Landau and Schottky Theorems for Holomorphic Curves

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The theorems of Landau and Schottky on holomorphic functions were generalized to holomorphic curves in the projective plane in a paper published in 1926 by Bloch [1]. A gap in Bloch's argument was filled by Cartan in his thesis published in 1928 [4]. The aim of this paper is to prove Bloch's versions of the Landau and Schottky theorems with explicit expressions for the constants. It appears that this has never been done before. In contrast, very precise expressions for the constants in the classical Landau and Schottky theorems have been obtained by a succession of authors, of whom the most recent is Hempel [9].

We shall write $B(a, r)$ for the open ball of centre a and radius r, and $\bar{B}(a,r)$ for the closed ball of centre a and radius r. The open annulus of centre a, inner radius r and outer radius R will be denoted by $A(a,r,R)$.

The classical Schottky theorem concerns a holomorphic function f: $B(0,1)\to\mathbf{C}$. Suppose that f omits the values 0 and 1 as well as the value ∞. Then, for any ρ satisfying $0<\rho<1$, there is an upper bound for $|f(z)|$ when $z\in\bar{B}(0,\rho)$, depending only on ρ and $f(0)$. It follows easily that under the same hypotheses there is an upper bound for $|f'(0)|$ depending only on $f(0)$, and this is the content of Landau's theorem.

To describe Bloch's version of the Landau and Schottky theorems, it is convenient to work in homogeneous coordinates on $\mathbf{CP}^2$. Adopting Bloch's notation, we let X, Y, Z, T be four holomorphic functions on $B(0,1)$. We write $X_0=X(0)$, $Y_0=Y(0)$, $Z_0=Z(0)$, and $T_0=T(0)$. We shall usually suppress the notation for the point at which one of these functions is evaluated. We assume that X, Y, Z, T have no zeros and satisfy the identity

$$X+Y+Z+T=0.$$

We may regard (X, Y, Z) as an expression in homogeneous coordinates (x^0, x^1, x^2) for a holomorphic curve $c\colon B(0,1)\to\mathbf{CP}^2$. The curve c omits four lines in general position with the equations

$$x^0=0,\quad x^1=0,\quad x^2=0,\quad x^0+x^1+x^2=0.$$

We seek bounds for the ratios of X, Y, Z, and T, or the derivatives of these ratios, in terms of X_0, Y_0, Z_0, and T_0. It is obvious that there are some cases

Received February 27, 1990.
Michigan Math. J. 38 (1991).

in which no bounds can be expected. Suppose that $X_0+Y_0=0$, so that (X_0, Y_0, Z_0) lies on the diagonal $x^0+x^1=0$ of the omitted quadrilateral. Consider the curve

$$X=-Y=X_0e^{Az}, \qquad Z=Z_0.$$

The value of X/Z can be made arbitrarily large on any disc $B(0, \rho)$ by suitably choosing the constant A. If (X_0, Y_0, Z_0) does not lie on a diagonal, it will turn out that we can bound all the ratios. This aspect of Bloch's results does not correspond to anything in the classical theorems.

The method of this paper is to go through the steps in Bloch's argument and obtain an explicit expression for each of the constants. It is pertinent to remark that most of the approaches to the classical Schottky and Landau theorems present severe problems when one tries to generalize them to holomorphic curves. The most common approach is to use the uniformization by the modular function, and in the present state of knowledge we do not have a corresponding tool available in higher dimensions. (Some remarks on the state of knowledge about uniformization in higher dimensions are given by Griffiths [7]; I am indebted to S.-T. Yau for this reference.) The so-called elementary method using the Bloch constant [12, Kapitel 7] actually involves a kind of uniformization by iterated logarithms, and again this would be hard to carry out in higher dimensions. Of the truly elementary methods, one due to Nevanlinna [14] and developed by Valiron ([2, Ch. XII], [18]) is very close in spirit to Bloch's ideas. The work of Carlson and Griffiths on equidimensional maps [3] has little relation to the present paper.

