Bergman Projections and Duality in Weighted Mixed-Norm Spaces of Analytic Functions

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Introduction

In [1], Békollé found a necessary and sufficient condition for weight functions such that the Bergman operator is bounded on the corresponding weighted L^p spaces in the unit ball of $\mathbf{C}^N$. In [6], Luecking characterized the dual of weighted Bergman spaces with weights satisfying Békollé's conditions. In [5] Jevtić proved that there are bounded projections from general mixed norm spaces onto the weighted mixed norm spaces of analytic functions with the normal-function weights. Using these projections, he characterized the dual of weighted mixed norm spaces of analytic functions with normal-function weights.

In this paper, we show that the Bergman operator is bounded on weighted L^p spaces on the boundary of the unit ball of $\mathbf{C}^N$ with normal-function weights. Then we determine the weighted mixed norm spaces as the interpolation spaces between weighted L^p spaces on the unit ball of $\mathbf{C}^N$ and the weighted L^p spaces on the boundary of the unit ball $\mathbf{C}^N$ with different weights. These facts enable us to prove that the Bergman operator is bounded on weighted mixed norm spaces with radial weights satisfying Békollé's conditions. The characterization of the dual of weighted mixed norm spaces of analytic functions then follows from the boundedness of the Bergman operator.

As applications, we extend Luecking's result to the mixed norm spaces with radial weights, and also show that Jevtić's results are corollaries of our results.

In Section 1, we present the notations and statements of the main results. The proofs of main results are given in Section 2 and Section 3. In the last two sections, applications of the main results are given to extend Luecking's result and to obtain Jevtić's results.

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1. Notations and Results

Let $\mathbf{U}$ denote the unit ball in $\mathbf{C}^N$, $N \geq 1$. Denote by m Lebesgue measure on $\mathbf{C}^N = \mathbf{R}^{2N}$ normalized so that $m(\mathbf{U}) = 1$. For $\alpha \geq -1$, let $dm_\alpha = c_\alpha(1-|z|^2)^\alpha \, dm$ with c_α chosen so that when $\alpha > -1$, $m_\alpha(\mathbf{U}) = 1$. Denote by ν_0 the surface measure on the boundary $\mathbf{S}$ of $\mathbf{U}$ normalized so that $\nu_0(S) = 1$. For $z = (z_1, z_2, \ldots, z_N)$ and $w = (w_1, w_2, \ldots, w_N)$ in $\mathbf{C}^N$, let $\langle z, w \rangle = \sum_{i=1}^N z_i \bar{w}_i$ so that $|z|^2 = \langle z, z \rangle$.

We say that a positive function ω satisfies Békollé's B_α^p condition $(1 < p < \infty)$ if

$$\int_K \omega \, dm_\alpha \left(\int_K \omega^{-(p'/p)} \, dm_\alpha \right)^{p/p'} \leq C m_\alpha^p(K),$$

where $K = K(a) = \{z \in \mathbf{U} : |1 - \langle z, |a|/a \rangle| < 1 - |a|\}$ for a in $\mathbf{U}$, $p' = p/(p-1)$, and C independent of $K(a)$.

For a radial function ω, the condition B_α^p can be written in the form

$$\begin{aligned} &\int_{1-h}^1 \omega(r)(1-r^2)^\alpha r^{2N-1} \, dr \\ &\times \left[\int_{1-h}^1 \omega(r)^{-(p'/p)} (1-r^2)^\alpha r^{2N-1} \, dr \right]^{p/p'} \leq C h^{(\alpha+1)p} \end{aligned} \tag{1}$$

for all $0 < h < 1$.

A positive continuous function $\varphi(r)$ on $[0,1)$ is normal if there exist a and b, $0 < a < b$, such that

$$\begin{aligned} &\text{(i)} \quad \frac{\varphi(r)}{(1-r)^a} \text{ is nonincreasing with } \lim_{r \to 1^-} \frac{\varphi(r)}{(1-r)^a} = 0, \quad \text{and} \\ &\text{(ii)} \quad \frac{\varphi(r)}{(1-r)^b} \text{ is nondecreasing with } \lim_{r \to 1^-} \frac{\varphi(r)}{(1-r)^b} = \infty. \end{aligned} \tag{2}$$

Let $\hat{b} = \inf\{b : b \text{ satisfies (ii) of (2)}\}$.

The functions $\{\varphi, \psi\}$ will be called a *normal pair* if φ is normal and there exists $\lambda > \hat{b}$ such that, for $0 \leq r < 1$,

$$\varphi(r)\psi(r) = (1-r^2)^\lambda. \tag{3}$$

If φ is normal, then there exists ψ such that $\{\varphi, \psi\}$ is a normal pair and ψ is normal (see [10]).

Let A be a nonnegative measurable function on $\mathbf{U}$, and let $B(r)$ and $C(r)$ be nonnegative radial measurable functions on $[0,1)$ such that the Lebesgue measure of $\{r \in [0,1) : C(r) = 0\}$ is zero. For measurable functions f on $\mathbf{U}$ and $\zeta \in \mathbf{S}$, let

$$\|f_r\|_{A,p}^p = \int_{\mathbf{S}} |f(r\zeta)|^p A(r\zeta) \, d\nu_0(\zeta) \quad \text{for } 0 \leq r < 1 \text{ and } 1 \leq p \leq \infty.$$

Since $|f(r\zeta)|^p A(r\zeta)$ is a measurable function on $\mathbf{U}$, $\|f_r\|_{A,p}^p$ is a measurable function on $[0,1)$. (See [7, p. 150].)

Denote

$$L^{p,q}(A,B)=\left\{f:\int_0^1\|f_r\|_{A,p}^q B(r)r^{2N-1}\,dr<\infty\right\},\quad 1\le q<\infty$$

and

$$L^{p,\infty}(A,C)=\{f:\sup_{r\in[0,1)}\|f_r\|_{A,p}C(r)<\infty\}.$$

Denote by $H(\mathbf{U})$ the space of analytic functions on $\mathbf{U}$, and let

$$H^{p,q}(A,B)=L^{p,q}(A,B)\cap H(\mathbf{U})\quad\text{and}\quad H^{p,\infty}=L^{p,\infty}(A,C)\cap H(\mathbf{U}).$$

In the case that A is a radial function, $B(r)=\omega(r)(1-r^2)^\alpha$, and $C(r)\equiv 1$, we will write

$$L^{p,q}(A^{q/p}\omega(1-r^2)^\alpha)=L^{p,q}(A,B),\quad L^{p,\infty}(A)=L^{p,\infty}(A,C),$$

and

$$H^{p,q}(A^{q/p}\omega(1-r^2)^\alpha)=H^{p,q}(A,B),\quad H^{p,\infty}(A)=H^{p,\infty}(A,C).$$

Let $K_\alpha(z,w)=(1-\langle z,w\rangle)^{-N-1-\alpha}$ for $\alpha>-1$ and z,w in $\mathbf{U}$. The Bergman operator T_α is defined by [8] as

$$T_\alpha f(z)=\binom{N+\alpha}{N}\int_{\mathbf{U}}K_\alpha(z,w)f(w)\,dm_\alpha(w).$$

Define

$$T_\alpha^* f(z)=\binom{N+\alpha}{N}\int_{\mathbf{U}}|K_\alpha(z,w)|f(w)\,dm_\alpha(w).$$

Note that T_α^* is a linear operator.

