Hankel and Toeplitz Operators on the Fock Space

KAREL STROETHOFF

1. Introduction

Throughout this paper let $n \in \mathbf{N}$ be fixed. Let μ be the Gaussian measure on $\mathbf{C}^n$ defined by $d\mu(z) = e^{-\|z\|^2/2} dV(z)/(2\pi)^n$, where V is the usual Lebesgue measure on $\mathbf{C}^n$. The *Fock space* $\mathfrak{F}$, also called the Segal–Bargmann space, is the set of holomorphic functions which are in $L^2(\mathbf{C}^n, \mu)$. The Fock space $\mathfrak{F}$ is a closed subspace of the Hilbert space $L^2(\mathbf{C}^n, \mu)$, with inner product given by $\langle f, g\rangle = \int_{\mathbf{C}^n} f(z)\overline{g(z)}\, d\mu(z)$ for $f, g \in L^2(\mathbf{C}^n, \mu)$. Let P denote the orthogonal projection of $L^2(\mathbf{C}^n, \mu)$ onto $\mathfrak{F}$. For a function $f \in L^\infty(\mathbf{C}^n)$, the *Toeplitz operator* $T_f \colon \mathfrak{F} \to \mathfrak{F}$ and the *Hankel operator* $H_f \colon \mathfrak{F} \to \mathfrak{F}^\perp$ are defined by

$$T_f g = P(fg), \quad g \in \mathfrak{F},$$

$$H_f g = (I-P)(fg), \quad g \in \mathfrak{F}.$$

It is clear that these are bounded operators for every function $f \in L^\infty(\mathbf{C}^n)$. Berger and Coburn [1] characterized the functions $f \in L^\infty(\mathbf{C}^n)$ for which H_f is compact, and also obtained the result that H_f is compact if and only if $H_{\bar{f}}$ is. In this paper we will give an alternate approach to Berger and Coburn's work. Our method is more elementary and furthermore it also gives the functions $f \in L^\infty(\mathbf{C}^n)$ for which T_f is compact. Writing τ_λ to denote the translation on $\mathbf{C}^n$ by λ, we will prove that the Hankel operator H_f is compact if and only if $\|f\circ\tau_\lambda - P(f\circ\tau_\lambda)\|_2 \to 0$ as $\|\lambda\| \to \infty$. This result is completely analogous to the author's characterization of the compact Hankel operators on the Bergman spaces of the unit disk [6], and the unit ball and polydisk in $\mathbf{C}^n$ [7]. We will show how this result implies Berger and Coburn's result.

The paper is arranged as follows. In Section 2 we give the preliminaries needed for the rest of the paper. In Section 3 we give the proof of our main result, characterizations of compact Hankel and Toeplitz operators. In Section 4 we obtain Berger and Coburn's result that H_f is compact if and only if $H_{\bar{f}}$ is. In Section 5 we consider Hankel operators with bounded continuous symbols. For a subclass of these Hankel operators we formulate a very useful criterium for compactness. As an immediate consequence we obtain another proof of Berger and Coburn's result mentioned above. In Section 6

Received December 21, 1988. Revision received January 3, 1991.
Michigan Math. J. 39 (1992).

we describe the essential spectrum of Toeplitz operators for which the corresponding Hankel operator is compact.

The author thanks Dechao Zheng for helpful discussions and the referee for suggesting the statement of Theorem 16.

2. Preliminaries

Point evaluation is a bounded linear functional on the Hilbert space $\mathfrak{F}$; thus for every $\lambda \in \mathbf{C}^n$ there exists a unique holomorphic function $k_\lambda \in \mathfrak{F}$ such that

$$f(\lambda) = \langle f, k_\lambda \rangle \quad \text{for all } f \in \mathfrak{F}.$$

These functions k_λ ($\lambda \in \mathbf{C}^n$) are called the *reproducing kernels* for $\mathfrak{F}$. They can be computed explicitly (see, e.g., [5]): Using $\langle \cdot, \cdot \rangle$ to denote the usual inner product on $\mathbf{C}^n$, for each $\lambda \in \mathbf{C}^n$ we have

$$k_\lambda(z) = e^{\langle z, \lambda \rangle / 2}, \quad z \in \mathbf{C}^n. \tag{1}$$

For $f \in L^\infty(\mathbf{C}^n)$, $g \in \mathfrak{F}$, and $z \in \mathbf{C}^n$ we have $(T_f g)(z) = \langle P(fg), k_z \rangle = \langle fg, k_z \rangle$, so we obtain the following formula for $T_f g$:

$$(T_f g)(z) = \int_{\mathbf{C}^n} f(w) g(w) \overline{k_z(w)} \, d\mu(w), \quad z \in \mathbf{C}^n. \tag{2}$$

Also using the reproducing property of k_z, we obtain the following formula for $H_f g$:

$$(H_f g)(z) = \int_{\mathbf{C}^n} (f(w) - f(z)) g(w) \overline{k_z(w)} \, d\mu(w), \quad z \in \mathbf{C}^n. \tag{3}$$

In our characterization the translations on $\mathbf{C}^n$ will play an important role. For $\lambda \in \mathbf{C}^n$ let the translation $\tau_\lambda : \mathbf{C}^n \to \mathbf{C}^n$ be defined by $\tau_\lambda(z) = z + \lambda$, $z \in \mathbf{C}^n$.

It is easy to check that for a Lebesgue integrable or nonnegative Lebesgue measurable function h on $\mathbf{C}^n$ we have the change-of-variable formula:

$$\int_{\mathbf{C}^n} h(\tau_\lambda(w)) \, d\mu(w) = \frac{1}{k_\lambda(\lambda)} \int_{\mathbf{C}^n} h(z) |k_\lambda(z)|^2 \, d\mu(z). \tag{4}$$

3. Compact Hankel and Toeplitz Operators

In this section we will give our characterization of compact Toeplitz and Hankel operators. The following proposition, which gives formulas for the images of the reproducing kernels k_λ ($\lambda \in \mathbf{C}^n$) under the operators T_f and H_f, will play an important role in our characterization.

