Rigidity Theorems for Foliations by Surfaces and Spin Manifolds

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1. Introduction

This paper is devoted to applications of the Lefschetz fixed point theorem for foliations [HL1; HL2]. The applications presented here are generalizations of two classical theorems: the finiteness of the automorphism group of a compact oriented surface of genus greater than 1, and the rigidity of compact spin manifolds with nonvanishing $\hat{A}$ genus [AH].

A foliation of a compact connected oriented Riemannian manifold is rigid if no compact connected Lie group acts nontrivially as a group of isometries taking each leaf to itself. If the foliation admits an invariant transverse measure then we can define the foliation Euler number and $\hat{A}$ genus. Proposition 3.1 says that a foliation by oriented surfaces with negative foliation Euler number is rigid. Proposition 3.2 says that a foliation by spin manifolds with nonzero foliation $\hat{A}$ genus is rigid if we insist that the Lie group of isometries preserves the leafwise spin structure. The proofs involve the application of our foliation Lefschetz theorem to the leafwise de Rham and Dirac complexes.

We also give examples of foliations which satisfy the hypotheses of Propositions 3.1 and 3.2 but whose leaf-preserving isometry group (resp., leaf-preserving isometry group preserving the leafwise spin structure) is still not finite.

In Section 2 we review the Lefschetz theorem for foliations. In Section 3 we state and prove our main theorems, and in Section 4 we present our examples. Section 5 contains the proof of Proposition 2.2 and some remarks about the leafwise de Rham and signature complex. We would like to thank John Wood for a number of helpful conversations.

2. Review of the Lefschetz Theorem

We recall some of the material from [HL1]. Let M be a compact connected oriented Riemannian manifold of dimension m and F a codimension-q oriented foliation.

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A *Dirac complex* (E, d) *along* F consists of:

(1) A family $E = (E_0, E_1, \ldots, E_k)$ of Hermitian vector bundles over M with Hermitian connection ∇^i on E_i.

(2) A family $d = (d_0, \ldots, d_{k-1})$ of differential operators, where

$$d_i : C^\infty(E_i) \to C^\infty(E_{i+1}).$$

The d_i differentiate only in leaf directions and the complex (E, d) is a generalized Dirac complex [GL] along the leaves of F. This means that $E = \bigoplus E_i$ is a module over the bundle of Clifford algebras $C(T^*F)$ and the connection $\nabla = \bigoplus \nabla^i$ on E is compatible with Clifford multiplication and the Riemannian connection on T^*F. Furthermore, the operator

$$D = \bigoplus (d_i + d_{i-1}^*) : C^\infty(E) \to C^\infty(E)$$

is given by the composition

$$C^\infty(E) \xrightarrow{\nabla} C^\infty(T^*M \otimes E) \to C^\infty(T^*F \otimes E) \xrightarrow{m} C^\infty(E),$$

where m is Clifford multiplication.

Let f be a diffeomorphism of M which takes each leaf of F to itself. A *geometric endomorphism* of (E, d) over f is a family $T = (T_0, \ldots, T_k)$ of complex linear maps

$$T_i : C^\infty(E_i) \to C^\infty(E_i)$$

such that $d_i T_i = T_{i+1} d_i$. We require that each T_i be given by

$$T_i(s)(x) = A_{i,x}(s(f(x))), \tag{2.1}$$

where $A_i : f^*E_i \to E_i$ is a smooth bundle map. Let E_i^L be the bundle E_i restricted to the leaf L. T_i induces a map T_i^L on $L^2(E_i^L)$ and a map

$$T_i^{L,\#} : H_L^i(E, d) \to H_L^i(E, d),$$

where $H_L^i(E, d)$ is reduced L^2 cohomology $\ker(d_i^L)/\overline{\operatorname{im}}(d_{i-1}^L)$. Here d_i^L is the restriction of d_i to sections of E_i^L. Let $d_i^{L,*}$ be the adjoint of d_i^L, and let

$$\Delta_i^L = d_{i-1}^L d_{i-1}^{L,*} + d_i^{L,*} d_i^L.$$

Using Hodge theory we can identify $H_L^i(E, d)$ with $\ker(\Delta_i^L)$, and $T_i^{L,\#}$ induces a map $T_i^{L,*}$ on $\ker(\Delta_i^L)$ given by compression of T_i^L to this kernel. We denote the family $\{T_i^{L,*}\}$ by T_i^*.

A holonomy invariant Radon transverse measure ν for F determines a trace tr_ν on certain families of operators on $\{L^2(E_i^L)\}$. In particular, for a Dirac complex along F and a geometric endomorphism T, $\operatorname{tr}_\nu(T_i^*)$ is defined. Then we define $L_\nu(T)$ by

$$L_\nu(T) = \sum_{i=1}^{k} (-1)^i \operatorname{tr}_\nu(T_i^*).$$

We note here the following proposition.

PROPOSITION 2.2. *For each i, $\mathrm{tr}_\nu(T_i^*)$ is independent of the metrics used to define it.*

See Section 5 for the proof.