In [1], Békollé proved the following theorem.

THEOREM (Békollé). *T_α^* is bounded on $L^{p,p}(\omega(z),(1-r^2)^\alpha)$ if and only if ω satisfies the condition B_α^p.*

The main results of this paper are contained the following two theorems.

THEOREM 1.1. *Suppose $p\le q\le\infty$ and $1<p<\infty$. Suppose that φ is a normal function and $\alpha-\hat{b}>-1$, where $\hat{b}$ is as defined after* (2). *If ω is a radial function and if $\varphi^p\omega$ satisfies the condition B_α^p, that is,*

$$\int_{1-h}^1\omega(r)\varphi^p(r)(1-r^2)^\alpha r^{2N-1}\,dr \times\left(\int_{1-h}^1\omega^{-(p'/p)}(r)\varphi^{-p'}(r)(1-r^2)^\alpha r^{2N-1}\,dr\right)^{p/p'}\le Ch^{(\alpha+1)p}\tag{4}$$

for all $0<h<1$, then

(i) *T_α^* is bounded on $L^{p,q}(\varphi^q\omega(1-r^2)^\alpha)$, and*
(ii) *T_α^* is bounded on $L^{p',q'}(\varphi^{-q'}\omega^{-(q'/q)}(1-r^2)^\alpha)$.*

THEOREM 1.2. *Suppose $p\le q\le\infty$ and $1<p<\infty$. If $\{\varphi,\psi\}$ is a normal pair with $\alpha+\lambda-\hat{b}>-1$, where $\hat{b}$ is as defined after* (2) *and λ is as in* (3), *and if $\omega(r)\ge 0$ satisfies*

$$\int_{1-h}^{1}\omega(r)\varphi^{p}(r)(1-r^2)^{\alpha}r^{2N-1}\,dr \times\left(\int_{1-h}^{1}\omega^{-(p'/p)}(r)\varphi^{-p'}(r)(1-r^2)^{\alpha}r^{2N-1}\,dr\right)^{p/p'}\le Ch^{(\alpha+\lambda+1)p} \tag{5}$$

for all $0<h<1$, *then the dual of* $H^{p,q}(\varphi^{q}\omega(1-r^2)^{\alpha})$ *can be identified with* $H^{p',q'}(\psi^{q'}\omega^{-(q'/q)}(1-r^2)^{\alpha})$ *under the pairing*

$$\langle f,g\rangle=\int_{\mathbf{U}}f(z)\bar{g}(z)\,dm_{\alpha+\lambda}(z). \tag{6}$$

More precisely, if g belongs to $H^{p',q'}(\psi^{q'}\omega^{-(q'/q)}(1-r^2)^{\alpha})$, *then the functional on* $H^{p,q}(\varphi^{q}\omega(1-r^2)^{\alpha})$ *defined by*

$$L_g(f)=\langle f,g\rangle$$

belongs to $[H^{p,q}(\varphi^{q}\omega(1-r^2)^{\alpha})]^*$ *and* $\|L_g\|\le C\|g\|_{H^{p',q'}(\psi^{q'}\omega^{-(q'/q)}(1-r^2)^{\alpha})}$.

Conversely, given a linear function L in $[H^{p,q}(\varphi^{q}\omega(1-r^2)^{\alpha})]^*$, *there is a unique g in* $H^{p',q'}(\psi^{q'}\omega^{-(q'/q)}(1-r^2)^{\alpha})$ *such that* $L_g=L$ *and*

$$\|g\|_{H^{p',q'}(\psi^{q'}\omega^{-(q'/q)}(1-r^2)^{\alpha})}\le C\|L\|.$$

Throughout this paper, C will be used to denote a positive constant, not necessarily the same at each occurrence. The symbols φ,ψ and a,b,λ will always be as in (2) and (3).

2. Proof of Theorem 1.1

We will use Békollé's theorem and an interpolation theorem, Theorem 2.5, to prove Theorem 1.1. We first prove that the Bergman operator is bounded on the weighted Hardy type spaces $L^{p,\infty}(\varphi^{p})$, $1\le p\le\infty$.

THEOREM 2.1. *Let* φ *be a normal function and let* $\hat{b}>0$ *be as defined after* (2). *If* $\alpha-\hat{b}>-1$, *then* T_α^* *is bounded on* $L^{p,\infty}(\varphi^{p})$.

Proof. Let $a>0$ as in (2). Suppose $\zeta,\xi\in\mathbf{S}$ and that $z=\rho\zeta$ and $w=r\xi$, where $0\le\rho<1$ and $0\le r<1$. Let $k+l=N+\alpha-1$, where k and l will be determined later.

For $1<p<\infty$,

$$\begin{aligned}
&\int_{\mathbf{S}}|T_\alpha^*f(z)|^{p}\varphi^{p}(\rho)\,d\nu_0(\zeta)\\
&\le\int_{\mathbf{S}}\left[\int_{\mathbf{U}}\frac{|f(w)|}{|1-\langle z,w\rangle|^{N+\alpha+1}}\,dm_\alpha(w)\right]^{p}\varphi^{p}(\rho)\,d\nu_0(\zeta)\\
&=\int_{\mathbf{S}}\left[\int_{\mathbf{U}}\frac{|f(w)|\varphi(r)}{|1-\langle z,w\rangle|^{N+\alpha+1}\varphi(r)}\,dm_\alpha(w)\right]^{p}\varphi^{p}(\rho)\,d\nu_0(\zeta)\\
&\le\int_{\mathbf{S}}\left[\int_{\mathbf{U}}\frac{|f(w)|^{p}\varphi(r)^{p}}{|1-\langle z,w\rangle|^{kp}\varphi(r)}\,dm_\alpha(w)\right]\\
&\quad\times\left[\int_{\mathbf{U}}\frac{dm_\alpha(w)}{|1-\langle z,w\rangle|^{lp'}\varphi(r)}\right]^{p/p'}\varphi^{p}(\rho)\,d\nu_0(\zeta).
\end{aligned}$$