There has not been much further work on the topics of Bloch's paper. Cartan [4] proved a result corresponding to Montel's theorem, an account of which may be found in a book by Lang [13, Ch. VIII]. Cartan's result is applied in papers by Dufresnoy [5], Fujimoto [6], and Kiernan and Kobayashi [10], but these authors say nothing about the proof.

Having obtained an inequality with an explicit constant, one may study its dependence on the data. As a function of the radius ρ, our Schottky theorem gives a bound for the logarithm of the maximum modulus that behaves like $(1-\rho)^{-(9+\epsilon)}$, where ϵ is any positive number, as ρ tends to 1. This compares with $(1-\rho)^{-1}$ in the classical theorem [9, Thm. 1]. The value $9+\epsilon$ could be improved without a major change in the argument by using a more precise version of Lemma 8. Here and elsewhere the author has aimed at simplicity rather than the sharpest possible results.

As for the dependence on the values at 0, unfortunately the bounds for all the ratios tend to infinity as (X_0, Y_0, Z_0) tends towards the omitted lines. This is a familiar difficulty with this type of estimate [2, p. 457]. The observation that it occurs in Bloch's paper was made by Kiernan and Kobayashi [10, p. 209].

As (X_0, Y_0, Z_0) approaches the diagonals, again the estimate tends to infinity. This may be viewed in terms of the intrinsic Kobayashi pseudometric ([10], [11], [13], [16]). Between points that lie on the diagonals, the Kobayashi pseudometric is zero, and it decays as the diagonals are approached. In Theorem 3 we use our explicit constant to obtain a bound on the rate of decay of Royden's infinitesimal form of the Kobayashi pseudometric. We assume that the diagonals are approached within a compact subset of the complement of the omitted lines, and prove that the rate of decay is polynomial, of degree at most 180,000.

We follow Bloch in writing $|f_1 f_2 \cdots f_n|$ for the Wronskian determinant

$$|f_1 f_2 \cdots f_n| = \begin{vmatrix} f_1 & f_2 & \cdots & f_n \\ f_1' & f_2' & \cdots & f_n' \\ \vdots & \vdots & \ddots & \vdots \\ f_1^{(n-1)} & f_2^{(n-1)} & \cdots & f_n^{(n-1)} \end{vmatrix}$$

of n meromorphic functions $f_1, \ldots, f_n$. The following identities involving Wronskians will be central to the argument.

LEMMA 1 ([1, p. 324], [4, p. 45], [13, Ch. VIII, Lemma 4.2]). *If f, g, h are meromorphic functions of z, then*

(1)
$$\frac{d}{dz} \log \frac{g}{f} = \frac{|fg|}{fg},$$

(2)
$$\frac{d}{dz} \log \frac{|gh|}{|fh|} = \frac{h|fgh|}{|fh||gh|}.$$

Proof. The first statement is obvious. The second may be verified directly; alternatively, express the logarithmic derivative in terms of minors of $|fgh|$ and apply the algebraic identity known as Sylvester's law of compound determinants [19, article 32]. □

Following Lang [13, p. 251], we call the right-hand member of (1) a *two-term derived fraction;* we call the right-hand member of (2) a *three-term derived fraction*. This is a translation of Cartan's terminology [4, p. 46]. For our purposes, only the modulus of a derived fraction is of interest, and we shall generally identify derived fractions that differ only in sign.

The estimates employed by Bloch are mainly those of Nevanlinna theory, in versions adapted to a different problem. For a meromorphic function f defined on $B(0,1)$ we introduce the *proximity function*

$$m(r, f) = \frac{1}{2\pi} \int_0^{2\pi} \overset{+}{\log} |f(re^{i\theta})| \, d\theta,$$

defined for r in $[0,1)$. When f is holomorphic, the proximity function is an increasing function equal to the Nevanlinna characteristic, which will not appear explicitly in our arguments. We state an elementary lemma.

LEMMA 2 ([8, p. 5], [13, p. 168], [15, p. 14]). *If $f_1, \ldots, f_n$ are meromorphic functions on $B(0,1)$, then*

$$m(r, f_1 \cdots f_n) \le m(r, f_1) + \cdots + m(r, f_n)$$

and

$$m(r, f_1 + \cdots + f_n) \le m(r, f_1) + \cdots + m(r, f_n) + \log n.$$

When f is holomorphic, we use $M(r,f)$ to denote the *maximum modulus* of f on $\bar{B}(0,r)$. The maximum modulus can be estimated by the proximity function.