In order to have a Lefschetz theorem for Dirac complexes and geometric endomorphisms, we must make assumptions about the fixed point set N of f and the behaviour of f near N. We assume that $N = \bigcup_\alpha N_\alpha$ is the finite disjoint union of closed connected submanifolds N_α each transverse to F, and that f is nondegenerate along N. Let L be a leaf and $N^L = N \cap L$. Nondegeneracy means that, for the map $df_{\eta,x}$ induced from the differential of f on $\eta_x = TL_x/TN_x^L$, we have

$$\det(I_x - df_{\eta,x}) \neq 0.$$

Finally we remark that, given a family $a = \{a^L\}$ of smooth measures on $\{N^L\}$ that vary smoothly transversely, we can define a measure $a\,d\nu$ on M using the invariant transverse measure. The main theorem of [HL1] is the following.

LEFSCHETZ FIXED POINT THEOREM. *Let M, F, (E,d), ν, f, A, T, and N be as above. To each N^L we can associate a smooth measure a^L which depends only on f, A, the symbols of Δ_i, the metrics, and their derivatives to a finite order on N^L. The family $a = \{a^L\}$ varies smoothly transversely and*

$$L_\nu(T) = \int_N a\,d\nu.$$

REMARK. For the applications that follow, we consider a compact connected Lie group G acting by isometries on M taking each leaf of F to itself. An element $g \in G$ acts, via the differential, on the cotangent bundle along the leaves and thus on the associated bundle Q of oriented orthonormal frames of this cotangent bundle. We will assume that the action of G on Q lifts to a principal bundle P which is a reduction of Q. Then G acts on vector bundles E which are associated to P via representations of the structure group of P. In the notation above, we have bundle maps $A_g: E_{gx} \to E_x$. It is convenient to define the action of g on E to be given by $A_{g^{-1}}$ so that $g: E_x \to E_{gx}$. We define $L_\nu(g)$ to be $L(T_{g^{-1}})$. With this convention G acts as a group of bundle maps of E and as a group on $C^\infty(E)$ with the action given by $(gs)(x) = gs(g^{-1}x)$.

We will be interested in the following complexes.

2.3. (E,d) is the de Rham complex along the leaves of F, the leaves of F are even-dimensional, and each component of N has dimension q so that each component of N^L is a single point x. If g is a topological generator of a maximal torus of G, then the measure on N^L associated to the geometric endomorphism $g = T_{g^{-1}}$ is given by

$$a^L(x) = \operatorname{sign}\det(I_x - g_x) = +1.$$

(See [ASe, p. 542] and [AB, I, p. 245].)

2.4. Each leaf has dimension $2k$, and the principal $\mathrm{SO}(2k)$ bundle Q has a $\mathrm{Spin}(2k)$ reduction P. We assume that the action of G on Q lifts to an action on P commuting with the projection $\mathrm{Spin}(2k) \to \mathrm{SO}(2k)$. Then G acts as a group of bundle maps on the vector bundles $\Delta^+(P), \Delta^-(P)$. G acts as a group of geometric endomorphisms on the complex $E_0 = \Delta^+(P)$, $E_1 = \Delta^-(P)$, $d_0 = D^+$, $d_1 = D^-$, where D^+, D^- are the ordinary Dirac operators along the leaves. (See [AS, III, pp. 569–571], [AB, II, p. 474], and [HL1, pp. 21–23].)

We conclude this section with a discussion of the characteristic numbers $\chi_\nu(F)$ and $\hat{A}_\nu(F)$. Let M, F, ν be as above, and assume that each leaf has dimension $2k$. For each leaf L let K^L be the curvature form of the Riemannian connection on L, and let $\mathrm{Pf}(K^L)$ be the associated Pffafian form. $\{\mathrm{Pf}(K^L)\}$ is a smooth family of measures.

DEFINITION 2.5. $\chi_\nu(F) = \int_M \mathrm{Pf}(K^L)\, d\nu$.

We note that the Connes index theorem [C] implies that $\chi_\nu(F)$ is equal to the ν-index of the leafwise de Rham complex, and by [HL1, §6] $\chi_\nu(F)$ is also equal to $L_\nu(I)$.

Let $\hat{A}_k(p_1, \ldots, p_k)$ be the $\hat{A}$ polynomial [AS, III, p. 570] and $\hat{A}_k(K^L)$ the differential form we get by replacing $p_1, \ldots, p_k$ by the Pontrjagin polynomials in the curvature K^L.

DEFINITION 2.6. $\hat{A}_\nu(F) = \int_M \hat{A}_k(K^L)\, d\nu$.

Again, the Connes index theorem implies that $\hat{A}_\nu(F)$ is equal to the ν-index of the leafwise Dirac complex, which is in turn equal to $L_\nu(I)$.

3. The Main Theorem

Let M, F, and ν be as in Section 2.

PROPOSITION 3.1. *Suppose F is a foliation by surfaces and ν is a nonnegative measure. If $\chi_\nu(F) < 0$ then no compact connected Lie group can act nontrivially as a group of isometries of M taking each leaf to itself.*

For the statement of the next proposition, let F, Q, P, and G be as in §2.4. In this case, we say that F has a spin structure and that G preserves the spin structure. The following is a generalization of the main theorem of [AH].