The second factor of the integrand is

$$I=\left[\int_{\mathrm{U}}\frac{dm_\alpha(w)}{|1-\langle z,w\rangle|^{lp'}\varphi(r)}\right]^{p/p'}$$
$$=\left[\int_0^1\left(\int_{\mathrm{S}}\frac{c_\alpha\,d\nu_0(\xi)}{|1-\langle z,w\rangle|^{lp'}}\right)\frac{(1-r^2)^\alpha}{\varphi(r)}r^{2N-1}\,dr\right]^{p/p'}.$$

If $lp'-N>0$, then (see [8, p. 17])

$$\int_{\mathrm{S}}\frac{d\nu_0(\xi)}{|1-\langle z,w\rangle|^{lp'}}=O\left(\frac{1}{(1-|r\rho|)^{lp'-N}}\right).$$

This implies

$$I\le C\left[\int_0^1\frac{(1-r^2)^\alpha\,dr}{\varphi(r)(1-r\rho)^{lp'-N}}\right]^{p/p'}.$$

By the assumption $\alpha-\hat{b}>-1$, there is a $b>0$ such that $\alpha-b>-1$. We can write

$$I\le C\left[\int_0^\rho\frac{(1-r)^a(1-r)^{\alpha-a}\,dr}{\varphi(r)(1-r\rho)^{lp'-N}}+\int_\rho^1\frac{(1-r)^b(1-r)^{\alpha-b}\,dr}{\varphi(r)(1-r\rho)^{lp'-N}}\right]^{p/p'}$$
$$\le C\left[\frac{(1-\rho)^a}{\varphi(\rho)}\int_0^\rho\frac{(1-r)^{\alpha-a}\,dr}{(1-r\rho)^{lp'-N}}+\frac{(1-\rho)^b}{\varphi(\rho)}\int_\rho^1\frac{(1-r)^{\alpha-b}\,dr}{(1-r\rho)^{lp'-N}}\right]^{p/p'}.$$

The last inequality follows because $(1-r)^a/\varphi(r)$ is nondecreasing whereas $(1-r)^b/\varphi(r)$ is nonincreasing. A simple calculation shows that, if $lp'-N>\alpha-a+1$, then (see [10, p. 291])

$$I\le C\left[\frac{(1-\rho)^a}{\varphi(\rho)}(1-\rho)^{-lp'+N+\alpha-a+1}+\frac{(1-\rho)^b}{\varphi(\rho)}(1-\rho)^{-lp'+N+\alpha-b+1}\right]^{p/p'}$$
$$=C\left[\frac{(1-\rho)^{N+1+\alpha-lp'}}{\varphi(\rho)}\right]^{p/p'}.$$

Therefore

$$\int_{\mathrm{S}}|T_\alpha^*f(z)|^p\varphi^p(\rho)\,d\nu_0(\zeta)$$
$$\le C\left[\frac{(1-\rho)^{N+1+\alpha-lp'}}{\varphi(\rho)}\right]^{p/p'}\int_{\mathrm{S}}\left[\int_{\mathrm{U}}\frac{|f(w)|^p\varphi^p(r)}{|1-\langle z,w\rangle|^{kp}\varphi(r)}\,dm_\alpha(w)\right]\varphi^p(\rho)\,d\nu_0(\zeta)$$
$$\le C\left[\frac{(1-\rho)^{N+1+\alpha-lp'}}{\varphi(\rho)}\right]^{p/p'}$$
$$\times\int_{\mathrm{U}}\left(\int_{\mathrm{S}}\frac{d\nu_0(\zeta)}{|1-\langle z,w\rangle|^{kp}}\right)\frac{|f(w)|^p\varphi^p(\rho)\varphi^p(r)}{\varphi(r)}\,dm_\alpha(w)$$
$$\le C\left[\frac{(1-\rho)^{N+1+\alpha-lp'}}{\varphi(\rho)}\right]^{p/p'}\int_{\mathrm{U}}\frac{1}{|1-\rho r|^{kp-N}}\frac{|f(w)|^p\varphi^p(\rho)\varphi^p(r)}{\varphi(r)}\,dm_\alpha(w)$$
$$\le C\left[\frac{(1-\rho)^{N+1+\alpha-lp'}}{\varphi(\rho)}\right]^{p/p'}\|f\|_{L^{p,\infty}(\varphi^p)}\int_0^1\frac{(1-r)^\alpha\varphi^p(\rho)}{(1-r\rho)^{kp-N}\varphi(r)}\,dr\le$$

$$\leq C\|f\|_{L^{p,\infty}(\varphi^p)}\left[\frac{(1-\rho)^{N+1+\alpha-lp'}}{\varphi(\rho)}\right]^{p/p'}\frac{(1-\rho)^{N+1+\alpha-kp}}{\varphi(\rho)}\varphi^p(\rho)$$
$$= C\|f\|_{L^{p,\infty}(\varphi^p)}.$$

The third inequality holds if $kp-N>0$, and the fifth inequality holds if $kp-N>\alpha-a+1$. Therefore if we choose k and l so that

$$kp-N>0, \quad kp-N>\alpha-a+1,$$

and

$$lp'-N>0, \quad lp'-N>\alpha-a+1,$$

or, more simply, so that

$$l+\frac{a}{p'}>\frac{N+1+\alpha}{p'} \quad \text{and} \quad k+\frac{a}{p}>\frac{N+1+\alpha}{p},$$

then $\|T_\alpha^* f\|_{L^{p,\infty}(\varphi^p)}\leq C\|f\|_{L^{p,\infty}(\varphi^p)}$. Since $k+l=N+1+\alpha$, we can choose $l=(N+1+\alpha)/p'$ and $k=(N+1+\alpha)/p$ so that all requirements are satisfied.

For $p=1$ and $p=\infty$, the arguments are similar. □

REMARKS. 1. T_α^* is not bounded on "unweighted Hardy type" spaces $L^{p,\infty}(1)$. In fact, $f(x)\equiv 1$ belongs to $L^{p,\infty}(1)$ but $T_\alpha^*(1)$ is not in $L^{p,\infty}(1)$.

2. The condition $\alpha-\hat{b}>-1$ may not be weakened. In fact, take $\varphi(r)=(1-r)^c$ for some $c>0$; then $\hat{b}=c$. If $\alpha-c=-1$, take $f=(1-r)^{-c}$; then f is in $L^{p,\infty}(\varphi^p)$ but $T_\alpha^*(f)$ is undefined.