LEMMA 3 ([8, Thm. 1.6], [13, p. 170], [15, p. 24]). *Let f be holomorphic on $B(0,1)$ and let $0 \le r < R < 1$. Then*

$$\log M(r,f) \le \left(\frac{R+r}{R-r}\right) m(R,f).$$

In the role of the First Main Theorem we shall use the following consequence of the Poisson–Jensen formula.

LEMMA 4 ([1, Lemme 4], [4, formula (5)]). *Let f be meromorphic in $B(0,1)$ and let $0 \le r < R < 1$. If $f(x) \ne 0, \infty$ for some $x = re^{i\theta}$, then*

$$\sum_{\lambda} \log\left|\frac{R^2 - \bar{a}_\lambda x}{R(x-a_\lambda)}\right| + \left(\frac{R-r}{R+r}\right) m\left(R, \frac{1}{f}\right) \le \log \frac{1}{|f(x)|} + \sum_{\nu} \log\left|\frac{R^2 - \bar{b}_\nu x}{R(x-b_\nu)}\right| + \left(\frac{R+r}{R-r}\right) m(R,f),$$

where the first sum is taken over all zeros a_λ of f in $B(0,R)$ and the second sum is taken over all poles b_ν of f in $B(0,R)$.

In the role of the Second Main Theorem, we shall use the following version of the lemma of the logarithmic derivative. The explicit constants in our version may be obtained from Nevanlina's original proof ([8, Lemma 2.3], [13, Ch. VI, Lemma 3.2], [15, p. 61]).

LEMMA 5. *Let f be a holomorphic function on $B(0,1)$ with no zeros. Then for $0 \le r < R < 1$ we have*

$$m\left(r, \frac{f'}{f}\right) < 3 + \overset{+}{\log}\,\overset{+}{\log} \frac{1}{|f(0)|} + 2\log\frac{1}{R-r} + \overset{+}{\log}\, m(R,f)$$

and

$$m\left(r, \frac{f''}{f}\right) < 5 + 2\,\overset{+}{\log}\,\overset{+}{\log} \frac{1}{|f(0)|} + 4\log\frac{1}{R-r} + 2\,\overset{+}{\log}\, m(R,f).$$

An elementary calculation from Lemma 5 yields the following estimates for homogenized Wronskians.

LEMMA 6. *Let f, g, and h be holomorphic functions on $B(0,1)$ with no zeros. Then for $0 \le r < R < 1$ we have*

$$m\left(r, \frac{|fg|}{fg}\right) < 7 + 2\,\overset{+}{\log}\,\overset{+}{\log}\max\left(\frac{1}{|f(0)|}, \frac{1}{|g(0)|}\right) + 4\log\frac{1}{R-r}$$
$$+2\,\overset{+}{\log}\max(m(R,f), m(R,g)),$$

$$m\left(r, \frac{|1fg|}{fg}\right) < 17 + 6\,\overset{+}{\log}\,\overset{+}{\log}\max\left(\frac{1}{|f(0)|}, \frac{1}{|g(0)|}\right) + 12\log\frac{1}{R-r}$$
$$+6\,\overset{+}{\log}\max(m(R,f), m(R,g)),$$

where 1 *denotes the constant function* 1, *and*

$$m\left(r, \frac{|fgh|}{fgh}\right) < 50 + 18\,\overset{+}{\log}\,\overset{+}{\log}\max\left(\frac{1}{|f(0)|}, \frac{1}{|g(0)|}, \frac{1}{|h(0)|}\right)$$
$$+36\log\frac{1}{R-r} + 18\,\overset{+}{\log}\max(m(R,f), m(R,g), m(R,h)).$$

The next lemma is of a different character. It was stated by Bloch [1, Lemme 6] but first proved by Cartan ([4, p. 32], [13, Ch. VIII, Thm. 3.3]).