PROPOSITION 1. *Let $f \in L^\infty(\mathbf{C}^n)$. For each $\lambda \in \mathbf{C}^n$ we have:*

$$T_f(k_\lambda) = (P(f \circ \tau_\lambda) \circ \tau_{-\lambda}) k_\lambda \tag{5}$$

and

$$H_f(k_\lambda) = (f - P(f \circ \tau_\lambda) \circ \tau_{-\lambda}) k_\lambda. \tag{6}$$

Proof. Take $f\in L^\infty(\mathbf{C}^n)$ and $\lambda\in\mathbf{C}^n$. Clearly it suffices to prove (5). Let $z\in\mathbf{C}^n$. By (2) we have

$$(T_f k_\lambda)(z)=\int_{\mathbf{C}^n} f(u)k_\lambda(u)\overline{k_z(u)}\,d\mu(u).$$

Now, using (1) and the properties of the exponential function, we have

$$k_z(u)=k_z(\lambda)k_z(u-\lambda)=\overline{k_\lambda(z)}k_{z-\lambda}(u-\lambda)\frac{k_\lambda(u)}{k_\lambda(\lambda)},$$

thus

$$\begin{aligned}(T_f k_\lambda)(z)&=\frac{k_\lambda(z)}{k_\lambda(\lambda)}\int_{\mathbf{C}^n} f(u)\overline{k_{z-\lambda}(u-\lambda)}|k_\lambda(u)|^2\,d\mu(u)\\ &=k_\lambda(z)\int_{\mathbf{C}^n} f(\tau_\lambda(w))\overline{k_{z-\lambda}(w)}\,d\mu(w) \qquad \text{(by (4))}\\ &=k_\lambda(z)P(f\circ\tau_\lambda)(z-\lambda).\end{aligned}$$

□

In the sequel we will frequently encounter weighted integrals of the reproducing kernels. By the reproducing property of k_λ we have $\|k_\lambda\|_2^2=\langle k_\lambda,k_\lambda\rangle=k_\lambda(\lambda)=e^{\langle\lambda,\lambda\rangle/2}$. Using that $|k_\lambda(cw)|=|k_{c\lambda/2}(w)|^2$ for real c, a simple change of variables yields the following formula:

$$\int_{\mathbf{C}^n}|k_\lambda(z)|^p e^{-(a/2)\langle z,z\rangle}\frac{dV(z)}{(2\pi)^n} =\left(\frac{1}{a}\right)^n e^{p^2\langle\lambda,\lambda\rangle/(8a)},\quad \lambda\in\mathbf{C}^n,\ a>0,\ p\ge 0. \tag{7}$$

An immediate consequence is that for a function $f\in L^\infty(\mathbf{C}^n)$ we have the estimate

$$|(Pf)(z)|\le\|f\|_\infty e^{\langle z,z\rangle/8},\quad z\in\mathbf{C}^n. \tag{8}$$

The following lemma gives an estimate that will be used in the proofs of Theorems 5 and 6, our characterization of compact Toeplitz and Hankel operators on the Fock space $\mathfrak{F}$.

LEMMA 2. *Let F be a nonnegative measurable function on $\mathbf{C}^n\times\mathbf{C}^n$. Assume that B is a constant such that $F(w,z)\le Be^{\langle z,z\rangle/8}$ for all $z,w\in\mathbf{C}^n$. Then there exists a constant C (depending only on B and n) such that, for every $w\in\mathbf{C}^n$,*

$$\int_{\mathbf{C}^n}F(w,\tau_{-w}(z))|k_w(z)|k_z(z)^{1/2}\,d\mu(z)\le Ck_w(w)^{1/2}\left(\int_{\mathbf{C}^n}F(w,z)^2\,d\mu(z)\right)^{1/4}.$$

Proof. Let F be a nonnegative measurable function on $\mathbf{C}^n\times\mathbf{C}^n$ and B a constant such that $F(w,z)\le Be^{\langle z,z\rangle/8}$ for all $z,w\in\mathbf{C}^n$. In the integral at the left make the change of variable $z=\tau_w(u)$. We obtain

$$\int_{\mathbf{C}^n} F(w, \tau_{-w}(z))|k_w(z)|k_z(z)^{1/2}\,d\mu(z) = k_w(w)^{1/2}\int_{\mathbf{C}^n} F(w,u)e^{-\langle u,u\rangle/4}\frac{dV(u)}{(2\pi)^n}.$$

Now, using Hölder's inequality with conjugate exponents 4 and 4/3, we have

$$\begin{aligned}
&\int_{\mathbf{C}^n} F(w,u)e^{-\langle u,u\rangle/4}\frac{dV(u)}{(2\pi)^n}\\
&= \int_{\mathbf{C}^n} F(w,u)e^{-(3/16)\langle u,u\rangle}e^{-(1/16)\langle u,u\rangle}\frac{dV(u)}{(2\pi)^n}\\
&\le \left(\int_{\mathbf{C}^n} F(w,u)^4 e^{-(3/4)\langle u,u\rangle}\frac{dV(u)}{(2\pi)^n}\right)^{1/4}\left(\int_{\mathbf{C}^n} e^{-\langle u,u\rangle/12}\frac{dV(u)}{(2\pi)^n}\right)^{3/4}\\
&\le \left(B^2\int_{\mathbf{C}^n} F(w,u)^2 e^{-\langle u,u\rangle/2}\frac{dV(u)}{(2\pi)^n}\right)^{1/4}\times 6^{3n/4} \qquad \text{(using (7))}\\
&= C\left(\int_{\mathbf{C}^n} F(w,z)^2\,d\mu(z)\right)^{1/4}.
\end{aligned}$$

□

In the proofs of Theorems 5 and 6 we will also need the estimates contained in the following lemma.

LEMMA 3. *Let $f \in L^\infty(\mathbf{C}^n)$. Then, for every $z \in \mathbf{C}^n$,*

$$\int_{\mathbf{C}^n}|P(f\circ\tau_w)(\tau_{-w}(z))||k_w(z)|k_w(w)^{1/2}\,d\mu(w) \le 2^{2n}\|f\|_\infty k_z(z)^{1/2} \tag{9}$$

and

$$\int_{\mathbf{C}^n}|f(z) - P(f\circ\tau_w)(\tau_{-w}(z))||k_w(z)|k_w(w)^{1/2}\,d\mu(w) \le 2^{2n+1}\|f\|_\infty k_z(z)^{1/2}. \tag{10}$$

Proof. Let $f \in L^\infty(\mathbf{C}^n)$ and $z \in \mathbf{C}^n$. It is easy to see that inequality (10) follows from (9), so it is enough to show (9). Then, using (5) and (2):

$$|P(f\circ\tau_w)(\tau_{-w}(z))||k_w(z)| = |(T_f k_w)(z)| = \left|\int_{\mathbf{C}^n} f(u)k_w(u)\overline{k_z(u)}\,d\mu(u)\right| \le \|f\|_\infty\int_{\mathbf{C}^n}|k_w(u)||k_z(u)|\,d\mu(u).$$

Thus

$$\int_{\mathbf{C}^n}|P(f\circ\tau_w)(\tau_{-w}(z))||k_w(z)|k_w(w)^{1/2}\,d\mu(w) \le \|f\|_\infty\int_{\mathbf{C}^n}|k_z(u)|\left(\int_{\mathbf{C}^n}|k_w(u)|k_w(w)^{1/2}\,d\mu(w)\right)d\mu(u). \tag{11}$$

As a consequence of (7) we have $\int_{\mathbf{C}^n}|k_w(u)|k_w(w)^{1/2}\,d\mu(w) = 2^n k_u(u)^{1/2}$. Applying this identity twice in (11) we get (9). □

To estimate the norms of certain (integral) operators we will make use of the Schur test as stated in the following proposition, a proof of which can be found in [4].

