On the Valence Structure of Analytic Functions

BORIS SOLOMYAK

1

Let f be an analytic function in the open unit disk $\mathbf{D}$. The *valence function* is defined by

$$\nu_f(w) = \operatorname{card}[f^{-1}(w) \cap \mathbf{D}], \quad w \in \mathbf{C},$$

where pre-images are counted with multiplicities. In this article we give a negative answer to the following question posed by Stephenson [5, Question 2]. We say that a function is *analytic on the closed unit disc* $cl\mathbf{D}$ if it is analytic in some neighbourhood of $cl\mathbf{D}$.

QUESTION. If f and g are analytic on $cl\mathbf{D}$ with identical valence functions, does there exist an algebraic homeomorphism ψ of $\partial\mathbf{D}$ with $f \circ \psi \equiv g$?

THEOREM 1. *There exist f and g analytic on $cl\mathbf{D}$ such that $\nu_f(w) \equiv \nu_g(w)$, $w \in \mathbf{C}$, but $f \circ \psi \not\equiv g$ for any homeomorphism ψ of the unit circle $\partial\mathbf{D}$.*

The proof is based on the following theorem.

THEOREM 2. *There exists a function ϕ analytic on $cl\mathbf{D}$ and two disjoint arcs $I_1, I_2 \subset \partial\mathbf{D}$ such that ϕ maps each of them homeomorphically onto the same arc but with opposite orientations.*

Note that if ϕ is not required to be analytic across $\partial\mathbf{D}$ then such a function can be constructed easily. (Take, for example, $\phi(z) = \omega^2(z)$, where $\omega(z)$ is a conformal map of $\mathbf{D}$ onto $\{z \in \mathbf{D} : \operatorname{Re} z > 0\}$.) To some extent this is also true for Theorem 1: Stephenson [5] produced two analytic functions in $\mathbf{D}$, piecewise analytic and continuous on $\partial\mathbf{D}$, with all other properties of f and g in Theorem 1.

2

Before we proceed to the proof, let us indicate some relations between Theorem 2 and the multiplicity of analytic Toeplitz operators. Recall that for $\phi \in H^\infty$ the Toeplitz operator T_ϕ is the multiplication operator on the Hardy

Received May 24, 1990. Revision received January 3, 1991.
Michigan Math. J. 39 (1992).

space H^2: $T_\phi f=\phi f$. The multiplicity μ_T of an operator T is the least dimension of a subspace X such that $\{T^nX: n\geq 0\}$ is total. By μ_ϕ we denote the multiplicity μ_{T_ϕ}. Let us introduce the following characteristics.

The *maximal valence* $n(\phi)$ is defined by

$$n(\phi)=\sup\{\nu_\phi(w),\ w\in \mathbf{C}\}.$$

It is noteworthy that $\nu_\phi(w)=\dim\ker(T_\phi^*-\bar{w}I)$. Now let ϕ be analytic in $\mathbf{D}$ and continuous on $cl\mathbf{D}$. We put

$$\tilde{\nu}_\phi(w)=\operatorname{card}[\phi^{-1}(w)\cap cl\mathbf{D}] \quad\text{and}\quad \Omega^k(\phi)=\{w\in\mathbf{C}: \tilde{\nu}_\phi(w)=k\}.$$

The *essential maximal valence* is by definition

$$n_e(\phi)=\sup\{k\in N: \operatorname{card}\Omega^k(\phi)=\infty\}.$$

Clearly $n(\phi)\leq n_e(\phi)$. It was proved in [2] that "for good ϕ" $n_e(\phi)\leq\mu_\phi\leq n_e(\phi)+1$, and in [3] (see also [4]) μ_ϕ was completely determined for ϕ analytic on $cl\mathbf{D}$. Since it is more convenient to work with $n(\phi)$, a question arises under what conditions is $n(\phi)=n_e(\phi)$. It is readily seen that for a function ϕ analytic on $cl\mathbf{D}$ one has $n_e(\phi)=\sup\{k\in N: \Omega^k(\phi)$ contains an arc$\}$. Hence the inequality $n_e(\phi)>n(\phi)$ can occur only if two arcs of $\partial\mathbf{D}$ are mapped onto one arc with opposite orientations. It is not hard to derive from the reasoning of [1, Thm. 1] that for an *entire* ϕ such behaviour is impossible and thus $n_e(\phi)=n(\phi)$. Theorem 2 shows that for ϕ analytic on $cl\mathbf{D}$ this is not the case. Moreover, one can show that for such a ϕ

$$n(\phi)\leq n_e(\phi)\leq 2n(\phi)-1$$

is a sharp inequality.