PROPOSITION 3.2. *Let F be a foliation with a spin structure. If $\hat{A}_\nu(F) \neq 0$ then no compact connected Lie group can act nontrivially on M as a group of isometries taking each leaf to itself and preserving the spin structure on F.*

Before proving these, we establish a lemma about fixed point sets of leaf-preserving group actions.

LEMMA 3.3. *Let G be a nontrivial compact connected Lie group acting on M by isometries taking each leaf of F to itself. The fixed point set of this action is a closed submanifold which is transverse to F.*

Proof. The fixed point set of a compact Lie group action is always a closed submanifold, so we must show transversality. Let p be a fixed point, L the leaf through p, and V the orthogonal complement of $T_p(L)$ in $T_p(M)$. We will show that the representation of G on V is trivial.

In a neighborhood of p the local leaves of F are given by $y_1=\epsilon_1,\dots,y_q=\epsilon_q$, where $\{x_1,\dots,x_r,y_1,\dots,y_q\}$ is a foliation chart. We write this as $y=\epsilon$. The map $\exp_p$ is a diffeomorphism on a neighborhood of $0\in T_p(M)$, so we get a foliation of this neighborhood with leaves $\Sigma_\epsilon=\exp_p^{-1}(y=\epsilon)$. Because the differential $d\exp_{p,0}$ is the identity, it follows that $\Sigma_0=\exp_p^{-1}(y_1=0,\dots,y_q=0)$ is tangent to $T_p(L)$ at $0\in T_p(M)$. Since V is transverse to Σ_0 at 0 it follows that V is transverse to Σ_ϵ for small $|\epsilon|$.

Let $u\in V$ (u sufficiently small), and assume that $Gu\neq u$. Then there is a path θ_t in G with $\theta_0=\mathrm{id}$, such that the path $u\to\theta_t u$ is a nontrivial path in V and hence is transverse to Σ_ϵ for small $|\epsilon|$. Since there are only a countable number of local leaves corresponding to a given leaf of F, it follows that this path will intersect some Σ_ϵ corresponding to a leaf of F other than the leaf through $\exp_p(u)$. Since $\exp_p$ commutes with the action of G, this contradicts the assumption that G takes each leaf to itself. Thus G acts trivially on V, so $\exp_p(V)$ is contained in the fixed point set. Thus the fixed point set is transverse to F. □

Proof of Proposition 3.1. We may assume that $G=S^1$. Suppose G acts nontrivially, and let g be a topological generator of G. As a leaf-preserving diffeomorphism of M, g is homotopic to the identity, and hence (by [HL1, 6.3]) the map $g^{L,*}$ induced on the L^2 harmonic forms on each leaf is the identity. Thus $L_\nu(g)=L_\nu(I)=\chi_\nu(F)$.

The action of g on M must have a nontrivial fixed point set, since otherwise the Lefschetz theorem implies that $\chi_\nu(F)=0$. Let p be a fixed point on the leaf L. dg_p^L is orientation-preserving because it is homotopic to the identity, and hence it must be rotation through an angle θ. $\theta\neq 0$, for otherwise g^L would be the identity on L, and from Lemma 3.3 it follows (since M is connected) that g is the identity on M. Thus p is isolated on L and from §2.3 we have $a^L(p)=+1$. We apply the Lefschetz theorem to g acting on the de Rham complex along the leaves to conclude that

$$\chi_\nu(F)=L_\nu(g)=\int_N(+1)\,d\nu=\nu(N)\geq 0,$$

which is a contradiction. □

We remark that in the case where there is only one leaf M, a proof using the Lefschetz theorem of [AB] was given by Atiyah (lecture).

For the proof of Proposition 3.2 we need the following result. Assume that the compact Lie group G acts as a group of geometric endomorphisms of (E,d), where $g: E_{i,x} \to E_{i,gx}$. Again, it will be sufficient to consider $G = S^1$.

THEOREM 3.4. *The Lefschetz number $L_\nu(g)$ is a continuous function on G.*

Proof. From [HL1, Thm. 5.1] we have

$$L_\nu(g) = \sum_{i=0}^{k} (-1)^i \operatorname{tr}_\nu(T_{g^{-1},i} e^{-t\Delta_i}).$$

Let $E = E_i$, $\Delta = \Delta_i$, and $T_g = T_{g,i}$. Let $k_{t,L}(x,y)$ be the Schwartz kernel of $\exp(-t\Delta^L)$ and let $B_{g,x}: E_x \to E_{gx}$ be the map induced by g. The Schwartz kernel $k_{t,L}^g(x,y)$ of $T_{g^{-1}}\exp(-t\Delta^L)$ is $B_{g,x}k_{t,L}(x,gy)$. From the definition of tr_ν it will be enough to show that for fixed t, $\operatorname{tr} k_{t,L}^g(x,x)$ is continuous in g, uniformly in x; that is,

$$|\operatorname{tr} k_{t,L}^g(x,x) - \operatorname{tr} k_{t,L}^h(x,x)| \le C|g-h|,$$

where C is independent of $x \in M$. Note that we use the more suggestive notation $g-h$ instead of gh^{-1}. We will suppress the sub- and superscript L in the subsequent discussion.