PROPOSITION 4. *Let (X, ν) be a measure space and K a measurable function on $X \times X$. Suppose there are positive measurable functions p and q on X and positive numbers α and β such that*

$$\int_X |K(x,y)|q(y)\,d\nu(y) \leq \alpha p(x) \quad \text{for } [\nu]\text{-a.e. } x \text{ in } X \tag{12}$$

and

$$\int_X |K(x,y)|p(x)\,d\nu(x) \leq \beta q(y) \quad \text{for } [\nu]\text{-a.e. } y \text{ in } X. \tag{13}$$

Then

$$(Af)(x) = \int_X K(x,y) f(y)\,d\nu(y),$$

with $f \in L^2(X, \nu)$ and $x \in X$, defines a bounded linear operator from $L^2(X, \nu)$ into itself. Moreover, $\|A\|^2 \leq \alpha\beta$.

We are now ready to state and prove our main result, contained in Theorems 5 and 6. The proofs of these theorems will be combined into one proof.

THEOREM 5. *Let $f \in L^\infty(\mathbf{C}^n)$. The following statements are equivalent:*

(a) *T_f is compact;*
(b) *$\|P(f \circ \tau_\lambda)\|_2 \to 0$ as $\|\lambda\| \to \infty$.*

THEOREM 6. *Let $f \in L^\infty(\mathbf{C}^n)$. The following statements are equivalent:*

(a) *H_f is compact;*
(b) *$\|f \circ \tau_\lambda - P(f \circ \tau_\lambda)\|_2 \to 0$ as $\|\lambda\| \to \infty$.*

Proof of Theorems 5 and 6. Fix a function $f \in L^\infty(\mathbf{C}^n)$. Let M_f be the multiplication operator $\mathfrak{F} \to L^2(\mathbf{C}^n, \mu)$ defined by $M_f(g) = fg$ for $g \in \mathfrak{F}$. Writing Q for either P or $I - P$, we note that both H_f and T_f are of the form QM_f; the proofs of Theorems 5 and 6 will be combined into one proof.

Proof that (a)$\Rightarrow$(b). Suppose that the operator QM_f is compact. We have already observed that $\|k_\lambda\|_2^2 = e^{\|\lambda\|^2/2}$. If g is a polynomial on $\mathbf{C}^n$, then

$$\langle g, k_\lambda / \|k_\lambda\|_2 \rangle = e^{-\|\lambda\|^2/4} g(\lambda) \to 0 \quad \text{as } \|\lambda\| \to \infty.$$

Since the polynomials are dense in $\mathfrak{F}$, this shows that $k_\lambda / \|k_\lambda\|_2 \to 0$ weakly in $\mathfrak{F}$ as $\|\lambda\| \to \infty$. A compact operator maps weakly null sequences to norm null sequences, so we have $\|QM_f(k_\lambda / \|k_\lambda\|_2)\|_2 \to 0$ as $\|\lambda\| \to \infty$. By Proposition 1, $QM_f(k_\lambda) = (Q(f \circ \tau_\lambda) \circ \tau_{-\lambda}) k_\lambda$. We have

$$\left\|QM_f\frac{k_\lambda}{\|k_\lambda\|_2}\right\|_2^2=\frac{1}{k_\lambda(\lambda)}\int_{\mathbf{C}^n}|Q(f\circ\tau_\lambda)(\tau_{-\lambda}(z))|^2|k_\lambda(z)|^2\,d\mu(z)$$
$$=\int_{\mathbf{C}^n}|Q(f\circ\tau_\lambda)(w)|^2\,d\mu(w);$$

thus $\|Q(f\circ\tau_\lambda)\|_2=\|QM_f(k_\lambda/\|k_\lambda\|_2)\|_2\to 0$ as $\|\lambda\|\to\infty$.

Proof that (b)$\Rightarrow$(a). Suppose that $\|Q(f\circ\tau_\lambda)\|_2\to 0$ as $\|\lambda\|\to\infty$. We will show that the operator $(QM_f)^*$ is compact by showing that $(QM_f)^*$ can be approximated—in the operator norm—by compact operators.

Let $h\in Q(L^2(\mathbf{C}^n,\mu))$. Then $(QM_f)^*h\in\mathfrak{F}$, so that for $w\in\mathbf{C}^n$ we have $((QM_f)^*h)(w)=\langle(QM_f)^*h,k_w\rangle=\langle h,(Q(f\circ\tau_w)\circ\tau_{-w})k_w\rangle$; hence

$$((QM_f)^*h)(w)=\int_{\mathbf{C}^n}h(z)\overline{Q(f\circ\tau_w)(\tau_{-w}(z))\,k_w(z)}\,d\mu(z). \tag{14}$$

For each number $R\in(0,\infty)$ define the operator $S_R\colon Q(L^2(\mathbf{C}^n,\mu))\to L^2(\mathbf{C}^n,\mu)$ by

$$(S_Rh)(w)=\chi_{R\mathbf{B}_n}(w)\int_{\mathbf{C}^n}h(z)\overline{Q(f\circ\tau_w)(\tau_{-w}(z))\,k_w(z)}\,d\mu(z)$$

for $h\in Q(L^2(\mathbf{C}^n,\mu))$, $w\in\mathbf{C}^n$ (where $\mathbf{B}_n$ denotes the unit ball in $\mathbf{C}^n$). Then, using Fubini's theorem and change-of-variable formula (4), we have

$$\int_{\mathbf{C}^n}\left(\int_{\mathbf{C}^n}\chi_{R\mathbf{B}_n}(w)|Q(f\circ\tau_w)(\tau_{-w}(z))|^2|k_w(z)|^2\,d\mu(z)\right)d\mu(w)$$
$$=\int_{R\mathbf{B}_n}k_w(w)\|Q(f\circ\tau_w)\|_2^2\,d\mu(w)<\infty,$$

and it follows that S_R is Hilbert-Schmidt. Using (14) and the definition of S_R we see that, for $h\in Q(L^2(\mathbf{C}^n,\mu))$ and $w\in\mathbf{C}^n$,

$$(((QM_f)^*-S_R)h)(w)=\int_{\mathbf{C}^n}K(w,z)\,h(z)\,d\mu(z),$$

where $K(w,z)=\chi_{\mathbf{C}^n\setminus R\mathbf{B}_n}(w)\overline{Q(f\circ\tau_w)(\tau_{-w}(z))\,k_w(z)}$. We will apply Proposition 4 to obtain an estimate on the operator norm $\|(QM_f)^*-S_R\|$. Let $p(w)=k_w(w)^{1/2}$ and $q(z)=k_z(z)^{1/2}$ for $w,z\in\mathbf{C}^n$. By Lemma 3, inequality (13) in Proposition 4 is satisfied with $\beta=2^{2n+1}\|f\|_\infty$. For $w,z\in\mathbf{C}^n$ put $F(w,z)=\chi_{\mathbf{C}^n\setminus R\mathbf{B}_n}(w)|Q(f\circ\tau_w)|$. By (8) we have $F(w,z)\le 2\|f\|_\infty e^{\langle z,z\rangle/8}$, so we can apply Lemma 2. Noting that $|K(w,z)|=F(w,\tau_{-w}(z))|k_w(z)|$, this lemma then tells us that there is a constant C, depending only on f and n, such that