3. Proof of Theorem 2

Fix $0<z_0<1$, and consider the function $f(z)=-i(z+z_0)^2$. The curve $\gamma=f(\partial\mathbf{D})$ consists of two loops: the "exterior" loop γ_e, and the "interior" loop γ_i. Denote the intersection points of γ with coordinate axes by A,B,C,D,E,F,G (see Figure 1). By $[AD]$, $[AB)$, etc. will be denoted the smallest arcs of γ with respective (included or excluded) endpoints. We shall construct a function ϕ analytic on $cl\mathbf{D}$ such that $\phi(\partial\mathbf{D})=f(\partial\mathbf{D})$ but the curve $t\mapsto\phi(e^{it})$ traces the interior loop γ_i as follows:

$$[AE),[EF),[FG),[GF),[FE),[EF),[FG),[GA).$$

Thus the arc $\alpha:=[EF)\cup[FG]$ is the image of three different circle arcs, one of which is mapped with the reverse orientation. The values E,G are the "returning points" in which the curve starts to trace its path in the opposite direction.

Let us observe that the behaviour of the boundary curve described above is not peculiar in itself. For example, consider the function $z\mapsto(\omega^3-\omega)(z)$, where ω is a conformal map of the unit disk onto the domain $\{z\in\mathbf{C}: |z|<5, \operatorname{Im} z>0\}$. The problem is to obtain such a behaviour for a function analytic across $\partial\mathbf{D}$.

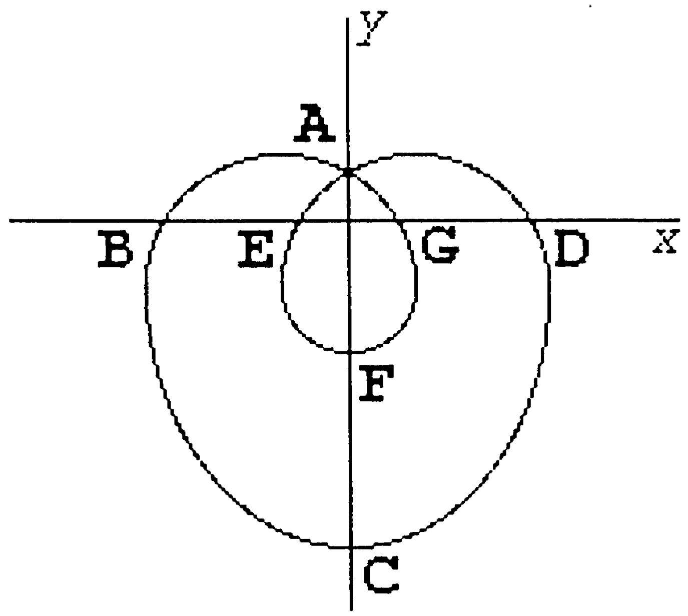

Figure 1

Now we pass to the construction of ϕ. Consider a polynomial of third degree with critical values E, G, and let R be the Riemann surface of its inverse. The required function ϕ will be the composition of a univalent function $\omega: \mathbf{D} \to R$ with the projection map on the Riemann surface.

We shall need a concrete geometric representation of the surface R. Take three copies of the complex plane P_I, P_{II}, P_{III} and apply a standard "cut and paste" operation. Namely, cut P_I and P_{II} along the ray $(-\infty, E]$ and then cut P_{II}, P_{III} along $[G, +\infty)$. Then identify the opposite edges of the cuts belonging to different sheets to obtain a surface R (see Figure 2).