We first take the case where E is a trivial 1-dimensional bundle. Relative to an orthonormal section of E, $k_t(x,y)$ and $B_{g,x}$ are functions. We first show that $k_t(x,gx)$ is continuous in g, uniformly in x. It follows from the uniform continuity of G acting on M that for $|g-h|$ small and any x, hx will lie in a good coordinate system [R, p. 93] centered at gx on the leaf through x. The disjoint union of the leaves of the foliation is a manifold of bounded geometry, so there is an integer $s > 0$ independent of x and the leaf such that δ_x, the Dirac function, lies in the Sobolev space $H_{-s}(L)$. Let ψ be a C^∞ function of compact support on L. For fixed x, g, h let φ be the L^2 function $(1+\Delta)^{-s}(\delta_{gx} - \delta_{hx})$, and let $\gamma = (1+\Delta)^{-s}\psi$. Then

$$|\langle \varphi, \psi \rangle_{L^2(L)}| = |\gamma(gx) - \gamma(hx)| = \left| \sum_i \frac{\partial \gamma}{\partial x_i}(\zeta_i)((gx)_i - (hx)_i) \right|$$

in normal coordinates on L centered at gx. From the Sobolev inequalities it follows that

$$|\langle \varphi, \psi \rangle_{L^2(L)}| \le \sum \sup_{\zeta} \left| \frac{\partial \gamma}{\partial x_i}(\zeta_i) \right| |(gx)_i - (hx)_i| \le C_K \|\gamma\|_K |gx - hx|,$$

where K is a positive integer greater than s and independent of x, $\| \ \|_K$ is the Sobolev K norm, C_K is a constant depending only on K, and $|gx-hx|$ is the distance between gx and hx determined by the metric on L. $\|\gamma\|_K = \|\psi\|_{K-s}$, so

$$|\langle \varphi, \psi \rangle|_{L^2(L)} \le C_K \|\psi\|_{K-s} |gx - hx|$$

and it follows that $\|\varphi\|_{-(K-s)} \le C_K|gx-hx|$. Now

$$\|\varphi\|_{-(K-s)} = \|(1+\Delta)^{-(K-s)}(1+\Delta)^{-s}(\delta_{gx} - \delta_{hx})\|_0 = \|\delta_{gx} - \delta_{hx}\|_{-K}.$$

Thus $\|\delta_{gx}-\delta_{hx}\|_{-K} \leq C_K|gx-hx|$, which is bounded by a constant times $|g-h|$ independent of x, because of the uniform continuity of the action of G. Now

$$|k_t(x,gx)-k_t(x,hx)| = |\langle e^{-t\Delta}(\delta_{gx}-\delta_{hx}), \delta_x\rangle|$$
$$\leq \|e^{-t\Delta}\|_{-K,K}\|\delta_x\|_{-K}\|\delta_{gx}-\delta_{hx}\|_{-K}.$$

Bounded geometry of the leaves implies that $\|\delta_x\|_{-K}$ is bounded independent of $x\in M$, and $\|\exp(-t\Delta^L)\|_{-K,K}$ is bounded for fixed t; hence $|k_t(x,gx)-k_t(x,hx)|$ is bounded by a constant times $|g-h|$ independent of $x\in L$ and of the leaf L.

Now

$$|B_{g,x}k_t(x,gx)-B_{h,x}k_t(x,hx)| \leq |B_{g,x}||k_t(x,gx)-k_t(x,hx)|$$
$$+|B_{g,x}-B_{h,x}||k_t(x,hx)|.$$

It follows from the boundedness and uniform continuity of the map $(g,x)\to B_{g,x}$, the boundedness of $|k_t(x,y)|$ for fixed t [HL1, 2.3.13], and the previous result that the right-hand side is bounded by a constant times $|g-h|$ independent of x and L.

More generally, when E is not a trivial bundle choose synchronous orthonormal framings $\{v_1,\ldots,v_r\}$ near x and $\{w_1,\ldots,w_r\}$ near gx on the leaf L through x. For v having length 1, let $\delta^{v,x}$ be the distributional section of E^L given by $\delta^{v,x}(s)=(s(x),v)_x$. $\delta^{v,x}\in H_{-s}(E^L)$ for s independent of x and v. As above, $\|\delta^{v,x}\|_{-K}$ is bounded independently of v, x, and L for $K>s$. As in the previous case, let $\varphi=(1+\Delta)^{-s}(\delta^{w_j,gx}-\delta^{w_j,hx})$, let ψ be a C^∞ compactly supported section of E^L, and let $\gamma=(1+\Delta^L)^{-s}\psi$. Then

$$\langle\varphi,\psi\rangle_{L^2(E^L)} \leq C_K\|\gamma\|_K\|gx-hx\|$$

and

$$\|\varphi\|_{-(K-s)} = \|\delta^{w_j,gx}-\delta^{w_j,hx}\|_{-K} \leq C_K\|gx-hx\|,$$

so that $|k_t^{i,j}(x,gx)-k_t^{i,j}(x,hx)|$ and hence also $\|k_t(x,gx)-k_t(x,hx)\|$ is bounded by a constant times $|g-h|$ independent of x and L. Now a calculation in normal coordinates and synchronous framings identical to the case of E trivial shows that $|\mathrm{tr}\,B_{g,x}k_t(x,gx)-\mathrm{tr}\,B_{h,x}k_t(x,hx)|$ is also bounded by a constant times $|g-h|$ independent of x and L. □