$$\int_{\mathbf{C}^n}|K(w,z)|k_z(z)^{1/2}\,d\mu(z)\le Ck_w(w)^{1/2}\left(\int_{\mathbf{C}^n}F(w,z)^2\,d\mu(z)\right)^{1/4}$$
$$=Ck_w(w)^{1/2}\chi_{\mathbf{C}^n\setminus R\mathbf{B}_n}(w)\|Q(f\circ\tau_w)\|_2^{1/2},$$

so that inequality (12) in Proposition 4 holds with

$$\alpha=C\sup\{\|Q(f\circ\tau_\lambda)\|_2^{1/2}\colon\|\lambda\|\ge R\}.$$

It follows from Proposition 4 that, with a possibly different constant C, $\|(QM_f)^* - S_R\| \le C \sup\{\|Q(f\circ\tau_\lambda)\|_2^{1/4} : \|\lambda\| \ge R\}$. Since $\|Q(f\circ\tau_\lambda)\|_2 \to 0$ as $\|\lambda\| \to \infty$ it follows that $S_R \to (QM_f)^*$ in operator norm as $R \to \infty$. Since the S_R are Hilbert–Schmidt and thus compact, it follows that $(QM_f)^*$ is compact and therefore QM_f is compact. □

For $f \in L^\infty(\mathbf{C}^n)$ define $\tilde{f}$, the *Berezin symbol of* f, by

(15) $$\tilde{f}(\lambda) = \int_{\mathbf{C}^n} f(z) e^{-\|\lambda - z\|^2/2} \frac{dV(z)}{(2\pi)^n}, \quad \lambda \in \mathbf{C}^n.$$

Although we will not make use of this, we note that the function $\tilde{f}$ is the solution of the heat equation on $\mathbf{C}^n = \mathbf{R}^{2n}$ at time $t = 1/2$ with initial value f (see [3]). Observe that, by change-of-variable formula (4),

$$\tilde{f}(\lambda) = \int_{\mathbf{C}^n} f(\tau_\lambda(w))\, d\mu(w).$$

An immediate consequence of this formula is that the Berezin symbol is invariant under translations: $\widetilde{f\circ\tau_\lambda} = \tilde{f}\circ\tau_\lambda$.

As in the preceding proof, for $f \in L^\infty(\mathbf{C}^n)$ let M_f be the multiplication operator $\mathfrak{F} \to L^2(\mathbf{C}^n, \mu)$ defined by $M_f(g) = fg$ for $g \in \mathfrak{F}$. As an easy corollary of Theorems 5 and 6 we get the following result of Berger and Coburn [1, Thm. C].

COROLLARY 7. *Let $f \in L^\infty(\mathbf{C}^n)$. The following statements are equivalent:*

(a) *M_f is compact;*

(b) *$\widetilde{|f|^2}(\lambda) \to 0$ as $\|\lambda\| \to \infty$.*

Proof. It is easily verified that, for $\lambda \in \mathbf{C}^n$,

(16) $$\widetilde{|f|^2}(\lambda) = \|f\circ\tau_\lambda\|_2^2 = \|P(f\circ\tau_\lambda)\|_2^2 + \|f\circ\tau_\lambda - P(f\circ\tau_\lambda)\|_2^2.$$

So if $\widetilde{|f|^2}(\lambda) \to 0$ as $\|\lambda\| \to \infty$, then both

$$\|P(f\circ\tau_\lambda)\|_2 \to 0 \quad \text{and} \quad \|f\circ\tau_\lambda - P(f\circ\tau_\lambda)\|_2 \to 0$$

as $\|\lambda\| \to \infty$, so that by Theorems 5 and 6 both T_f and H_f are compact; thus, M_f is compact. The converse is obvious. □

4. More on Compact Hankel Operators

In this section we will obtain Berger and Coburn's result [1] that H_f is compact if and only if $H_{\bar{f}}$ is. We start with a result that gives estimates on the norms of the Hankel operators.

PROPOSITION 8. *Let $f \in L^\infty(\mathbf{C}^n)$. Suppose that $|f(z) - f(w)| \le L\|z - w\|$ for all $z, w \in \mathbf{C}^n$. Then $\|H_f\| \le 4L$.*

Proof. Let $f \in L^\infty(\mathbf{C}^n)$ and suppose that $|f(z) - f(w)| \le L\|z - w\|$ for all $z, w \in \mathbf{C}^n$. Making a change of variable, it is easy to verify that

$$\int_{\mathbf{C}^n}\|z-w\||k_z(w)|k_w(w)^{1/2}\,d\mu(w)=4k_z(z)^{1/2}.$$

Hence

$$\int_{\mathbf{C}^n}|f(z)-f(w)||k_z(w)|k_w(w)^{1/2}\,d\mu(w)\leq 4Lk_z(z)^{1/2}.$$

The statement follows immediately from Proposition 4. □

The following corollary is perhaps not surprising in view of the fact that repeatedly taking Berezin symbols leads to smoother and smoother functions. For a function $g\in L^\infty(\mathbf{C}^n)$ and $m\in\mathbf{N}$, let $\tilde{g}^{(m)}$ denote the mth Berezin symbol of g.

COROLLARY 9. *Let $g\in L^\infty(\mathbf{C}^n)$. Then $\|H_{\tilde{g}^{(m)}}\|\to 0$ as $m\to\infty$.*

Proof. For $g\in L^\infty(\mathbf{C}^n)$ and $m\in\mathbf{N}$, the function $\tilde{g}^{(m)}$ is Lipschitz with Lipschitz constant at most $2(2\pi m)^{-1/2}$ (see [1, Lemma 2]). □

The following lemma will make the connection with our results in the previous section.

