measures of linear dependence between the respective associated pairs of spaces. In [5], the existence of such a trace was termed a "complete regularity" condition, and exact criteria in terms of μ were found. For the present situation we have the following theorem.

3.11. THEOREM. *Let $f\in H^2(\mathbf{T}^2)$ and $d\mu=|f|^2\,d\sigma_2$. Suppose that f is weakly (strongly) outer. Then the trace of $P_1P_2P_1$ ($Q_1Q_2Q_1$) is finite if and only if*

$$\sum_{j=1}^{\infty}\sum_{k=1}^{\infty} jk|(f/\bar{f})\hat{\ }(j,k)|^2<\infty.$$

Proof. The trace of $P_1P_2P_1$ is given by

$$\sum_{(j,k)\in R\cap T} \langle P_1P_2P_1(f^{-1}e^{ijs+ikt}), f^{-1}e^{ijs+ikt}\rangle_{L^2(\mu)}$$
$$= \sum_{(j,k)\in R\cap T} \langle P_2 f^{-1}e^{ijs+ikt}, f^{-1}e^{ijs+ikt}\rangle_{L^2(\mu)}$$
$$= \sum_{(j,k)\in R\cap T} \left\langle \overline{f^{-1}} \sum_{(m,n)\in R^c\cap T^c} (\bar{f}/f)\hat{\ }(m-j,n-k)e^{ims+int}, f^{-1}e^{ijs+ikt} \right\rangle_{L^2(\mu)}$$
$$= \sum_{(j,k)\in R\cap T}\ \sum_{(m,n)\in R^c\cap T^c} [(\bar{f}/f)\hat{\ }(m-j,n-k) \cdot\langle |f|^2\overline{f^{-1}}e^{ims+int}, f^{-1}e^{ijs+ikt}\rangle_{L^2(\sigma_2)}]$$
$$= \sum_{(j,k)\in R\cap T}\ \sum_{(m,n)\in R^c\cap T^c} |(f/\bar{f})\hat{\ }(j-m,k-n)|^2$$
$$= \sum_{j=1}^{\infty}\sum_{k=1}^{\infty} jk|(f/\bar{f})\hat{\ }(j,k)|^2.$$

The second equality is established with the help of [1, Lemma 2.4], using the weak outerness assumption. The strong case can be proved in a similar way. □

Theorem 3.11 asserts that complete regularity holds if and only if associated coefficients satisfy a weighted square-summability condition. This condition is a restriction on the "argument" of f, and hence on $\log|f|$. In particular, this suggests that if $d\sigma = w\,d\mu_2$, where w is bounded away from zero and infinity, then complete regularity would be equivalent to a similar condition on the Fourier coefficients of w. This is indeed the case.

3.12. THEOREM. *Let* $d\mu = w\,d\sigma_2$, *where* $0 < k_1 \le w(e^{is}, e^{it}) \le k_2 < \infty$. *For any subsets* U *and* V *of* $\mathbf{Z}^2$, *let* $\tau(U, V)$ *be the trace of* $P(U)P(V)P(U)$. *Then*

$$k_1^2 \cdot \tau(U, V) \le \sum_{\alpha \in U} \sum_{\beta \in V} |\hat{w}_{\alpha-\beta}|^2 \le k_2^2 \cdot \tau(U, V).$$

Note that no factorization assumption on w is made: the role played by the outerness of f in Theorem 3.11 is here taken by the boundedness of $\log w$. Accordingly, the proof of Theorem 3.12 is more elementary. The argument below is simplified from [2, VI. 6].

To begin, let us identify $L^2(\sigma_2)$ and $l^2(\mathbf{Z}^2)$ in the usual way, and treat operators as matrices acting on $l^2(\mathbf{Z}^2)$. Similarly, we shall identify $L^2(\mu)$ with a weighted sequence space. To do this, let multiplication by w in $l^2(\mathbf{Z}^2)$ have the block structure

$$\begin{bmatrix} A & B^* \\ B & C \end{bmatrix}$$

with respect to $l^2(U) \oplus l^2(V)$. Note that A and C are positive, and a Neumann series argument shows that they are invertible. For $\alpha \in \mathbf{Z}^2$, write $e_\alpha = e^{i\alpha_1 s + i\alpha_2 t}$. Then we have the next lemma.

3.13. LEMMA. *The matrices for* $P(U)P(V)$ *and* $P(V)P(U)$ *are* $A^{-1}B^*$ *and* $C^{-1}B$, *respectively.*

Proof. Let P be the matrix for $P(V)P(U)$. For all $\alpha \in U$ and $\rho \in V$,

$$\langle [1 - P(V)P(U)]e_\alpha, e_\rho \rangle_{L^2(\mu)} = 0,$$

$$B_{\alpha\rho} - \left\langle \sum_{\sigma \in V} [P(V)P(U)]_{\alpha\sigma} e_\sigma, e_\rho \right\rangle_{L^2(\mu)} = 0,$$

$$B - CP = 0,$$

$$P = C^{-1}B.$$

The other case is similar. □

Now if $A^{-1/2}$ is the positive square root of A^{-1}, then $\{A^{-1/2}e_\alpha\}_{\alpha \in U}$ is an orthonormal basis for $\mathcal{L}^2_\mu(U)$ (identified as $l^2(U)$ with the inner product $\langle A, \cdot, \cdot \rangle_{l^2(U)}$). Hence

$$\begin{aligned} \tau(U, V) &= \sum_{\alpha \in U} \langle [P(U)P(V)P(U)]A^{-1/2}e_\alpha, A^{-1/2}e_\alpha \rangle_{L^2(\mu)} \\ &= \sum_{\alpha \in U} \langle A(A^{-1}B^*)(C^{-1}B)A^{-1/2}e_\alpha, A^{-1/2}e_\alpha \rangle_{l^2(U)} \\ &= \sum_{\alpha \in U} \|C^{-1/2}BA^{-1/2}e_\alpha\|^2_{l^2(U)} \\ &= \|C^{-1/2}BA^{-1/2}\|_2^2 \end{aligned}$$

Finally,

$$\|C^{-1/2}BA^{-1/2}\|_2^2 \le \|C^{-1/2}\|^2 \|B\|_2^2 \|A^{-1/2}\|^2$$
$$\le k_1^2 \cdot \sum_{\sigma\in U} \sum_{\beta\in V} |\hat{w}_{\alpha-\beta}|^2;$$

$$\sum_{\sigma\in U} \sum_{\beta\in V} |\hat{w}_{\alpha-\beta}|^2 = \|B\|_2^2$$
$$\le \|C^{1/2}\|^2 \|C^{-1/2}BA^{-1/2}\|_2^2 \|A^{1/2}\|^2$$
$$\le k_2^2 \cdot \tau(U, V).$$

This verifies Theorem 3.12. □

3.14. COROLLARY. *Let* $d\mu = w\,d\sigma_2$, *where* $0 < k_1 \le w(e^{is}, e^{it}) \le k_2 < \infty$. *Let* Q_1 *and* Q_2 *be the orthogonal projection operators of* $L^2_{(\mu)}$ *onto* $L^2_{(\mu)}(R \cap T)$ *and* $L^2_{(\mu)}(R^c \cap T^c)$, *respectively. Then the trace of* $Q_1 Q_2 Q_1$ *is finite if and only if*

$$\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} mn |\hat{w}_{mn}|^2 < \infty.$$

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