Proof of Proposition 3.2. We can assume that $G=S^1$. The fixed point set N of G is a closed submanifold transverse to the foliation F. Let N_α be a connected component of N and let $y\in N_\alpha\cap L$. The normal bundle to $N_\alpha\cap L$ in L at y can be written as $\bigoplus V_y^j$, where G acts on V_y^j by the representation $\theta\to\exp(2\pi i m_j\theta)$ for $m_j>0$. It follows that the V^j are complex G vector bundles on $N_\alpha\cap L$.

Let z be a complex number, $z\neq 1$. The function $1/(1-ze^{-x})$ can be written as $R(x,z)$, where $R(x,z)$ is a formal power series in x whose coefficients are rational functions in z having a pole only at $z=1$ (and no pole at $z=\infty$).

Let $A^r(z)=(z^r)^{1/2}\prod_{j=1}^r e^{-x_j/2}R(x_j,z)$. Because of the factor $(z^r)^{1/2}$ this is defined only up to sign. It defines a formal power series in $c_1,\ldots,c_r$ the elementary symmetric functions in $x_1,\ldots,x_r$.

Now let $z \in G$ be a generator. z acts on V^j by multiplication by z^{m_j}. Let $A(V^j, z) = A^{d_j}(z^{m_j})$, where d_j is the (complex) dimension of V.

The Riemannian connection on $N_\alpha \cap L$ preserves the bundles V^j and is a complex connection on each V^j. We can replace $c_1, \dots, c_r$ by the Chern polynomials in the curvature of this connection. In this way $A(V^j, z)$ becomes a differential form on $N_\alpha \cap L$. Let $\hat{A}(N_\alpha \cap L)$ be the differential form we get by replacing the Pontrjagin classes in the expression for the $\hat{A}$ class of a real vector bundle [AS, III, p. 570] by the Pontrjagin polynomials in the curvature of the Riemannian connection on $N_\alpha \cap L$. Then we define

$$A(N_\alpha \cap L, z) = \hat{A}(N_\alpha \cap L) \prod_j A(V^j, z).$$

Now $A(N_\alpha \cap L, z)$ is a differential form on $N_\alpha \cap L$ that has the factor $(z^d)^{1/2}$, where $d = \sum m_j d_j$ and $d_j = \dim_C V^j$. The choice of sign is determined by the argument of [AH, p. 21]. Namely, let $y \in N_\alpha$ and choose $(z^d)^{1/2}$ so that

$$2^{n_\alpha} z^d \prod_j (1 + z^{m_j})^{d_j} = \operatorname{trace}(z \mid (\Delta^+(P)_y \oplus \Delta^-(P)_y)),$$

where L is the leaf through y and $2n_\alpha = \dim(N_\alpha \cap L)$. $A(N_\alpha \cap L, z)$ is the leafwise measure on the fixed point set given in the foliation Lefschetz theorem for z acting on the leafwise Dirac complex, and

$$L_\nu(z) = \sum_\alpha \int_{N_\alpha} A(N_\alpha \cap L, z)\, d\nu$$

(see [HL1] and [G, Thm. 0.4, 2d]).

Now consider the function on the complex plane given by

$$A(F, z) = \sum_\alpha \int_{N_\alpha} A(N_\alpha \cap L, z)\, d\nu.$$

Since $A(N_\alpha \cap L, z)$ is a differential form whose coefficients are rational functions having poles only at the roots of unity, the same is true of $A(F, z)$ and, because of the factor $(z^d)^{1/2}$, $A(F, 0) = 0$ and $A(F, z)$ has no pole at $z = \infty$. For $z \in S^1$ (but z not a root of unity), z is a topological generator of S^1 and thus the fixed point set of z acting on M is the same as the fixed point set of S^1 acting on M. For such a z acting on the leafwise Dirac complex we have

$$L_\nu(z) = A(F, z).$$

The left-hand side of this equation, however, is defined for all $z \in S^1$ and, by Theorem 3.4, $L_\nu(z)$ is a continuous function on S^1. Thus $A(F, z)$ has no poles and so is analytic and bounded, and thus is a constant and hence zero. Therefore $L_\nu(z) = 0$ for all $z \in S^1$. But for $z = 1$, $L_\nu(z) = \hat{A}(F)$. □

4. Examples

Let M be a compact Riemannian manifold, F an oriented foliation, and ν an invariant transverse measure. In [HL1] we have introduced the group

$O(M;F)$ of isometries of M taking each leaf to itself. Proposition 3.1 implies that if F is a foliation by surfaces, ν is nonnegative, and $\chi_\nu(F)<0$, then $O(M;F)$ contains no nontrivial compact connected Lie group.