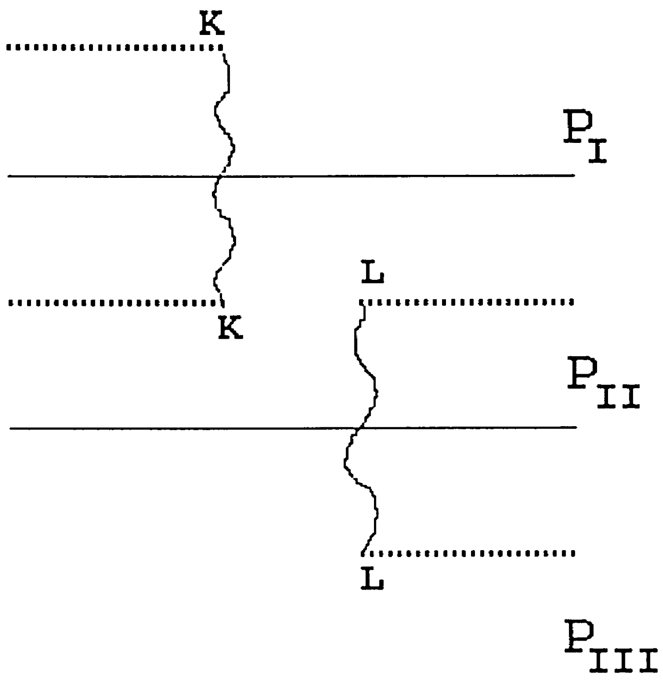

Figure 2

Define the projection $\pi : R \to \mathbf{C}$ to be the identity map for each sheet. Our construction has resulted in a branched covering with two simple branch points that will be denoted K, L: $\pi(K) = E$, $\pi(L) = G$. The conformal structure is lifted to the set $R \setminus \{K, L\}$ by means of π^{-1}. The functions $(z - E)^{1/2}$ and $(z - G)^{1/2}$ can be lifted in a single-valued manner to the neighbourhoods of K and L, and provide the conformal structure there.

Now we proceed lifting the curve $\gamma = f(\partial \mathbf{D})$ to R, starting with the point K and moving in the sheet P_I. Put

$$\Gamma_I = \pi^{-1}([EFGAB]) \cap P_I, \qquad \Gamma_{II}^{(1)} = \pi^{-1}([BCD]) \cap P_{II},$$

and

$$\Gamma_{III} = \pi^{-1}([DAEFG]) \cap P_{III}.$$

As long as we do not hit a branch point, the curve γ is lifted by π^{-1} without any problem. Since Γ_{III} arrives at the branch point L we must decide where to go next. We choose to continue in the sheet P_{II}, returning to K along the arc $\Gamma_{II}^{(2)} := \pi^{-1}(\alpha) \cap P_{II}$ (see Figure 3).