Next we give a real interpolation theorem for mixed-norm spaces. We will follow the notation of [3].

DEFINITION 2.2. Let X_0, X_1 be two topological vector spaces. X_0 and X_1 are said to be *compatible* if there is a Hausdorff topological vector space U such that X_0 and X_1 are subspaces of U.

Let $\bar{X}=(X_0,X_1)$ denote a compatible couple of two quasi-normed spaces X_0 and X_1.

DEFINITION 2.3. For a in $\sum_{i=0}^1 X_i$, let

$$K(t,a)=K(t,a;X_0,X_1)$$
$$=\inf\{\|a_0\|_{X_0}+t\|a_1\|_{X_1}: a=a_0+a_1,\ a_0\in X_0,\ a_1\in X_1\},$$
$$\|a\|_{\theta,q,\bar{X}}^q=\int_0^\infty [t^\theta K(t,a)]^q\,\frac{dt}{t} \quad \text{for } 0<\theta<1 \text{ and } 0<q\leq\infty,$$

and let

$$(X_0,X_1)_{\theta,q,\bar{X}}=\left\{a\in\sum_{i=0}^1 X_i: \|a\|_{\theta,q,\bar{X}}<\infty\right\}.$$

LEMMA 2.4. *Let $\bar{X}=(X_0,X_1)$ and $\bar{Y}=(Y_0,Y_1)$. Suppose T is a linear map from $\sum_{i=0}^1 X_i$ into $\sum_{i=0}^1 Y_i$ such that, for each $a_i\in X_i$,*

$$\|Ta_i\|_{Y_i}\leq K_i\|a_i\|_{X_i}.$$

Then T is a bounded map from $(X_0,X_1)_{\theta,q,\bar{X}}$ into $(Y_0,Y_1)_{\theta,q,\bar{Y}}$.

For the proof, see [3].

Suppose that C is a nonnegative radial function on $[0,1)$ such that the Lebesgue measure of the set $\{r\in[0,1): C(r)=0\}$ is zero. Suppose B is a nonnegative function on $[0,1)$ and that $d\mu=B(r)C(r)^{-\gamma}r^{2N-1}\,dr$ is a σ-finite measure on $[0,1)$, where $0<\gamma<\infty$. Fix $1\le p<\infty$. Let

$$m(\rho,f)=\mu\{r\in[0,1): \|f_r\|_{A,p}C(r)>\rho\},$$

$$f^*(t)=\inf\{\rho: m(\rho,f)\le t\},$$

and

$$\|f\|_{\tau,q}^q=\int_0^\infty [t^{1/\tau}f^*(t)]^q\,\frac{dt}{t}\quad\text{for } 0<\tau\le\infty,\ 0<q\le\infty.$$

The vector-valued Lorentz space $L(p,\tau,q)$ is defined by

$$L(p,\tau,q)=\{f: \|f\|_{\tau,q}\le\infty\}.$$

For the properties of $m(\rho,f)$, f^*, and $L(p,\tau,q)$, see [4] and [3].

Assuming that $\bar{L}=(L^{p,q}(A,B),L^{p,\infty}(A,C))$ is a compatible couple, with a slight modification of its proof we can extend Theorem 5.2.1 of [3] to the following vector-valued version.

THEOREM 2.5. *Suppose f is in $L^{p,\gamma}(A,B)+L^{p,\infty}(A,C)$, where $1\le p<\infty$ and $0<\gamma<\infty$. Then*

(i) *$K(t,f;L^{p,\gamma}(A,B),L^{p,\infty}(A,C))\sim(\int_0^{t^\gamma}|f^*(s)|^\gamma\,ds)^{1/\gamma}$, and*

(ii) *for $\gamma<q\le\infty$, $1/\tau=(1-\theta)/\gamma$,*

$$(L^{p,\gamma}(A,B),L^{p,\infty}(A,C))_{\theta,q,L}=L(p,\tau,q).$$

The proof follows the argument in the proof of Theorem 5.2.1 in [3] with $|f(x)|$ replaced by $\|f_r\|_{A,p}C(r)$. We omit the details.

Let $\gamma=p$, $\theta=1-p/q$, and $C(r)\equiv1$ in (ii) of Theorem 2.5. We have the following corollary.

COROLLARY 2.6. *For $q>p$,*

$$L^{p,q}(A,B)=(L^{p,p}(A,B),L^{p,\infty}(A,1))_{1-p/q,q}.$$

In particular,

$$L^{p,q}(\varphi^q\omega(1-r^2)^\alpha)=(L^{p,p}(\varphi^p\omega(1-r^2)^\alpha),L^{p,\infty}(\varphi^p))_{1-p/q,q}.$$

Proof. It is easy to verify that $L(p,q,q)=L^{p,q}(A,BC^{q-r})$. The conclusion follows from this fact and (ii) of Theorem 2.5. □

In the proof of Theorem 1.1, we will use the following pairing between functions in $L^{p,q}(\varphi^q\omega(1-r^2)^\alpha)$ and functions in $L^{p',q'}(\varphi^{-q'}\omega^{-(q'/q)}(1-r^2)^\alpha)$:

$$\langle f,g\rangle=\int_U f(z)\bar{g}(z)\,dm_\alpha(z). \tag{7}$$

In [2, p. 304] Benedek and Panzone showed that the dual space of $L^{p,q}(\varphi^q\omega(1-r^2)^\alpha)$ can be identified with $L^{p',q'}(\varphi^q\omega(1-r^2)^\alpha)$ under the pairing

$$\langle f, g\rangle = \int_{\mathbf{U}} f(z)\bar{g}(z)\varphi^q\omega\, dm_\alpha(z).$$

LEMMA 2.7. *If* $1 \le p < \infty$ *and* $1 \le q < \infty$, *then under the pairing* (7) *the dual of* $L^{p,q}(\varphi^q\omega(1-r^2)^\alpha)$ *can be identified with* $L^{p',q'}(\varphi^{-q'}\omega^{-(q'/q)}(1-r^2)^\alpha)$.

Proof. For each linear function L on $L^{p,q}(\varphi^q\omega(1-r^2)^\alpha)$ there is a unique h in $L^{p',q'}(\varphi^q\omega(1-r^2)^\alpha)$ such that, for each f in $L^{p,q}(\varphi^q\omega(1-r^2)^\alpha)$,

$$L(f) = \int_{\mathbf{U}} f(z)\bar{h}(z)\varphi^q(r)\omega(r)\, dm_\alpha(z)$$

and $\|L\| = \|h\|_{L^{p',q'}(\varphi^q\omega(1-r^2)^\alpha)}$. (See [2].)