Similarly, let F be a foliation by Spin($2k$) manifolds and let P be a spin structure for F. We let $O(M;F,P)$ be the group of isometries of M which take each leaf of F to itself and which preserve the spin structure of F. With the hypothesis of Proposition 3.2, $O(M;F,P)$ contains no nontrivial compact connected Lie group.

We now give examples to show that $O(M;F)$ and $O(M;F,P)$ can still be nontrivial and not finite.

LEMMA 4.1. *Let M, F, ν be as above. Suppose that M is a flat bundle over the compact even-dimensional manifold X with fiber S, and that F is the foliation of M given by the flat structure. Then $\chi_\nu(F)=\chi[X]\nu(S)$ and $\hat{A}_\nu(F)=\hat{A}[X]\nu(S)$. In particular, if $\chi[X]<0$ and $\nu(S)>0$ then $\chi_\nu(F)<0$.*

Proof. Let $\pi: M\to X$ be the bundle projection. We can construct an open set U in X of full Lebesgue measure such that, for each leaf L and fixed $x_0\in U$,

$$\pi^{-1}(U)\cap L=\bigcup_{z\in\pi^{-1}(x_0)\cap L} U_z,$$

where $\{U_z\}$ are open and mutually disjoint, and where $\pi(U_z)=U$ and each U_z is a fundamental domain for the cover $\pi: L\to X$. Construct a measurable function $f: M\to S=\pi^{-1}(x_0)$ by

$$f(u)=\begin{cases} z & \text{for } u\in U_z,\\ z_0 & \text{for } u\in M-\pi^{-1}(U),\end{cases}$$

where z_0 is a fixed point in $\pi^{-1}(x_0)$. By the definition of χ_ν and $\hat{A}_\nu$,

$$\chi_\nu(F)=\int_{z\in S}\left(\int_{f^{-1}(z)}\mathrm{Pf}(K^L)\right)d\nu(z);$$

$$\hat{A}_\nu(F)=\int_{z\in S}\left(\int_{f^{-1}(z)}\hat{A}_k(K^L)\right)d\nu(z).$$

But $\mathrm{Pf}(K^L)$ and $\hat{A}_k(K^L)$ are given by local data on X: $\mathrm{Pf}(K^L)=\mathrm{Pf}(K^X)$ and $\hat{A}_k(K^L)=\hat{A}_k(K^X)$ on U_z. $\pi(U_z)$ has full measure in X, so the integrals become $\chi[X]\nu(S)$ and $\hat{A}[X]\nu(S)$, respectively. □

Let Γ be the fundamental group of a compact surface of genus 3. Let H be the hyperbolic disk. We take the complex structure on the surface given by a symmetric fundamental domain D in H. Let $\{a,a'\}$, $\{b,b'\}$, $\{c,c'\}$, $\{d,d'\}$, $\{e,e'\}$, and $\{f,f'\}$ be pairs of opposite sides of D. Let A, B, C, D, E, and F be orientation-preserving isometries of H without fixed points which take (resp.) a to a', b to b', c to c', ... Γ is the free group on these generators subject to

$$FE^{-1}DC^{-1}BA^{-1}F^{-1}ED^{-1}CB^{-1}A=I.$$

Let θ_1, θ_2 be independent irrational numbers in $S^1 = R/Z$. Define a homomorphism $h\colon \Gamma \to \text{translations}(S^1)$ by sending

$$A, E, C^{-1} \text{ to } \theta \to \theta + \theta_1, \quad A^{-1}, E^{-1}, C \text{ to } \theta \to \theta - \theta_1,$$

$$B, F, D^{-1} \text{ to } \theta \to \theta + \theta_2, \quad B^{-1}, F^{-1}, D \text{ to } \theta \to \theta - \theta_2.$$

The above relation is clearly preserved under this homomorphism. Let $M = H \times_\Gamma S^1$, where the identifications are given by $(x, \theta) \sim (\gamma x, h(\gamma)\theta)$ for $\gamma \in \Gamma$. M is a flat bundle over a surface of genus 3, and M is foliated, the leaves being given by the images of $H \times \{\text{point}\}$ in M. The action of Γ on S^1 preserves the standard measure and thus induces a nonnegative transverse invariant measure for the foliation. From Lemma 4.1 we have $\chi_\nu(F) < 0$.

We construct a leaf-preserving isometric action of $Z \times Z$ on M by

$$(m, n)[x, \theta] = [x, \theta + m\theta_1 + n\theta_2].$$

This action is clearly well defined and takes each leaf to itself. It is clear that a nontrivial element of this group cannot have any fixed points.

There are other leaf-preserving isometries of M which do have fixed points. Let $G = \mathrm{SL}(2)$ and $K = \mathrm{SO}(2)$ so that $H = G/K$. It is easy to check that the map of M given by $[gK, \theta] \to [g_0 gK, \theta]$ is well defined on M if

$$h(g_0 \gamma^{-1} g_0^{-1} \gamma) = 1 \quad \text{for all } \gamma \in \Gamma.$$

This map is clearly an isometry taking each leaf to itself. Let g_0 be the element of SL(2) which induces rotation by $-2\pi/3$. It is easy to see that the condition above is satisfied; thus the map induces an isometry of M with fixed points. In fact, one can readily show that $L_\nu(g_0) > 0$.