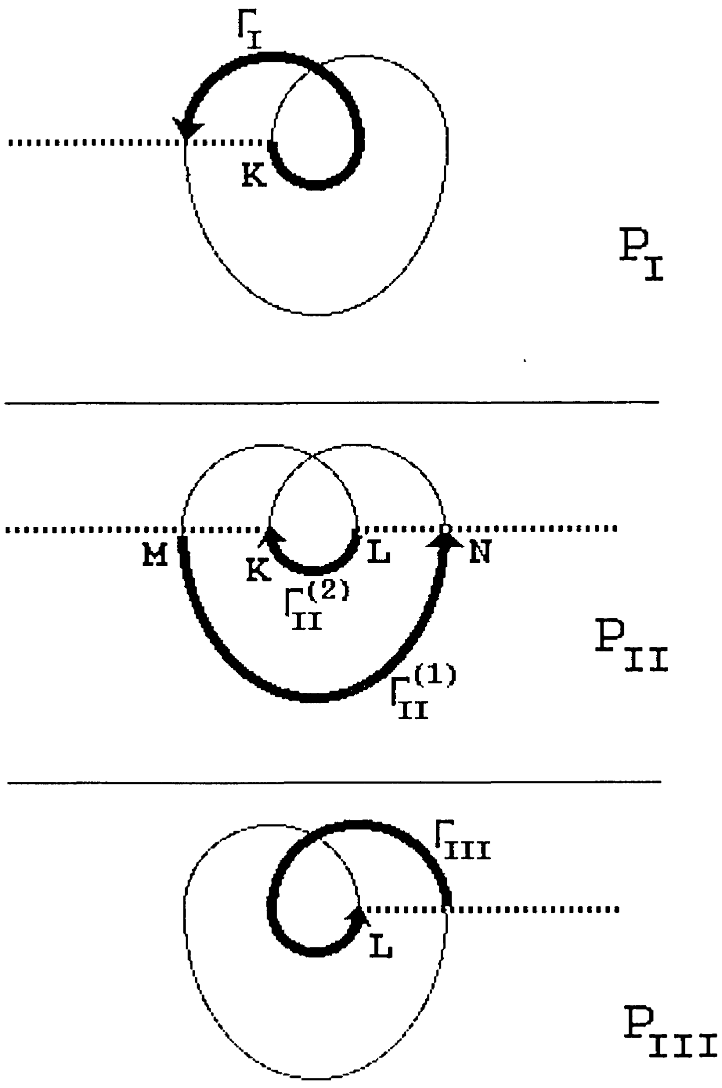

Figure 3

It is clear that $\Gamma=\Gamma_I\cup\Gamma_{II}^{(1)}\cup\Gamma_{III}\cup\Gamma_{II}^{(2)}$ is a simple closed curve on the Riemann surface R such that $\pi(\Gamma)=\gamma$. Moreover, Γ is contractible in R. Indeed, if we perform a homotopy of the arcs Γ_I, Γ_{III} to the cuts, we shall obtain a curve $[KM]\cup\Gamma_{II}^{(1)}\cup[NL]\cup\Gamma_{II}^{(2)}$ which is contractible in P_{II} (see Figure 3). Thus Γ is the boundary of a topological disk lying in R. By the Riemann mapping theorem there exists a univalent analytic function ω from the disk $\mathbf{D}$ onto the interior of Γ. Put $\phi=\pi\circ\omega$. Clearly the curve $t\mapsto\phi(e^{it})$ traces the arc α thrice as planned at the beginning of the proof. It remains to check that ϕ extends analytically to a neighbourhood of $cl\mathbf{D}$. This property may be verified for $\omega:\mathbf{D}\to R$. But this would follow if we prove that the curve Γ is analytic. In other words, we must show that any arc of Γ which is small enough can be represented as an image of a line segment under a suitable mapping analytic and univalent in a neighbourhood of the segment.

It is immediate from the construction of the curve that Γ is locally analytic outside the branch points K, L. Let U be a small disc centered at $z=E$ and let V be the component of $\pi^{-1}(U)$ such that $L\in V$. Then the intersection $V\cap\Gamma$ consists of two arcs α_+, α_- ending at L such that $\alpha_+\in P_{III}$ and $\alpha_-\in P_{II}$ (see Figure 3). The conformal structure in V is defined by the map $z\mapsto(z-E)^{1/2}$, which carries $\alpha_+\cup\alpha_-$ to a part of the arc $(\alpha-E)^{1/2}$ (both branches of the square root must be considered). It remains only to make use of the following elementary lemma.

LEMMA. *Let f be analytic in a neighbourhood of the origin, $f(0)=0$, $f'(0)\neq 0$, $\rho=f([0,\epsilon])$. Then for $\epsilon>0$ sufficiently small the set $\beta=\{z\in\mathbf{C}: z^2\in\rho\}$ is an analytic arc; that is, for some function g analytic at the origin, one has $g(0)=0$, $g'(0)\neq 0$, and $\beta=g([-\delta,\delta])$.*

Proof. One can write $f(z)=zh(z)$, where $h(z)=a_0+a_1z+\cdots$, $a_0\neq 0$. In a small neighbourhood of the origin there exists an analytic function $h_1(z)=b_0+b_1z+\cdots$ such that $h_1^2(z)=h(z)$. Thus $f(z^2)=(zh_1(z^2))^2$. We can put $g(z)=zh_1(z^2)$. Clearly $g'(0)=b_0\neq 0$ and $(g([-\delta,\delta]))^2=f([0,\delta^2])$. □

The local analyticity of Γ in the neighbourhood of K is verified in the same manner, and the proof of Theorem 2 is complete. □

4. Proof of Theorem 1

We employ the same ideas as in the proof of Theorem 2, so the exposition will be sketchier. Set $h(z)=(z+0.2)^3$ and $\gamma=h(\partial\mathbf{D})$. This curve has the form indicated in Figure 4.

We mark four points A, B, C, D on the interior loop of the curve. The functions f, g will be chosen so that $f(\partial\mathbf{D})=g(\partial\mathbf{D})=\gamma$. The arc $[AD]$ will be traced as follows:

$$t\mapsto f(e^{it})\colon [AD), [DA), [AC), [CB), [BD];$$

$$t\mapsto g(e^{it})\colon [AC), [CA), [AD), [DB), [BD].$$

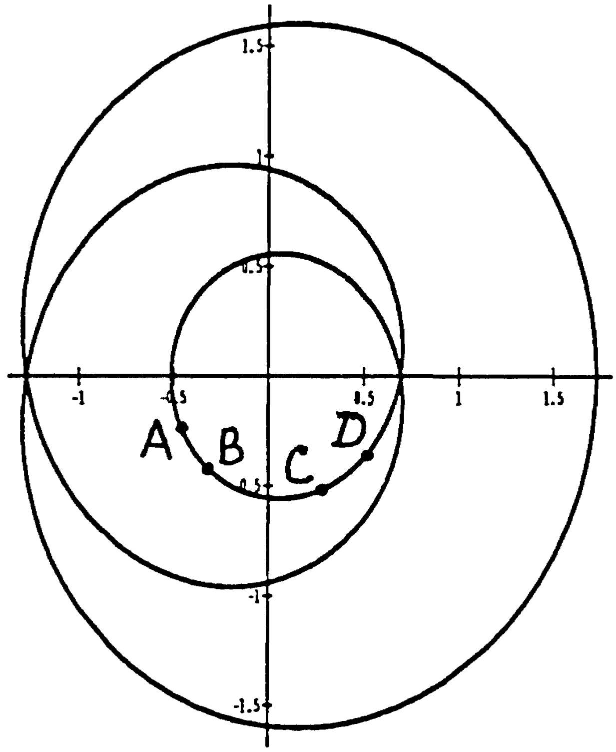

Figure 4

Because, for any homeomorphism ψ of the unit circle, the order in which the values A, B, C, D are taken by the functions $t \mapsto f(e^{it})$ and $t \mapsto f(\psi(e^{it}))$ coincides modulo a cyclic permutation, we see that $f \circ \psi \neq g$.

Now we pass to the construction of f, g. We consider two Riemann surfaces R_f, R_g, each obtained by a "cut and paste" procedure from five copies of the plane. Any pair of these planes can be pasted together to form a simple branch point by cutting them along a common ray and identifying opposite edges of the cuts, as was done in the earlier construction. Forming simple branch points over A, B, C, D in this way, using pairs of planes according to the two patterns in Figure 5, will determine Riemann surfaces denoted by R_f and R_g. Let π_f, π_g be the projection maps. We shall construct simple closed curves $\Gamma_f \subset R_f$ and $\Gamma_g \subset R_g$ such that $\pi_f(\Gamma_f) = \pi_g(\Gamma_g) = \gamma$. In Figure 5 one sees schematically the arcs of Γ_f, Γ_g projecting onto $[ABCD]$. All the rest of γ is lifted uniquely by means of π_f^{-1}, π_g^{-1}. As before, we can prove that Γ_f, Γ_g are contractible and then set $f = \pi \circ \omega_f$ and $g = \pi \circ \omega_g$, where ω_f, ω_g are the conformal mappings of the unit disk onto the interiors of Γ_f, Γ_g correspondingly. Using the lemma we obtain that f, g are analytic in a neighbourhood of $cl\mathbf{D}$. The construction is then complete.

It remains to check that f, g have identical valence functions. The argument principle yields that $\nu_f(w) = \nu_g(w)$, $w \in \mathbf{C} \setminus \gamma$. For the points $w \in \gamma \setminus [AD]$ the valence functions are also easily calculated to be equal. Over the arc $[BC]$ there are no *interior* points, since all its five pre-images are boundary arcs. So $\nu_f|_{[BC]} \equiv \nu_g|_{[BC]} \equiv 0$. Over the arcs $[AB), (CD]$ there are three boundary arcs. From Figure 5 it is clear that the remaining two arcs of $\pi_f^{-1}([AB))$