Let $g = h\varphi^q(r)\omega(r)$. Then a simple calculation shows that

$$\|g\|^{q'}_{L^{p',q'}(\varphi^{-q'}\omega^{-(q'/q)}(1-r^2)^\alpha)} = \|h\|_{L^{p',q'}(\varphi^q\omega(1-r^2)^\alpha)}.$$

Therefore g belongs to $L^{p',q'}(\varphi^{-q'}\omega^{-(q'/q)}(1-r^2)^\alpha)$ and

$$L(f) = \int_{\mathbf{U}} f(z)\bar{g}(z)\, dm_\alpha(z).$$

Conversely, for each g in $L^{p',q}(\varphi^{-q'}\omega^{-(q'/q)}(1-r^2)^\alpha)$ it follows from Hölder's inequality that

$$\int_{\mathbf{U}} f(z)\bar{g}(z)\, dm_\alpha(z) = L_g(f)$$

is a bounded linear functional on $L^{p,q}(\varphi^q\omega(1-r^2)^\alpha)$. □

Now we turn to the proof of Theorem 1.1.

(i) From Corollary 2.6,

$$L^{p,q}(\varphi^q\omega(1-r^2)^\alpha) = (L^{p,p}(\varphi^p\omega(1-r^2)^\alpha), L^{p,\infty}(\varphi^p))_{1-p/q,q}.$$

By Békollé's theorem, condition (4) in the statement of Theorem 1.1 implies that T_α^* is bounded on $L^{p,p}(\varphi^p\omega(1-r^2)^\alpha)$. It follows from Theorem 2.1 and Corollary 2.6 that T_α^* is bounded on $L^{p,q}(\varphi^q\omega(1-r^2)^\alpha)$.

(ii) For f in $L^{p',q'}(\varphi^{-q'}\omega^{-(q'/q)}(1-r^2)^\alpha)$ and g in $L^{p,q}(\varphi^q\omega(1-r^2)^\alpha)$,

$$\begin{aligned}
&\left|\int_{\mathbf{U}} g(z)\overline{T_\alpha^* f(z)}\, dm_\alpha(z)\right| \\
&\quad\le \int_{\mathbf{U}} |g(z)||T_\alpha^* f(z)|\, dm_\alpha(z) \\
&\quad\le \int_{\mathbf{U}} T_\alpha^*|g(z)||f(w)|\, dm_\alpha(w) \\
&\quad\le C\|T_\alpha^*|g(z)|\|_{L^{p,q}(\varphi^q\omega(1-r^2)^\alpha)}\|f\|_{L^{p',q'}(\varphi^{-q'}\omega^{-(q'/q)}(1-r^2)^\alpha)} \\
&\quad\le C\|g\|_{L^{p,q}(\varphi^q\omega(1-r^2)^\alpha)}\|f\|_{L^{p',q'}(\varphi^{-q'}\omega^{-(q'/q)}(1-r^2)^\alpha)}.
\end{aligned}$$

The last inequality holds because of part (i). Lemma 2.7 implies that T_α^* is in $L^{p',q'}(\varphi^{-q'}\omega^{-(q'/q)}(1-r^2)^\alpha)$, since

$$\|T_\alpha^* f\|_{L^{p',q'}(\varphi^{-q'}\omega^{-(q'/q)}(1-r^2)^\alpha)} = \sup\left|\int_{\mathbf{U}} g(z)\overline{T_\alpha^* f(z)}\, dm_\alpha(z)\right|,$$

where the sup is taken over all g in the unit ball of $L^{p,q}(\varphi^q\omega(1-r^2)^\alpha)$ (see [2, p. 303].) Hence T_α^* is bounded on $L^{p',q'}(\varphi^{-q'}\omega^{-(q'/q)}(1-r^2)^\alpha)$. □

It is easy to show that, if $1\le p<\infty$ and $1\le q<\infty$, then

$$\lim_{\rho\to 1^-}\|f_\rho-f\|_{H^{p,q}(\varphi^q\omega(1-r^2)^\alpha)}=0,$$

where $f_\rho(z)=f(\rho z)$ (see [9, Prop. 3.3]). It follows that $H^\infty(\mathbf{U})$ is dense in $H^{p,q}(\varphi^q\omega(1-r^2)^\alpha)$. Since $T_\alpha f=f$ for all f in H^∞ (see [7, p. 121]), we have the following corollary of Theorem 1.1.

COROLLARY 2.8. *Under the assumptions of Theorem 1.1, T_α is a bounded projection of $L^{p,q}(\varphi^q\omega(1-r^2)^\alpha)$ onto $H^{p,q}(\varphi^q\omega(1-r^2)^\alpha)$ and a bounded projection of $L^{p',q'}(\varphi^{-q'}\omega^{-(q'/q)}(1-r^2)^\alpha)$ onto $H^{p',q'}(\varphi^{-q'}\omega^{-(q'/q)}(1-r^2)^\alpha)$ for $p\le q\le\infty$ and $1<p<\infty$.*

REMARKS. 1. The example in Remark 2 after the proof of Theorem 2.1 shows that, in general (assuming that $\int_0^1\omega(1-r)^\alpha dr<\infty$), in order to make T_α^* well defined in $L^{p,q}(\varphi^q\omega(1-r)^\alpha)$ we must have $\alpha-\hat{b}>-1$.

2. In order to make T_α^* a bounded operator, it is not necessary that ω and φ satisfy the condition (4). For example, when $N=1$, fix q and p with $q>p$, take $c>0$ and $\alpha>-1$ so that $\alpha-c>-1$ and $\alpha-c(q-p)<-1$. If we let $\varphi(r)=(1-r)^c$ and $\omega(r)=(1-r)^{-cq}$, then $\int_{1-h}^1\omega\varphi^p(1-r^2)^\alpha dr=\infty$, and hence $\varphi^p\omega$ does not satisfy the condition B_α^p. Because $L^{p,q}(\varphi^q\omega(1-r^2)^\alpha)=L^{p,q}((1-r^2)^\alpha)$, by choosing $\gamma>0$ very small and taking $\hat{\varphi}(r)=(1-r)^\gamma$ and $\hat{\omega}(r)=(1-r)^{-c\gamma}$, it is easy to see that $\hat{\varphi}^p\hat{\omega}$ satisfies the condition B_α^p. Theorem 1.1 then implies that T_α^* is bounded on $L^{p,q}(\hat{\varphi}^q\hat{\omega}(1-r^2)^\alpha)=L^{p,q}((1-r^2)^\alpha)$.