Finally, we construct an example of a compact manifold foliated by spin manifolds with $\hat{A}_\nu(F) \neq 0$ and with an infinite group of leaf-preserving isometries which preserve the spin structure on the foliation. Let $X_2(d) \subset CP^3$ be the surface given by a polynomial of degree d. Then from [LW, p. 478] the characteristic classes of $X_2(d)$ are given by $c_1 = (4-d)x$ and $p_1 = (4-d^2)x^2$, where x is the restriction of the canonical generator of $H^2(CP^3)$. Thus $X_2(4)$ is a compact orientable 4-manifold with $c_1 = 0$ and hence $w_2 = 0$ and $p_1 = -12x^2$, so that $\hat{A}[X_2(4)] \neq 0$. Let $X = X_2(4) \# T^4$. Since

$$w_2(X) = w_2(X_2(4)) + w_2(T^4) = 0,$$

X is a spin manifold. $\hat{A}[X] = \hat{A}[X_2(4)] + \hat{A}[T^4] = \hat{A}[X_2(4)] \neq 0$ since the connected sum is oriented cobordant to the disjoint union. Let Γ be the fundamental group of X. $H_1(X) \cong H_1(X_2(4)) \oplus Z^4$, and thus Γ maps surjectively to Z^4. Choose a Z factor in Z^4 and let $\rho\colon \Gamma \to Z$ be the surjection onto this factor.

Let P be a spin(4) reduction of X, $\hat{X}$ the universal cover of X, and $\hat{P}$ the pullback of P to $\hat{X}$. It follows that $\hat{P}$ is a spin(4) reduction of $\hat{X}$. Choose an irrational number $\theta_0 \in S^1$ and let Γ act on S^1 by

$$\gamma\theta = 2\pi\rho(\gamma)\theta_0 + \theta.$$

We let $M=\hat{X}\times_\Gamma S^1$ with the foliation F having leaves given by the images of $\hat{X}\times\{\text{point}\}$ in M. F has an invariant transverse measure ν as in the previous example, and has a spin structure given by $\hat{P}\times_\Gamma S^1$. It follows from Lemma 4.1 that $\hat{A}_\nu(F)\neq 0$.

Let $\gamma_0\in\Gamma$ be an element which maps onto a generator of Z under ρ. We have an action of Z on M given by

$$n[x,\theta]=[x,2\pi\rho(\gamma_0^n)\theta_0+\theta].$$

Clearly, this is an isometric leaf-preserving action which has a lift to the spin structure on F given by

$$n[e,\theta]=[e,2\pi\rho(\gamma_0^n)\theta_0+\theta]$$

for $[e,\theta]\in\hat{P}\times_\Gamma S^1$.

5. Proof of Proposition 2.2 and Remarks

This section is essentially an addendum to [HL1]. We begin by proving the following.

PROPOSITION 2.2. *For each i, $\operatorname{tr}_\nu(T_i^*)$ is independent of the metric used to define it.*

Proof. Choose metrics on M and the E_i. For a leaf L we have E_i^L, d_i^L, and Δ_i^L as in Section 2. Denote by

$$Q^L: L^2(E_i^L)\to\ker(d_i^L) \quad\text{and}\quad P^L: L^2(E_i^L)\to\ker(\Delta_i^L)$$

the orthogonal projections. Let $\Delta_i^{L,1}$ be the Laplacian determined by another choice of metrics and $P_1^L: L^2(E_i^L)\to\ker(\Delta_i^{L,1})$ the associated projection. Note that P_1^L is not necessarily an orthogonal projection on $L^2(E_i^L)$ with respect to the inner product given by the original metrics (which we continue to use).

The Hodge theorem for Dirac complexes tells us that

$$\ker d_i^L=\overline{\operatorname{im}}\, d_{i-1}^L\oplus\ker\Delta_i^L=\overline{\operatorname{im}}\, d_{i-1}^L\oplus\ker\Delta_i^{L,1}.$$

In addition, both P^L and P_1^L, when restricted to $\ker d_i^L$, are zero on $\overline{\operatorname{im}}\, d_{i-1}^L$. Thus, on $\ker d_i^L$, $P^LP_1^L=P^L$ and $P_1^L=P_1^LP^L$. The facts that $T_i(\overline{\operatorname{im}}\, d_{i-1}^L)\subset\overline{\operatorname{im}}\, d_{i-1}^L$ and $T_i(\ker d_i^L)\subset\ker d_i^L$ then easily give that

$$P^LT_iP^L\,|\ker d_i^L=P^LP_1^LT_iP_1^L\,|\ker d_i^L.$$

Note that $(P^L)^2=P^L$, $(P_1^L)^2=P_1^L$, and $Q^LP^L=P^LQ^L=P^L$. Now let $P=\{P^L\}$, $P_1=\{P_1^L\}$, and $Q=\{Q^L\}$. Then

$$\begin{aligned}\operatorname{tr}_\nu(PT_iP)&=\operatorname{tr}_\nu(QPT_iPQP)=\operatorname{tr}_\nu(QPP_1T_iP_1QP)=\operatorname{tr}_\nu(PP_1T_iP_1P)\\&=\operatorname{tr}_\nu(P_1T_iP_1P^2)=\operatorname{tr}_\nu(P_1^2T_iP_1P)=\operatorname{tr}_\nu(P_1T_iP_1PP_1)=\operatorname{tr}_\nu(P_1T_iP_1).\end{aligned}$$