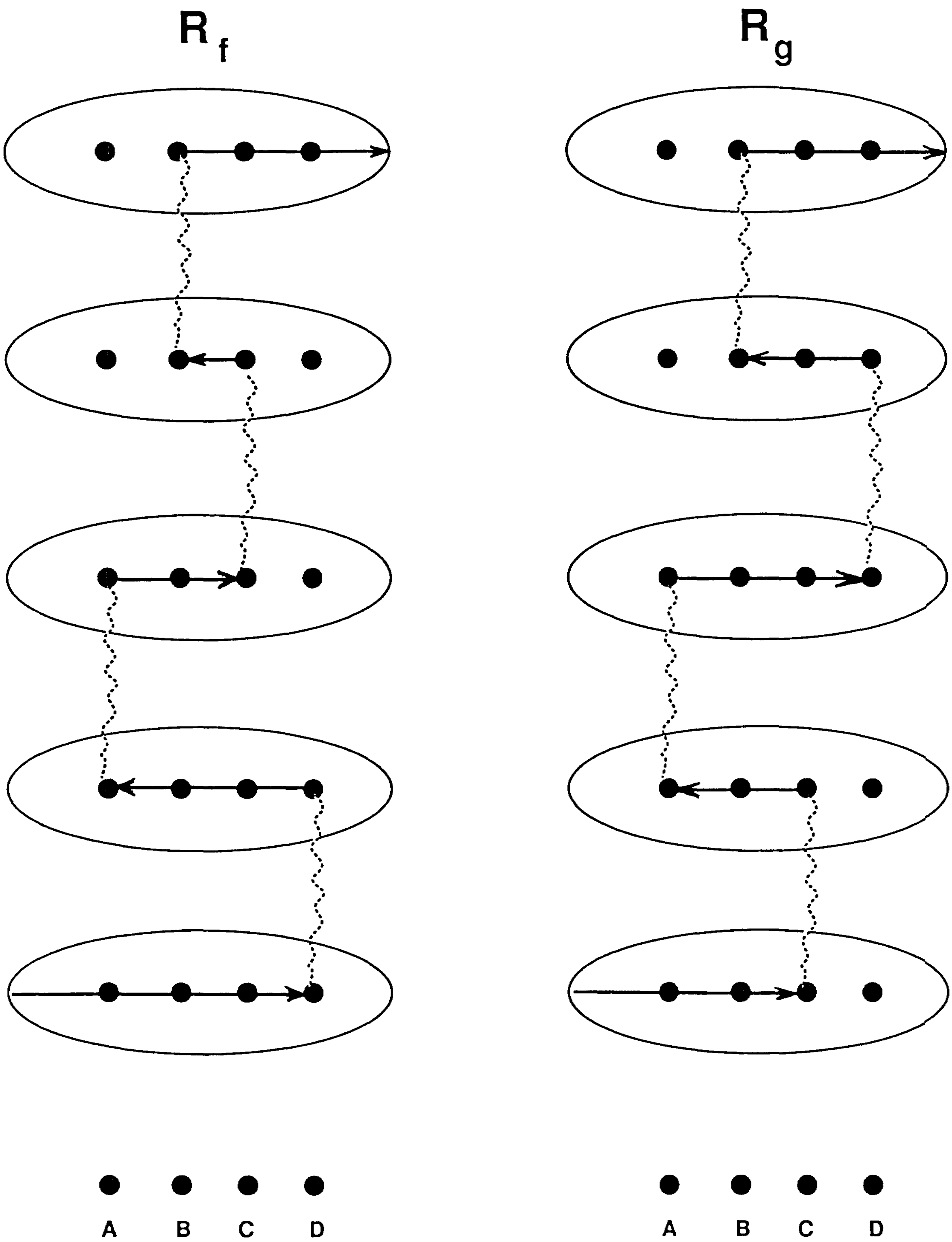

Figure 5

are separated by Γ_f. Therefore, only one pre-image of $[AB)$ lies in $\mathbf{D}$, and $\nu_f|_{[AB)} \equiv 1$. The same is true for the arc $(CD]$. Similarly, $\nu_g|_{[AB)} = \nu_g|_{(CD]} \equiv 1$. Thus $\nu_f \equiv \nu_g$ and we are done. □

I am grateful to Michael Lyubich for many helpful discussions on the subject of this article, and to the referee for valuable suggestions.

References

1. I. N. Baker, J. A. Deddens, and J. L. Ullman, *A theorem on entire functions with applications to Toeplitz operators,* Duke Math. J. 41 (1974), 739–745.
2. B. Solomyak, *On the multiplicity of the spectrum of analytic Toeplitz operators,* Soviet Math. Dokl. 33 (1986), 286–290.
3. B. Solomyak and A. Volberg, *Multiplicity of the spectrum of Toeplitz operators, weighted cocycles, and the vector Riemann problem,* Functional Anal. Appl. 21 (1987), 175–183.
4. ———, *Multiplicity of analytic Toeplitz operators,* Toeplitz operators and spectral function theory, pp. 87–192, Oper. Theory: Adv. Appl., 42, Birkhäuser, Basel, 1989.
5. K. Stephenson, *Analytic functions of finite valence, with applications to Toeplitz operators,* Michigan Math. J. 32 (1985), 5–19.

Department of Mathematics
University of Washington
Seattle, WA 98195