3. Suppose T_α is bounded on $L^{p,q}(\varphi^q\omega(1-r^2)^\alpha)$. Following the method Békollé used in [1, p. 311], if we put $f(z)=\omega^{-(q'/q)}(r)\varphi^{-q'}(r)\chi_{K(a)}(z)$, we can show that $\varphi^q\omega$ satisfies the condition B_α^q.

3. Proof of Theorem 1.2

Take $\tilde{\omega}(r)=\omega(r)(1-r^2)^{-\lambda}$. Then

$$\varphi^{-p'}\tilde{\omega}^{-(p'/p)}=\varphi^{-p'}\omega^{-(p'/p)}(1-r^2)^{\lambda(p'/p)}=\psi^{p'}\omega^{-(p'/p)},$$

$$L^{p,q}(\varphi^q\omega(1-r^2)^\alpha)=L^{p,q}(\varphi^q\tilde{\omega}(1-r^2)^{\alpha+\lambda}),$$

and

$$L^{p',q'}(\psi^{q'}\omega^{-(q'/q)}(1-r^2)^\alpha)=L^{p',q'}(\varphi^{-q'}\tilde{\omega}^{-(q'/q)}(1-r^2)^{\alpha+\lambda}).$$

Thus, if ω satisfies (5) then $\tilde{\omega}$ satisfies (4). Theorem 1.1 implies that $T_{\alpha+\lambda}^*$ is a bounded map on both

$$L^{p,q}(\varphi^q\tilde{\omega}(1-r^2)^{\alpha+\lambda}) \quad\text{and}\quad L^{p',q'}(\varphi^{-q'}\tilde{\omega}^{-(q'/q)}(1-r^2)^{\alpha+\lambda}).$$

Therefore, in order to prove Theorem 1.2, it suffices to prove that the dual of $L^{p,q}(\varphi^q\tilde{\omega}(1-r^2)^{\alpha+\lambda})$ can be identified with $L^{p',q'}(\varphi^{-q'}\tilde{\omega}^{-(q'/q)}(1-r^2)^{\alpha+\lambda})$ under the pairing (6) for $p\le q$.

Let g be in $L^{p',q'}(\varphi^{-q'}\tilde{\omega}^{-(q'/q)}(1-r^2)^{\alpha+\lambda})$. It follows from Hölder's inequality that

$$|\langle f, g\rangle| \le C\|f\|_{L^{p,q}(\varphi^q\tilde{\omega}(1-r^2)^{\alpha+\lambda})}\|g\|_{L^{p',q'}(\varphi^{-q'}\tilde{\omega}^{-(q'/q)}(1-r^2)^{\alpha+\lambda})}.$$

Hence g defines a bounded linear functional L on $H^{p,q}(\varphi^q\tilde{\omega}(1-r^2)^{\alpha+\lambda})$, and $\|L\| \le C\|g\|_{L^{p',q'}(\varphi^{-q'}\tilde{\omega}^{-(q'/q)}(1-r^2)^{\alpha+\lambda})}$.

Conversely, let L be a bounded linear functional on $H^{p,q}(\varphi^q\tilde{\omega}(1-r^2)^{\alpha+\lambda})$. Then L can be extended to be a linear functional on $L^{p,q}(\varphi^q\tilde{\omega}(1-r^2)^{\alpha+\lambda})$. By Lemma 2.7 there exists an h in $L^{p',q'}(\varphi^{-q'}\tilde{\omega}^{-(q'/q)}(1-r^2)^{\alpha+\lambda})$ such that $L(f)=\langle f, h\rangle$ and $\|L\| \sim \|h\|_{L^{p',q'}(\varphi^{-q'}\tilde{\omega}^{-(q'/q)}(1-r^2)^{\alpha+\lambda})}$.

By setting $g = T_{\alpha+\lambda}h$, from Theorem 1.1 we thereby have that g is in $H^{p',q'}(\varphi^{-q'}\tilde{\omega}^{-(q'/q)}(1-r^2)^{\alpha+\lambda})$. By the fact that $T_\alpha f = f$ for f in $H^\infty(\mathbf{U})$, Fubini's theorem implies that

$$L(f) = \langle f, h\rangle = \langle T_{\alpha+\lambda}f, h\rangle = \langle f, T_{\alpha+\lambda}(h)\rangle = \langle f, g\rangle$$

for every f in $H^\infty(\mathbf{U})$. Since H^∞ is dense in $H^{p,q}(\varphi^q\tilde{\omega}(1-r^2)^{\alpha+\lambda})$, the continuity of L implies $L(f)=\langle f, g\rangle$ for all f in $H^{p,q}(\varphi^q\tilde{\omega}(1-r^2)^{\alpha+\lambda})$. We also have

$$\|g\|_{L^{p',q'}(\varphi^{-q'}\tilde{\omega}^{-(q'/q)}(1-r^2)^{\alpha+\lambda})} \le C\|h\|_{L^{p',q'}(\varphi^{-q'}\tilde{\omega}^{-(q'/q)}(1-r^2)^{\alpha+\lambda})} \le C\|L\|.$$

Finally, suppose that g in $L^{p',q'}(\varphi^{-q'}\tilde{\omega}^{-(q'/q)}(1-r^2)^{\alpha+\lambda})$ determines a zero functional. Since $\overline{K_{\alpha+\lambda}(z,\cdot)}$ is in $H^\infty(\mathbf{U})$ for each fixed z in $\mathbf{U}$, we have $0 = \langle \overline{K_{\alpha+\lambda}(z,\cdot)}, g\rangle = Cg(z)$, $C>0$. Hence $g \equiv 0$. Therefore there is a one-to-one, continuous, linear transformation from $L^{p',q'}(\varphi^{-q'}\tilde{\omega}^{-(q'/q)}(1-r^2)^{\alpha+\lambda})$ onto the dual space of $L^{p,q}(\varphi^q\tilde{\omega}(1-r^2)^{\alpha+\lambda})$. □

4. Application of Theorem 1.1

In [5] Jevtić proved the following theorem.

THEOREM 4.1. *For $1 \le p \le \infty$ and $1 \le q \le \infty$, the transformation P defined by*

$$Pf(w) = \int_{\mathbf{U}} f(z)K_{\lambda-1}(z,w)\psi(r)(1-r^2)^{-(1/q')}\,dm(z)$$

for w in $\mathbf{U}$, is bounded from $L^{p,q}(r^{1-2N})$ onto $H^{p,q}(\varphi^q r^{1-2N}(1-r)^{-1})$.

We now show that if $1<p<\infty$ and $1<q<\infty$, Theorem 4.1 is a special case of Theorem 1.1.