LEMMA 7. *Let $0<r<R<1$ and let a_λ, where $\lambda = 1, \dots, \Lambda$, be numbers in $B(0,R)$, not necessarily distinct. Let $0<\gamma<1$. Then there exists a subset $\Gamma \subset B(0,r)$ which is a finite union of discs such that the sum of the radii is less than γ, and with the property that if $x, z \in B(0,r)$ and $x \notin \Gamma$ then*

$$\sum_\lambda \log\left|\frac{R^2-\bar{a}_\lambda x}{R(x-a_\lambda)}\right| < \frac{4R^2}{(R-r)^2}\log\frac{4e}{\gamma}\sum_\lambda \log\left|\frac{R^2-\bar{a}_\lambda z}{R(z-a_\lambda)}\right|.$$

The final lemma is based on a version of Borel's lemma on increasing functions given by Cartan ([4, p. 15], [13, Ch. VIII, Lemma 1.5]).

LEMMA 8. *Let $m(r)$ be positive and increasing for $0 \le a \le r < b \le 1$. Suppose that, for some positive constants A, B, C, we have*

$$m(r) \le A + B\log\frac{1}{R-r} + C\,\overset{+}{\log}\, m(R) \tag{3}$$

whenever $a \le r < R < b$ and r lies in a subset Σ of $[a,b)$ of measure $k(b-a)$, with $0<k\le 1$. Then

$$m(a) \le \max\left(2B\log\frac{2}{k(b-a)}, 16C^2, 4(A+2C+C\,\overset{+}{\log}\, B)\right). \tag{4}$$

Proof. If

$$m(a) \le 2B\log\frac{2}{k(b-a)}$$

then we have the first term in the right-hand member of (4). If

$$m(a) > 2B\log\frac{2}{k(b-a)} \tag{5}$$

then a standard argument ([8, Lemma 2.4], [13, Ch. VI, Lemma 3.7]) proves that

$$m\left(r+\exp\frac{2B}{m(r)}\right)\le m(r)+2B \tag{6}$$

for r in $[a,b)$, except on a set of measure at most $2\exp(2B/m(r))$. Because of (5), the set on which (3) holds must meet the set Σ mentioned in the statement of the lemma. For r in the intersection, we substitute

$$R=r+\exp\frac{2B}{m(r)}$$

in (3) to obtain

$$m(r)\le A+\frac{1}{2}m(r)+C\overset{+}{\log}\, m\left(r+\exp\frac{2B}{m(r)}\right)$$

so that

$$\begin{aligned} m(r) &\le 2A+2C\overset{+}{\log}\, m\left(r+\exp\frac{2B}{m(r)}\right)\\ &\le 2A+2C\overset{+}{\log}(m(r)+2B) \qquad \text{by (6)}\\ &\le 2A+4C+2C\overset{+}{\log}\, B+2C\overset{+}{\log}\, m(r), \end{aligned}$$

by Lemma 2. An elementary inequality gives

$$m(r)\le\max(16C^2, 4(A+2C+C\overset{+}{\log}\, B)),$$

and since m is increasing the lemma follows. □

THEOREM 1 (Bloch's version of Schottky's theorem [1, Thm. II]). *Let X, Y, Z, T be holomorphic functions without zeros on $B(0,1)$ satisfying the identity*

$$X+Y+Z+T=0. \tag{7}$$

Let $S=\{X,Y,Z,T\}$ and write $U_0=U(0)$ for $U\in S$.

I. *Suppose that $X_0+Y_0\ne 0$, $X_0+Z_0\ne 0$, and $X_0+T_0\ne 0$. Then, if $U,V\in S$ and $0<\rho<1$,*

$$\begin{aligned}\log M\left(\rho,\frac{U}{V}\right)<\max\Bigg\{&\frac{512}{27}\left(\frac{8}{1-\rho}\right)^9\left(8+\log\frac{1}{1-\rho}\right)^2+3\left(\frac{8}{1-\rho}\right)\delta,\\ &\frac{16}{3}\left(\frac{8}{1-\rho}\right)^4\left(8+\log\frac{1}{1-\rho}\right)\\ &\times\left(\log\frac{1}{\epsilon}+\left(\frac{8}{1-\rho}\right)\left(12+\delta+4\log\frac{1}{1-\rho}\right)\right)\Bigg\},\end{aligned} \tag{8}$$