LEMMA 10. *Let $g\in L^\infty(\mathbf{C}^n)$. Then there is a constant C, depending only on g and n, such that for every $\lambda\in\mathbf{C}^n$:*

$$\|\tilde{g}\circ\tau_\lambda-P(g\circ\tau_\lambda)\|_2\leq C\|g\circ\tau_\lambda-P(g\circ\tau_\lambda)\|_2^{1/4}.$$

Proof. It follows immediately from (15) and (8) that

$$|\tilde{g}(w)-(Pg)(w)|\leq 2\|g\|_\infty e^{\langle w,w\rangle/8}\quad\text{for all }\ w\in\mathbf{C}^n. \tag{17}$$

Using (7) and (8) it is easily seen that $(Pg)k_w\in\mathfrak{F}$ for fixed $w\in\mathbf{C}^n$; thus, by the reproducing property of k_w, $\int_{\mathbf{C}^n}(Pg)(z)|k_w(z)|^2\,d\mu(z)=e^{\langle w,w\rangle/2}(Pg)(w)$. Also, $\tilde{g}(w)=e^{-\langle w,w\rangle/2}\int_{\mathbf{C}^n}g(z)|k_w(z)|^2\,d\mu(z)$. Thus we have

$$|\tilde{g}(w)-(Pg)(w)|\leq e^{-\langle w,w\rangle/2}\int_{\mathbf{C}^n}|g(z)-(Pg)(z)||k_w(z)|^2\,d\mu(z). \tag{18}$$

Combining (17) and (18), we see that

$$\begin{aligned}\|\tilde{g}-Pg\|_2^2&=\int_{\mathbf{C}^n}|\tilde{g}(w)-(Pg)(w)|^2e^{-\langle w,w\rangle/2}\frac{dV(w)}{(2\pi)^n}\\&\leq 2\|g\|_\infty\int_{\mathbf{C}^n}|g(z)-(Pg)(z)|\left(\int_{\mathbf{C}^n}|k_w(z)|^2e^{-(7/8)\langle w,w\rangle}\frac{dV(w)}{(2\pi)^n}\right)d\mu(z)\\&=2\|g\|_\infty\int_{\mathbf{C}^n}|g(z)-(Pg)(z)|\left(\frac{4}{7}\right)^n e^{(2/7)\langle z,z\rangle}\,d\mu(z)\qquad\text{(by (7))}\\&=2\left(\frac{4}{7}\right)^n\|g\|_\infty\int_{\mathbf{C}^n}|g(z)-(Pg)(z)|e^{-(3/14)\langle z,z\rangle}\frac{dV(z)}{(2\pi)^n}.\end{aligned}$$

Proceeding as in the proof of Lemma 2, we see that there is a constant c_n such that

$$\int_{\mathbf{C}^n} |g(z)-(Pg)(z)| e^{-(3/14)\langle z,z\rangle} \frac{dV(z)}{(2\pi)^n}$$

$$\leq c_n \left(\int_{\mathbf{C}^n} |g(z)-(Pg)(z)|^4 e^{-(3/4)\langle z,z\rangle} \frac{dV(z)}{(2\pi)^n} \right)^{1/4}$$

$$\leq c_n \left(\int_{\mathbf{C}^n} 4\|g\|_\infty^2 |g(z)-(Pg)(z)|^2 e^{-\langle z,z\rangle/2} \frac{dV(z)}{(2\pi)^n} \right)^{1/4} \qquad \text{(using (17))}$$

$$= 2^{1/2} c_n \|g\|_\infty^{1/2} \|g-Pg\|_2^{1/2}.$$

Combining this with the inequality in the previous paragraph, we see that for a constant C_n:

$$\|\tilde{g}-Pg\|_2 \leq C_n \|g\|_\infty^{3/4} \|g-Pg\|_2^{1/4}.$$

Finally, replacing g by $g \circ \tau_\lambda$ and using the translation invariance of the Berezin symbol, we obtain the desired inequality. □

The following proposition is implicit in the work of Berger and Coburn [1], who proved it by an averaging operation over a representation of the Heisenberg group related to the operation of taking the Berezin symbol. In our set-up, an elementary proof is obtained by simply combining Lemma 10 with Theorems 5 and 6.

PROPOSITION 11. *Let $g \in L^\infty(\mathbf{C}^n)$ be such that H_g is compact. Then both operators $H_{\tilde{g}}$ and $T_{g-\tilde{g}}$ are compact.*

Proof. Suppose that $g \in L^\infty(\mathbf{C}^n)$ is such that H_g is compact. By Theorem 6 we have $\|g\circ\tau_\lambda - P(g\circ\tau_\lambda)\|_2 \to 0$ as $\|\lambda\| \to \infty$. Invoking Lemma 10, we see $\|\tilde{g}\circ\tau_\lambda - P(g\circ\tau_\lambda)\|_2 \to 0$ as $\|\lambda\|\to\infty$. By continuity of P this implies that also $\|P(\tilde{g}\circ\tau_\lambda) - P(g\circ\tau_\lambda)\|_2 \to 0$; thus, by Theorem 5, $T_{g-\tilde{g}}$ is compact. Also

$$\|\tilde{g}\circ\tau_\lambda - P(\tilde{g}\circ\tau_\lambda)\|_2 \to 0 \quad \text{as } \|\lambda\| \to \infty,$$

and Theorem 6 gives us that $H_{\tilde{g}}$ is compact. □

Now we are in a position to prove the following theorem. The equivalence of (a) and (b) was proved by Berger and Coburn [1]; condition (d) was obtained by Berger, Coburn and Zhu [2]. Berger and Coburn had to develop quite a lot of machinery to prove the equivalence of (a) and (b); our proof is very elementary.

THEOREM 12. *Let $f \in L^\infty(\mathbf{C}^n)$. The following statements are equivalent:*
(a) *H_f is compact;*
(b) *$H_{\bar{f}}$ is compact;*
(c) *$\|f\circ\tau_\lambda - \tilde{f}(\lambda)\|_2 \to 0$ as $\|\lambda\| \to \infty$;*
(d) *$\widetilde{|f|^2}(\lambda) - |\tilde{f}(\lambda)|^2 \to 0$ as $\|\lambda\| \to \infty$.*

Proof. For (b)$\Leftrightarrow$(a), first suppose that $H_{\bar{f}}$ is compact. By Theorem 6 we have $\|\bar{\tilde{f}}\circ\tau_\lambda - P(\bar{f}\circ\tau_\lambda)\|_2 \to 0$ as $\|\lambda\|\to\infty$. Invoking Lemma 10, we see that $\|\bar{f}\circ\tau_\lambda - P(\bar{f}\circ\tau_\lambda)\|_2\to 0$; thus $\|\bar{f}\circ\tau_\lambda - \bar{\tilde{f}}\circ\tau_\lambda\|_2\to 0$ as $\|\lambda\|\to\infty$. Taking complex conjugates (using that $\bar{\tilde{f}} = \tilde{\bar{f}}$) we see that $\|f\circ\tau_\lambda - \tilde{f}\circ\tau_\lambda\|_2 \to 0$ as $\|\lambda\|\to\infty$, and it follows that $H_{f-\tilde{f}}$ is compact. Using Proposition 11 and iteration, we conclude that $H_{f-\tilde{f}^{(m)}}$ is compact for every $m\in\mathbf{N}$. By Corollary 9, $H_{f-\tilde{f}^{(m)}}\to H_f$ in operator norm as $m\to\infty$. Hence H_f is compact. Reversing the role of f and $\bar{f}$, we also get that (a)$\Rightarrow$(b).