Take $\omega(r) = (1-r^2)^{-\lambda}r^{1-2N}$. For each f in $L^{p,q}(r^{1-2N})$, let $F(z) = f(z)\psi(r)(1-r^2)^{1/q-\lambda}$. Then F belongs to $L^{p,q}(\varphi^q\omega(1-r^2)^{\lambda-1})$ and

$$\|F\|_{L^{p,q}(\varphi^q\omega(1-r^2)^{\lambda-1})} = \|f\|_{L^{p,q}(r^{1-2N})}.$$

Thus $Pf(w) = T_{\lambda-1}F(w)$. Since

$$H^{p,q}(\varphi^q r^{1-2N}(1-r^2)^{-1}) = H^{p,q}(\varphi^q\omega(1-r^2)^{\lambda-1}),$$

Theorem 4.1 is equivalent to the statement: $T_{\lambda-1}$ is a bounded operator mapping $L^{p,q}(\varphi^q\omega(1-r^2)^{\lambda-1})$ onto $H^{p,q}(\varphi^q\omega(1-r^2)^{\lambda-1})$.

Since $H^\infty(\mathrm{U})$ is dense in $H^{p,q}(\varphi^q\omega(1-r^2)^{\lambda-1})$, it suffices to prove the boundedness of $T_{\lambda-1}$. By Theorem 1.1, for $q\geq p$ it suffices to verify (4). Since

$$\varphi^{-p'}\omega^{-(p'/p)}=\psi^{p'}r^{(2N-1)(p'-1)}(1-r^2)^{-\lambda},$$

we only need to verify that

$$\text{(8)}\quad \int_{1-h}^1\varphi^p(r)(1-r^2)^{-1}dr\left(\int_{1-h}^1\psi(r)^{p'}r^{(2N-1)p'}(1-r^2)^{-1}dr\right)^{p/p'}\leq Ch^{\lambda p}.$$

Condition (8) will be verified by the following lemma.

LEMMA 4.2. *For any normal pair* $\{\varphi,\psi\}$ *and every nonnegative real number* t,

$$\text{(9)}\quad \int_{1-h}^1\varphi^p(r)(1-r)^{t-1}dr\left(\int_{1-h}^1\psi(r)^{p'}(1-r)^{t-1}dr\right)^{p/p'}\leq Ch^{(\lambda+t)p}$$

for all $0<h<1$, *as long as each integral exists.*

Proof. Let $0<h<1$. Since $\varphi(r)/(1-r)^a$ is nonincreasing, for $a>0$ as in (2),

$$\begin{aligned}\int_{1-h}^1\varphi^p(r)(1-r)^{t-1}dr&\leq C\int_{1-h}^1\frac{\varphi^p(r)}{(1-r)^{ap}}(1-r)^{t-1+ap}dr\\&\leq C\frac{\varphi^p(1-h)}{h^{ap}}\int_{1-h}^1(1-r)^{t-1+ap}dr\\&=C\varphi^p(1-h)h^t.\end{aligned}$$

Similarly,

$$\left(\int_{1-h}^1\psi^{p'}(1-r)^{t-1}dr\right)^{p/p'}\leq C\psi^p(1-h)h^{(t)(p/p')}.$$

Since $\varphi^p(1-h)\psi^p(1-h)=h^{\lambda p}$, (9) follows. The proof of the lemma is complete.

It is clear that (9) implies (8). For $p\leq q$, Theorem 4.1 follows from Lemma 4.2 and Theorem 1.1. If $q<p$, then $q'>p'$. Note that in (9), the position of p,p' and φ,ψ are symmetric; (ii) of Theorem 1.1 implies that $T^*_{\lambda-1}$ is bounded on $L^{p,q}(\psi^{-q}\omega^{-(q/q')}(1-r^2)^{\lambda-1})$. Recalling that $\varphi(r)\psi(r)=(1-r^2)^\lambda$ and $\omega(r)=(1-r^2)^{-\lambda}r^{1-2N}$, we have

$$\psi^{-q}\omega^{-(q/q')}(1-r^2)^{\lambda-1}=\varphi^q\omega r^{(2N-1)(q/q')+1}(1-r^2)^{\lambda-1}.$$

Therefore

$$L^{p,q}(\psi^{-q}\omega^{-(q/q')}(1-r^2)^{\lambda-1})=L^{p,q}(\varphi^q\omega r^{(2N-1)(q/q')+1}(1-r^2)^{\lambda-1}).$$

On the other hand, it is easy to show that, if $0<p,q<\infty$ and $\omega_1\sim\omega_2$ when $0<r_0<r<1$, then $H^{p,q}(\omega_1)\sim H^{p,q}(\omega_2)$. Since $T_{\lambda-1}f$ is analytic if f is in $L^{p,q}(\varphi^q\omega(1-r^2)^{\lambda-1})$, we have

$$\begin{aligned}\|T_{\lambda-1}f\|_{L^{p,q}(\varphi^q\omega(1-r^2)^{\lambda-1})}&=\|T_{\lambda-1}f\|_{H^{p,q}(\varphi^q\omega(1-r^2)^{\lambda-1})}\\&\leq C\|T_{\lambda-1}f\|_{H^{p,q}(\varphi^q\omega r^{(2N-1)(q/q')+1}(1-r^2)^{\lambda-1})}\\&\leq C\|f\|_{L^{p,q}(\varphi^q\omega r^{(2N-1)(q/q')+1}(1-r^2)^{\lambda-1})}\\&\leq C\|f\|_{L^{p,q}(\varphi^q\omega(1-r^2)^{\lambda-1})}.\end{aligned}$$

Thus $T_{\lambda-1}$ is bounded on $L^{p,q}(\varphi^q\omega(1-r^2)^{\lambda-1})$. Therefore Theorem 4.1 is also true for $q<p$. □

5. Application of Theorem 1.2

In this section, we will give two applications of Theorem 1.2.

In [6] Luecking used Békollé's theorem to identify the dual of weighted Bergman spaces and proved the following theorem.

THEOREM 5.1. *Suppose* $\omega(z)$ *satisfies*

$$\int_K \omega(z)\,dm_\eta(z)\left(\int_K \omega^{-(p'/p)}(z)\,dm_\gamma(z)\right)^{p/p'} \leq Cm_\alpha^p(K) \tag{10}$$

for $1<p<\infty$, *where* $\eta>-1$, $\gamma>-1$, *and* $\alpha=\eta/p+\gamma/p'$. *Then the dual of* $H^{p,p}(\omega(z),(1-r^2)^\eta)$ *can be identified with* $H^{p',q'}(\omega(z)^{-(p'/p)},(1-r^2)^\gamma)$ *under the pairing* (7).