where

$$\delta=\max_{U,V\in S}\log\left|\frac{U_0}{V_0}\right| \tag{9}$$

and

$$\epsilon = \min\left\{ \min_{\substack{U,V\in S\\ U\neq V}} \log\left(1+\frac{1}{3}\left|\frac{U_0}{V_0}\right|\right), \min_{\substack{U,V\in S\\ U\neq V}} \log\left(1+\frac{1}{2}\left|\frac{U_0+V_0}{V_0}\right|\right)\right\}. \tag{10}$$

II. *Suppose that* $X_0+Y_0=0$, *but* $X_0+Z_0\neq 0$ *and* $X_0+T_0\neq 0$. *Then the inequality* (8) *holds for the four ratios* X/Y, Y/X, Z/T, *and* T/Z, *except that four terms containing* X_0+Y_0 *or* T_0+Z_0 *must be removed from the last minimum in the definition* (10) *of* ϵ. *Similarly for* $X_0+Z_0=0$ *or* $X_0+T_0=0$.

III. *Suppose that* $X_0+Y_0=X_0+Z_0=0$, *so that* $X_0+T_0\neq 0$. *Then the inequality* (8) *holds for either the four ratios* X/Y, Y/X, Z/T, T/Z *or the four ratios* X/Z, Z/X, Y/T, T/Y, *except that eight terms containing* X_0+Y_0, X_0+Z_0, Y_0+T_0, *or* Z_0+T_0 *must be removed from the last minimum in* (10); *and similarly for* $X_0+Y_0=X_0+T_0=0$ *or* $X_0+Z_0=X_0+T_0=0$.

Proof. Suppose first that $X_0+Y_0\neq 0$, $X_0+Z_0\neq 0$, and $X_0+T_0\neq 0$. The argument will be divided into two cases, each of which will be divided into two subcases.

The two-term derived fractions $|XY|/XY$ and $|YX|/YX$ differ only by a sign, and we shall regard them as the same. With the four variables X, Y, Z, T we can thus form six different two-term derived fractions, which may be regarded as corresponding to the vertices of the omitted quadrilateral.

Case 1. At least two of the two-term derived fractions are smaller in modulus than ϵ at every point of $B(0,\rho)$. There are two subcases to consider, according to whether the corresponding vertices define a side or a diagonal. Without loss of generality, the small derived fractions are either

$$|XY|/XY, |XZ|/XZ \quad \text{or} \quad |XY|/XY, |ZT|/ZT.$$

Case 1.1. On $B(0,\rho)$ we have

$$\left|\frac{|XY|}{XY}\right|<\epsilon \quad \text{and} \quad \left|\frac{|XZ|}{XZ}\right|<\epsilon.$$

Note that the definition (10) of ϵ implies that $e^{\epsilon}<2$ and $e^{2\epsilon}<2$. According to Lemma 1, $|XY|/XY$ is, up to a sign, the logarithmic derivative of either Y/X or X/Y. By integrating along a radius we obtain

$$\frac{Y}{X}=e^{\zeta}\frac{Y_0}{X_0},$$

where $|\zeta|<\epsilon$. Consequently,

$$\left|\frac{Y}{X}\right|<e^{\epsilon}\left|\frac{Y_0}{X_0}\right|<2\left|\frac{Y_0}{X_0}\right|,$$

and the corresponding inequalities hold for X/Y, Z/X, and X/Z. Then

$$\left|\frac{Y}{Z}\right|=\left|\frac{Y}{X}\right|\left|\frac{X}{Z}\right|<e^{2\epsilon}\left|\frac{Y_0}{Z_0}\right|<2\left|\frac{Y_0}{Z_0}\right|$$

and likewise for Z/Y.