The fourth and sixth equalities follow from [HL1, §2] and the facts that $P_1T_iP_1P$ is a bounded measurable leafwise smoothing operator,

$$P^L = \lim_{t\to\infty} \exp(-t\Delta_i^L) \quad \text{and} \quad P_1^L = \lim_{t\to\infty} \exp(-t\Delta_i^{L,1}). \qquad \square$$

COROLLARY 5.1. *The Betti numbers β_i of a foliation do not depend on the metric on M.*

Recall that $\beta_i = \mathrm{tr}_\nu(P_i)$. This corollary was first proven by Connes [C].

Consider now the signature operators D_+^L along the leaves of a foliated manifold where the foliation has dimension $2k$. Let σ be the usual involution on leafwise forms; let $P_k^{L,+}$ be projection onto the $+1$ eigenspace of σ in $\ker \Delta_k^L$, and similarly for $P_k^{L,-}$ (see [AS, III]). On each leaf L, consider the quadratic form on L^2 harmonic forms given by $(\alpha, \beta) = \int_L \alpha \wedge *\beta$. The usual argument of [AS, III] shows that $P_k^\pm$ are the projections onto the positive and negative definite spaces of this form. Let $P_k^\pm = \{P_k^{L,\pm}\}$, $D_+ = \{D_+^L\}$, and

$$S_\nu(M, F) = \mathrm{tr}_\nu(P_k^+) - \mathrm{tr}_\nu(P_k^-).$$

The argument of [AS, III] extends to show that the ν index of D_+, $I_\nu(D_+)$, satisfies

$$I_\nu(D_+) = S_\nu(M, F).$$

COROLLARY 5.2. *$S_\nu(M, F)$ does not depend on the metric on M.*

Proof. $I_\nu(D_+) = L_\nu(I)$, where I is the identity map of M. $\square$

We note the following.

THEOREM 5.3. *Suppose f_0 and f_1 are leaf-preserving isometries of the foliated manifold M which are homotopic through leaf-preserving diffeomorphisms (not necessarily isometries). Then $L_\nu(f_0) = L_\nu(f_1)$ for the signature complex.*

Proof. By (6.3) of [HL1], $f_0^* = f_1^*$ acting on harmonic forms. But for the signature complex, $\ker \Delta_i^L \subset$ harmonic forms. $\square$

Theorem 5.3 is also true for the de Rham complex, and we may drop the condition that f_0 and f_1 be isometries.

Finally, we remark that the results of [BT], originally conjectured by Witten, extend to foliated manifolds in much the same way as the result of [AH] on the $\hat{A}$ was extended here.

References

[AB] M. F. Atiyah and R. Bott, *A Lefschetz fixed point formula for elliptic complexes I and II,* Ann. of Math. (2) 86 (1967), 374–407; Ann. of Math. (2) 88 (1968), 451–491.

[AH] M. F. Atiyah and F. Hirzebruch, *Spin manifolds and group actions,* Essays on Topology and Related Topics (Mémoires dédiés à Georges de Rham), pp. 18–28, Springer, New York, 1970.

[AS] M. F. Atiyah and I. M. Singer, *The index of elliptic operators I, III,* Ann. of Math. (2) 87 (1968), 484–530 and 546–604.

[ASe] M. F. Atiyah and G. Segal, *The index of elliptic operators II,* Ann. of Math. (2) 87 (1968), 531–545.

[BT] R. Bott and C. Taubes, *On the rigidity theorems of Witten,* preprint, Harvard University.

[C] A. Connes, *Sur la théorie non-commutative de l'intégration,* Lecture Notes in Math., 725, pp. 19–143, Springer, Berlin, 1979.

[G] P. B. Gilkey, *Lefschetz fixed point formulas and the heat equation.* Partial Differential Equations and Geometry, Proceedings of the Park City Conference (C. Byrnes, ed.), pp. 91–147, Marcel Dekker, New York, 1979.

[GL] M. Gromov and H. B. Lawson, Jr., *Positive scalar curvature and the Dirac operator on complete Riemannian manifolds,* Publ. Math. IHES 58 (1983), 83–196.

[HL1] J. L. Heitsch and C. Lazarov, *A Lefschetz fixed point theorem for foliated manifolds,* Topology 29 (1990), 127–162.

[HL2] ———, *The Lefschetz fixed point theorem for foliated manifolds,* Contemp. Math., 105, pp. 83–89, Amer. Math. Soc., Providence, R.I., 1990.

[LW] A. Libgober and J. Wood, *Differentiable structures on complete intersections I,* Topology 21 (1982), 469–482.

[R] J. Roe, *An index theorem on open manifolds I,* J. Differential Geom. 27 (1988), 87–113.

[T] M. Taylor, *Pseudodifferential operators,* Princeton Univ. Press, Princeton, N.J., 1981.

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