In [5] Jevtić obtained the following result.

THEOREM 5.2. *Let* $1\leq p\leq\infty$ *and* $1\leq q<\infty$. *Then the dual space of* $H^{p,q}(\varphi^q r^{1-2N}(1-r^2)^{-1})$ *can be identified with* $H^{p',q'}(\psi^{q'}r^{1-2N}(1-r^2)^{-1})$ *under the pairing*

$$\langle f,g\rangle=\int_U f(z)\bar{g}(z)\,dm_{\lambda-1}.$$

We first show that, for a radial function ω, Theorem 1.2 gives a generalization of Theorem 5.1 in mixed-norm space.

To see this, let $\varphi=(1-r^2)^i$ and $\psi=(1-r^2)^j$, where $i,j>0$, and apply Theorem 1.2 with $\alpha=-1$ and $i+j=\lambda$. Then (5) becomes

$$\begin{aligned}&\int_{1-h}^1 \omega(r)(1-r^2)^{ip-1}r^{2N-1}\,dr\\ &\times\left(\int_{1-h}^1 \omega^{-(p'/p)}(r)(1-r^2)^{jp'-1}r^{2N-1}\,dr\right)^{p/p'}\leq Ch^{\lambda p}.\end{aligned} \tag{11}$$

It is easy to see that

$$H^{p,q}(\varphi^q\omega(1-r^2)^{-1})=H^{p,q}(\omega(1-r^2)^{iq-1})$$

and

$$H^{p',q'}(\psi^{q'}\omega^{-(q'/q)}(1-r^2)^{-1})=H^{p',q'}(\omega^{-(q'/q)}(1-r^2)^{jq'-1}).$$

By Theorem 1.2, for $q\geq p$, if ω satisfies the condition (11) then the dual space of $H^{p,q}(\omega(1-r^2)^{iq-1})$ can be identified with $H^{p',q'}(\omega^{-(q'/q)}(1-r^2)^{jq'-1})$ under the pairing (6). Let $i=(\eta+1)/q$, $j=(\gamma+1)/q'$, and $t=\eta/q+\gamma/q'$. Then $\eta=iq-1$, $\gamma=jq-1$, and $\lambda=i+j=t+1$. Hence Theorem 1.2 gives the following theorem.

THEOREM 5.3. *If* $1<p\leq q<\infty$, $\eta>-1$, $\gamma>-1$, *and* $t>-1$ *satisfy* $t=\eta/q+\gamma/q'$, *and if a radial function* ω *satisfies*

$$\text{(12)}\qquad \int_{1-h}^{1} \omega(r)(1-r^2)^{(\eta+1)(p/q)-1} r^{2N-1}\,dr \times\left(\int_{1-h}^{1} \omega^{-(p'/p)}(r)(1-r^2)^{(\gamma+1)(p'/q')-1} r^{2N-1}\,dr\right)^{p/p'} \le Ch^{(t+1)p}$$

for all $0<h<1$, *then under the pairing*

$$\text{(13)}\qquad \langle f,g\rangle = \int_{U} f\bar{g}\,dm_t,$$

the dual of $H^{p,q}(\omega(1-r^2)^{\eta})$ *can be identified with* $H^{p',q'}(\omega^{-(q/q')}(1-r^2)^{\gamma})$.

Taking $q=p$ in Theorem 5.3, we immediately get Theorem 5.1.

Next we show that Theorem 5.2 is a special case of Theorem 1.2 if $1<p<\infty$ and $1<q<\infty$.

Indeed, it suffices to show that the dual of $H^{p,q}(\varphi^q(1-r^2)^{-1})$ can be identified with $H^{p',q'}(\psi^{q'}(1-r^2)^{-1})$ under the pairing

$$\langle f,g\rangle = \int_{U} f(z)\bar{g}(z)\,dm_{\lambda-1}.$$

Taking $\omega=1$ and $\alpha=-1$ in Theorem 1.2, (5) becomes

$$\int_{1-h}^{1} \varphi^p(r)(1-r^2)^{-1} r^{2N-1}\,dr\left(\int_{1-h}^{1} \psi^{p'}(1-r^2)^{-1} r^{2N-1}\,dr\right)^{p/p'} \le Ch^{\lambda p}.$$

By Lemma 4.2 and the discussion after Lemma 4.2, every normal pair satisfies this inequality. Therefore, for $q\ge p$, Theorem 5.2 follows from Theorem 1.2 immediately.

If $q<p$, then $q'>p'$. Using the duality argument, it can be shown that the dual space of $H^{p',q'}(\psi^{q'}(1-r^2)^{-1})$ is $H^{p,q}(\varphi^q(1-r^2)^{-1})$. This implies that $H^{p,q}(\varphi^q(1-r^2)^{-1})$ is a closed subspace of $L^{p,q}(\varphi^q(1-r^2)^{-1})$. Because $L^{p,q}(\varphi^q(1-r^2)^{-1})$ is reflexive (see [4, p. 306]), it follows that the dual of $H^{p,q}(\varphi^q(1-r^2)^{-1})$ is $H^{p',q'}(\psi^{q'}(1-r^2)^{-1})$. We also get Theorem 5.2.

References

1. D. Békollé, *Inégalités à poids pour le projecteur de Bergman dans la boule unité de* $\mathbf{C}^N$, Studia Math. 71 (1982), 305–323.
2. A. Benedek and R. Panzone, *The spaces* L^p, *with mixed norm,* Duke Math. J. 28 (1961), 301–324.
3. J. Bergh and J. Löfström, *Interpolation spaces,* Springer, New York, 1976.
4. R. Hunt, *On* $L(p,q)$ *space,* Enseign. Math. (2) 12 (1966), 249–276.
5. M. Jevtić, *Bounded projections and duality in mixed-norm spaces of analytic functions,* Complex Variables Theory Appl. 8 (1987), 293–301.
6. D. Luecking, *Representation and duality in weighted spaces of analytic functions,* Indiana Univ. Math. J. 34 (1985), 319–336.
7. W. Rudin, *Real and complex analysis,* 2nd ed., McGraw-Hill, New York, 1974.
8. ———, *Function theory in the unit ball of* $\mathbf{C}^N$, Springer, New York, 1980.

9. J. Shapiro, *Linear functions on non-locally convex spaces,* Ph.D. dissertation, Univ. of Michigan, 1968.
10. A. Shields and D. Williams, *Bounded projections, duality and multipliers in spaces of analytic functions,* Trans. Amer. Math. Soc. 162 (1971), 287–302.

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