Proof that (a)$\Rightarrow$(c): Using that $P(\overline{P(\bar{f}\circ\tau_\lambda)}) = \overline{P(\bar{f}\circ\tau_\lambda)(0)} = \tilde{f}(\lambda)$, it is easy to verify that $\|P(f\circ\tau_\lambda) - \tilde{f}(\lambda)\|_2 \le \|\bar{f}\circ\tau_\lambda - P(\bar{f}\circ\tau_\lambda)\|_2$. It is then readily verified that

$$\|f\circ\tau_\lambda - \tilde{f}(\lambda)\|_2^2 \le \|f\circ\tau_\lambda - P(f\circ\tau_\lambda)\|_2^2 + \|\bar{f}\circ\tau_\lambda - P(\bar{f}\circ\tau_\lambda)\|_2^2,$$

and by using Theorem 6 for both H_f and $H_{\bar{f}}$ the statement follows.

Proof that (c)$\Rightarrow$(a): This follows immediately from the inequality

$$\|f\circ\tau_\lambda - P(f\circ\tau_\lambda)\|_2 \le \|f\circ\tau_\lambda - \tilde{f}(\lambda)\|_2$$

and Theorem 6.

That (c)$\Leftrightarrow$(d) follows from the identity $\|f\circ\tau_\lambda - \tilde{f}(\lambda)\|_2^2 = \widetilde{|f|^2}(\lambda) - |\tilde{f}(\lambda)|^2$. This completes the proof of Theorem 12. □

5. Hankel Operators with Bounded Continuous Symbols

In this section we will give a description for compactness of the Hankel operator associated with bounded continuous symbols. This description will then be used to give yet another proof of Berger and Coburn's result that for f in $L^\infty(\mathbf{C}^n)$, H_f is compact if and only if $H_{\bar{f}}$ is.

Let BC denote the algebra of bounded continuous functions on $\mathbf{C}^n$. Let $\beta\mathbf{C}^n$ denote the Stone–Čech compactification of $\mathbf{C}^n$. Every function f in BC has a unique continuous extension to $\beta\mathbf{C}^n$ which we will denote by f^β. Let $\mathfrak{J}$ be the set of all possible limits in the product space $(\beta\mathbf{C}^n)^{\mathbf{C}^n}$ of nets $\{\tau_{\lambda_\alpha}\}$ for which $\|\lambda_\alpha\|\to\infty$. Note that by Tychonoff's theorem the space $(\beta\mathbf{C}^n)^{\mathbf{C}^n}$ is compact, so that every net $\{\lambda_\alpha\}$ for which $\|\lambda_\alpha\|\to\infty$ has a subnet $\{w_\gamma\}$ such that $\{\tau_{w_\gamma}\}$ converges in $(\beta\mathbf{C}^n)^{\mathbf{C}^n}$ to some $\tau\in\mathfrak{J}$. We refer to the subsets $\tau(\mathbf{C}^n)$ of $\beta\mathbf{C}^n$ as the *$\mathfrak{J}$-parts of $\beta\mathbf{C}^n$*. Define the algebra CO$\mathfrak{J}$, which stands for "Constant On $\mathfrak{J}$-parts," by

$$\mathrm{CO}\mathfrak{J} = \{f\in BC: f^\beta\circ\tau \text{ is constant on } \mathbf{C}^n \text{ for every } \tau\in\mathfrak{J}\}.$$

It is easily seen that for a function f in $L^\infty(\mathbf{C}^n)$ its Berezin transform $\tilde{f}$ is in BC. In the following theorem we will characterize the compact Hankel operators whose symbol is the Berezin transform of a bounded measurable function.

THEOREM 13. *Let $f\in L^\infty(\mathbf{C}^n)$. Then $H_{\tilde{f}}$ is compact if and only if $\tilde{f}\in$ CO$\mathfrak{J}$.*

Proof. Let $f\in L^\infty(\mathbf{C}^n)$ and $\tau\in\mathfrak{J}$. Let $\{\lambda_\alpha\}$ be a net in $(\beta\mathbf{C}^n)^{\mathbf{C}^n}$ such that $\{\tau_{\lambda_\alpha}\}$ converges to τ. Since $\tilde{f}\in BC$ we have $\tilde{f}\circ\tau_{\lambda_\alpha}\to\tilde{f}^\beta\circ\tau$ pointwise on $\mathbf{C}^n$.

We claim that this convergence is uniform on each compact subset of $\mathbf{C}^n$, and consequently $\tilde{f}\circ\tau_{\lambda_\alpha}\to\tilde{f}^\beta\circ\tau$ in $L^2(\mathbf{C}^n,\mu)$. To prove this claim it suffices to show that the family $\{\tilde{f}\circ\tau_{\lambda_\alpha}\}$ is equicontinuous. But this follows from the fact that $\tilde{f}$ is Lipschitz with constant at most $(2/\pi)^{1/2}$ [1, Lemma 2], so that

$$|(\tilde{f}\circ\tau_{\lambda_\alpha})(z)-(\tilde{f}\circ\tau_{\lambda_\alpha})(w)|\le(2/\pi)^{1/2}\|\tau_{\lambda_\alpha}(z)-\tau_{\lambda_\alpha}(w)\|=(2/\pi)^{1/2}\|z-w\|,$$

and the claim is proved.

By Theorem 6, $H_{\tilde{f}}$ is compact if and only if $\|\tilde{f}\circ\tau_\lambda-P(\tilde{f}\circ\tau_\lambda)\|_2\to 0$ as $\|\lambda\|\to\infty$. Using the above claim and the definition of $\mathfrak{J}$, this is easily seen to be equivalent to $\|\tilde{f}^\beta\circ\tau-P(\tilde{f}^\beta\circ\tau)\|_2=0$; that is, $\tilde{f}^\beta\circ\tau=P(\tilde{f}^\beta\circ\tau)$ for every $\tau\in\mathfrak{J}$, which by Liouville's theorem is equivalent to $\tilde{f}^\beta\circ\tau$ is constant for every $\tau\in\mathfrak{J}$; thus $\tilde{f}\in CO\mathfrak{J}$. □

The above theorem gives more insight into Berger and Coburn's result that, for a bounded measurable function f on $\mathbf{C}^n$, H_f is compact if and only if $H_{\bar{f}}$ is compact: This fact follows from the absence of nonconstant bounded holomorphic functions on $\mathbf{C}^n$. With the help of Theorem 13 this is now easy to prove.

Second Proof of (a)$\Leftrightarrow$(b) *in Theorem 12.* Let $f\in L^\infty(\mathbf{C}^n)$, and suppose that H_f is compact. By Proposition 11, $H_{\tilde{f}}$ is compact, so that by Theorem 13, $\tilde{f}\in CO\mathfrak{J}$. But then also $\bar{\tilde{f}}\in CO\mathfrak{J}$, hence $H_{\bar{\tilde{f}}}$ is compact. As a consequence of Proposition 11 we also have that $H_{\bar{f}-\bar{\tilde{f}}}$ is compact, and it follows that $H_{\bar{f}}$ is compact. □

Apparently the compactness of H_f implies "holomorphic-type" behavior of the bounded function f, which must then be constant in some sense. This idea can be made to yield yet another proof of the equivalence of (a) and (b) in Theorem 12.

Third Proof of (a)$\Leftrightarrow$(b) *in Theorem 12.* Let $f\in L^\infty(\mathbf{C}^n)$ be such that H_f is compact. We must show that $H_{\bar{f}}$ is compact, which by Theorem 6 means that $\|\bar{f}\circ\tau_\lambda-P(\bar{f}\circ\tau_\lambda)\|_2\to 0$ as $\|\lambda\|\to\infty$. Let $\{\lambda_m\}$ be a sequence in $\mathbf{C}^n$ such that $\|\lambda_m\|\to\infty$. As in [1, Thm. 20], it can be shown that operator $P|_{L^\infty(\mathbf{C}^n)}$: $L^\infty(\mathbf{C}^n)\to\mathfrak{F}$ is compact. Since the $f\circ\tau_{\lambda_m}$ are uniformly bounded, the sequence $\{P(f\circ\tau_{\lambda_m})\}$ has a further subsequence $\{P(f\circ\tau_{\lambda_j})\}$ which converges to some h in $\mathfrak{F}$. Because H_f is compact we have $f\circ\tau_{\lambda_j}-P(f\circ\tau_{\lambda_j})\to 0$, and thus $f\circ\tau_{\lambda_j}\to h$ in $L^2(\mathbf{C}^n,\mu)$. By going to yet another subsequence—which we will not relabel—we may assume that $f\circ\tau_{\lambda_j}\to h$ $[V]$-a.e. on $\mathbf{C}^n$. It follows that the holomorphic function h is bounded on $\mathbf{C}^n$, and by Liouville's theorem h is constant. Since $\bar{h}$ is constant and $\bar{f}\circ\tau_{\lambda_j}\to\bar{h}$ in $L^2(\mathbf{C}^n,\mu)$, we have $(I-P)(\bar{f}\circ\tau_{\lambda_j})\to 0$ in $L^2(\mathbf{C}^n,\mu)$. We conclude that indeed

$$\|\bar{f}\circ\tau_\lambda-P(\bar{f}\circ\tau_\lambda)\|_2\to 0 \quad \text{as } \|\lambda\|\to\infty,$$

and thus $H_{\bar{f}}$ is compact. □

6. The Essential Spectrum of Toeplitz Operators

Let $\mathcal{L}(\mathfrak{F})$ denote the Banach algebra of the bounded linear operators on $\mathfrak{F}$, and let $\mathcal{K}$ denote the ideal of compact operators in $\mathcal{L}(\mathfrak{F})$. For an operator T in $\mathcal{L}(\mathfrak{F})$, the essential spectrum of T, denoted by $\sigma_e(T)$, is by definition the spectrum of the operator $T+\mathcal{K}$ on the Calkin algebra $\mathcal{L}(\mathfrak{F})/\mathcal{K}$; that is, $\sigma_e(T)$ is the set of all complex numbers ζ such that $T-\zeta+\mathcal{K}$ is not invertible in $\mathcal{L}(\mathfrak{F})/\mathcal{K}$. In this section we describe the essential spectrum of the Toeplitz operator T_f for those bounded measurable functions f for which the Hankel operator H_f is compact.

It will be convenient to make use of the following well-known identity, which gives a simple relationship between Toeplitz and Hankel operators:

$$T_{fg}-T_gT_f=H_{\bar{g}}^*H_f, \tag{19}$$

for $f,g\in L^\infty(\mathbf{C}^n)$.

For $E\subset\mathbf{C}^n$, let $\operatorname{cl} E$ denote the closure of the set E in $\mathbf{C}^n$. The following theorem and corollary are due to Berger and Coburn [1].

THEOREM 14. *Let $f\in L^\infty(\mathbf{C}^n)$ and assume that H_f is compact. Then*

$$\sigma_e(T_f)=\bigcap_{R>0}\operatorname{cl}\tilde{f}(\mathbf{C}^n\backslash R\mathbf{B}_n).$$

Proof. Let $f\in L^\infty(\mathbf{C}^n)$ and assume that H_f is compact. The proof will be divided into several steps.

Step 1. Suppose that $\zeta\notin\operatorname{cl} f(\mathbf{C}^n\backslash R\mathbf{B}_n)$ for some $R>0$. Define the function g on $\mathbf{C}^n$ by

$$g(z)=\begin{cases}(f(z)-\zeta)^{-1} & \text{if } z\in\mathbf{C}^n\backslash R\mathbf{B}_n,\\ 1 & \text{if } z\in R\mathbf{B}_n.\end{cases}$$

Then $g\in L^\infty(\mathbf{C}^n)$, and using identity (19) it is readily verified that $T_gT_{f-\zeta}=I-H_{\bar{g}}^*H_f-T_{(f-\zeta-1)\chi_{R\mathbf{B}_n}}$. Since both operators H_f and $T_{(f-\zeta-1)\chi_{R\mathbf{B}_n}}$ are compact, it follows that $T_{f-\zeta}+\mathcal{K}$ is left-invertible in the Calkin algebra $\mathcal{L}(\mathfrak{F})/\mathcal{K}$. Using that also $H_{\bar{f}}$ is compact and $T_{f-\zeta}=T_{\bar{f}-\bar{\zeta}}^*$, we see that $T_{f-\zeta}+\mathcal{K}$ is also right-invertible in $\mathcal{L}(\mathfrak{F})/\mathcal{K}$. Thus $T_f-\zeta+\mathcal{K}=T_{f-\zeta}+\mathcal{K}$ is invertible in $\mathcal{L}(\mathfrak{F})/\mathcal{K}$, so that $\zeta\notin\sigma_e(T_f)$. We conclude that $\sigma_e(T_f)\subset\operatorname{cl} f(\mathbf{C}^n\backslash R\mathbf{B}_n)$ for all $R>0$.

Step 2. We have already seen that $T_{f-\tilde{f}}$ is compact (Proposition 11). Thus $\sigma_e(T_f)=\sigma_e(T_{\tilde{f}})$, so that by Step 1, $\sigma_e(T_f)\subset\operatorname{cl}\tilde{f}(\mathbf{C}^n\backslash R\mathbf{B}_n)$ for all $R>0$.

Step 3. Suppose that $\zeta\in\operatorname{cl}\tilde{f}(\mathbf{C}^n\backslash R\mathbf{B}_n)$ for all $R>0$. Pick a sequence $(\lambda_j)_j$ in $\mathbf{C}^n$ such that $\tilde{f}(\lambda_j)\to\zeta$ and $\|\lambda_j\|\to\infty$ as $j\to\infty$. It follows from the identity

$$\widetilde{|f-\zeta|^2}(\lambda_j)=\widetilde{|f|^2}(\lambda_j)-|\tilde{f}(\lambda_j)|^2+|\tilde{f}(\lambda_j)-\zeta|^2$$

and Theorem 12 that $\widetilde{|f-\zeta|^2}(\lambda_j)\to 0$ as $j\to\infty$. Then by (16), using (5), we have $\|T_{f-\zeta}(k_{\lambda_j}/\|k_{\lambda_j}\|_2)\|_2=\|P((f-\zeta)\circ\tau_{\lambda_j})\|_2\to 0$ as $j\to\infty$, and $T_{f-\zeta}+\mathcal{K}$ cannot be invertible in the Calkin algebra $\mathcal{L}(\mathfrak{F})/\mathcal{K}$. Hence $\zeta\in\sigma_e(T_f)$. □

COROLLARY 15. *Let $f \in L^\infty(\mathbf{C}^n)$ and assume that H_f is compact. Then the essential spectrum $\sigma_e(T_f)$ is connected.*

Proof. The function $\tilde{f}$ is a bounded continuous function on $\mathbf{C}^n$, so all the sets $\mathrm{cl}\,\tilde{f}(\mathbf{C}^n \setminus R\mathbf{B}_n)$ are compact and connected. By Theorem 14, $\sigma_e(T_f)$ is the intersection of a nested family of compact connected sets. □

If for $f \in L^\infty(\mathbf{C}^n)$ the operator H_f is compact, then also $H_{\bar{f}}$ is compact and by Theorem 13 the function $\tilde{f}$ is constant on $\mathfrak{J}$-parts. The following theorem states that the essential spectrum of T_f is then equal to the set of constant values taken on by $\tilde{f}$ on the $\mathfrak{J}$-parts.

THEOREM 16. *Let $f \in L^\infty(\mathbf{C}^n)$ and assume that H_f is compact. Then*

$$\sigma_e(T_f) = \{\tilde{f}^\beta(\tau(0)) : \tau \in \mathfrak{J}\}.$$

Proof. Let $\tau \in \mathfrak{J}$ and $\zeta = \tilde{f}^\beta(\tau(0))$. Pick a net $\{\lambda_\alpha\}$ in $\mathbf{C}^n$ with $\|\lambda_\alpha\| \to \infty$ and $\tau_{\lambda_\alpha} \to \tau$ in $(\beta\mathbf{C}^n)^{\mathbf{C}^n}$. Then $\tilde{f}(\lambda_\alpha) = \tilde{f} \circ \tau_{\lambda_\alpha}(0) \to \tilde{f}^\beta \circ \tau(0) = \zeta$, so that

$$\zeta \in \bigcap_{R>0} \mathrm{cl}\,\tilde{f}(\mathbf{C}^n \setminus R\mathbf{B}_n) = \sigma_e(T_f).$$

Conversely, if $\zeta \in \sigma_e(T_f)$, then there is a net $\{\lambda_\alpha\}$ in $\mathbf{C}^n$ with $\|\lambda_\alpha\| \to \infty$ and $\tilde{f}(\lambda_\alpha) \to \zeta$. By passing to a subnet we may furthermore assume that $\tau_{\lambda_\alpha} \to \tau$ in $(\beta\mathbf{C}^n)^{\mathbf{C}^n}$. Then $\zeta = \lim_\alpha \tilde{f}(\lambda_\alpha) = \lim_\alpha \tilde{f} \circ \tau_{\lambda_\alpha}(0) = \tilde{f}^\beta \circ \tau(0) \in \tilde{f}^\beta(\tau(\mathbf{C}^n))$. □

The essential norm of an operator is its distance to the compact operators; that is, if T is an operator in $\mathcal{L}(\mathfrak{F})$ then its essential norm, denoted by $\|T\|_e$, is by definition the norm of the operator $T + \mathcal{K}$ in the Calkin algebra $\mathcal{L}(\mathfrak{F})/\mathcal{K}$. The following corollary gives the essential norm for certain Toeplitz operators on the Fock space.

COROLLARY 17. *Let $f \in L^\infty(\mathbf{C}^n)$ and assume that H_f is compact. Then*

$$\|T_f\|_e = \max\{|\tilde{f}^\beta(\tau(0))| : \tau \in \mathfrak{J}\}.$$

Proof. It is an easy consequence of identity (19), together with the fact that both H_f and $H_{\bar{f}}$ are compact, that $T_f + \mathcal{K}$ is a normal element of the Calkin algebra $\mathcal{L}(\mathfrak{F})/\mathcal{K}$. Thus $\|T_f + \mathcal{K}\|$ is equal to the spectral radius of $T_f + \mathcal{K}$, $\max\{|\zeta| : \zeta \in \sigma_e(T_f)\}$, which by Theorem 16 is equal to $\max\{|\tilde{f}^\beta(\tau(0))| : \tau \in \mathfrak{J}\}$. □

To state another corollary of Theorem 16, we recall that an operator in $\mathcal{L}(\mathfrak{F})$ is called *Fredholm* if its kernel has finite dimension and its range has finite co-dimension. It is a standard fact that T in $\mathcal{L}(\mathfrak{F})$ is Fredholm if and only if the operator $T + \mathcal{K}$ is invertible in the Calkin algebra $\mathcal{L}(\mathfrak{F})/\mathcal{K}$.

COROLLARY 18. *Let $f \in L^\infty(\mathbf{C}^n)$ and assume that H_f is compact. Then T_f is a Fredholm operator if and only if $\tilde{f}^\beta(\tau(0)) \neq 0$ for all $\tau \in \mathfrak{J}$.*

Proof. Observing that T_f is Fredholm if and only if $0 \notin \sigma_e(T_f)$, the statement follows immediately from Theorem 16. □

References

1. C. A. Berger and L. A. Coburn, *Toeplitz operators on the Segal–Bargmann space,* Trans. Amer. Math. Soc. 301 (1987), 813–829.
2. C. A. Berger, L. A. Coburn and K. H. Zhu, *Toeplitz operators and function theory in n-dimensions,* Pseudo-differential operators (H. O. Cordes, B. Gramsch, H. Widom, eds.), Lecture Notes in Math., 1256, Springer, Berlin, 1987, pp. 28–35.
3. V. Guillemin, *Toeplitz operators in n dimensions,* Integral Equations Operator Theory 7 (1984), 145–205.
4. P. R. Halmos and V. S. Sunder, *Bounded integral operators on L^2 spaces,* Springer, Berlin, 1978.
5. S. Janson, J. Peetre, and R. Rochberg, *Hankel forms and the Fock space,* Rev. Mat. Iberoamericana 3 (1987), 61–138.
6. K. Stroethoff, *Compact Hankel operators on the Bergman space,* Illinois J. Math. 34 (1990), 159–174.
7. ———, *Compact Hankel operators on the Bergman spaces of the unit ball and polydisk in* $\mathbf{C}^n$, J. Operator Theory 23 (1990), 153–170.

Department of Mathematical Sciences
University of Montana
Missoula, MT 59812