We now wish to estimate X/T. We have

$$\frac{Y}{X}=e^{\zeta}\frac{Y_0}{X_0} \quad\text{and}\quad \frac{Z}{X}=e^{\eta}\frac{Z_0}{X_0},$$

where $|\zeta|<\epsilon$ and $|\eta|<\epsilon$. With these substitutions, the identity

$$X+Y+Z+T=0 \tag{7}$$

becomes

$$\left(-\frac{T_0}{X_0}+(e^{\zeta}-1)\frac{Y_0}{X_0}+(e^{\eta}-1)\frac{Z_0}{X_0}\right)X+T=0.$$

The first term in the definition (10) of ϵ now ensures that

$$|e^{\zeta}-1|<e^{\epsilon}-1\leq\frac{1}{3}\left|\frac{T_0}{Y_0}\right| \quad\text{and}\quad |e^{\eta}-1|<\frac{1}{3}\left|\frac{T_0}{Z_0}\right|,$$

so that

$$\left|\frac{T}{X}\right|<\left|\frac{T_0}{X_0}\right|+\frac{1}{3}\left|\frac{T_0}{X_0}\right|+\frac{1}{3}\left|\frac{T_0}{X_0}\right|=\frac{5}{3}\left|\frac{T_0}{X_0}\right|$$

and

$$\left|\frac{T}{X}\right|>\left|\frac{T_0}{X_0}\right|-\frac{1}{3}\left|\frac{T_0}{X_0}\right|-\frac{1}{3}\left|\frac{T_0}{X_0}\right|=\frac{1}{3}\left|\frac{T_0}{X_0}\right|,$$

which yields

$$\left|\frac{X}{T}\right|<3\left|\frac{X_0}{T_0}\right|.$$

Then

$$\left|\frac{T}{Y}\right|=\left|\frac{T}{X}\right|\left|\frac{X}{Y}\right|<\frac{10}{3}\left|\frac{T_0}{Y_0}\right| \quad\text{and}\quad \left|\frac{Y}{T}\right|=\left|\frac{Y}{X}\right|\left|\frac{X}{T}\right|<6\left|\frac{Y_0}{T_0}\right|,$$

and likewise for T/Z and Z/T.

Case 1.2. On $B(0,\rho)$ we have

$$\left|\frac{|XY|}{XY}\right|<\epsilon \quad\text{and}\quad \left|\frac{|ZT|}{ZT}\right|<\epsilon.$$

As before, we have

$$\left|\frac{Y}{X}\right|<2\left|\frac{Y_0}{X_0}\right|$$

and the corresponding inequalities for X/Y, T/Z, and Z/T.

We now estimate X/Z. We have

$$\frac{Y}{X}=e^{\zeta}\frac{Y_0}{X_0} \quad\text{and}\quad \frac{T}{Z}=e^{\eta}\frac{T_0}{Z_0},$$

where $|\zeta|<\epsilon$ and $|\eta|<\epsilon$. The identity

$$X+Y+Z+T=0 \tag{7}$$

now becomes

$$\left(\frac{X_0+Y_0}{X_0}+(e^{\zeta}-1)\frac{Y_0}{X_0}\right)X+\left(\frac{Z_0+T_0}{Z_0}+(e^{\eta}-1)\frac{T_0}{Z_0}\right)Z=0.$$

The second term in the definition (10) of ϵ ensures that

$$\left|\frac{X_0+Y_0}{X_0}+(e^{\zeta}-1)\frac{Y_0}{X_0}\right|>\left|\frac{X_0+Y_0}{X_0}\right|-\frac{1}{2}\left|\frac{X_0+Y_0}{X_0}\right|=\frac{1}{2}\left|\frac{X_0+Y_0}{X_0}\right|$$

and that

$$\left|\frac{Z_0+T_0}{Z_0}+(e^{\eta}-1)\frac{T_0}{Z_0}\right|<\left|\frac{Z_0+T_0}{Z_0}\right|+\frac{1}{2}\left|\frac{Z_0+T_0}{Z_0}\right|=\frac{3}{2}\left|\frac{Z_0+T_0}{Z_0}\right|,$$

so that

$$\frac{X}{Z}<3\left|\frac{Z_0+T_0}{Z_0}\right|\left|\frac{X_0}{X_0+Y_0}\right|=3\left|\frac{X_0}{Z_0}\right|.$$

The remaining seven ratios can be estimated in the same way.

Recalling the definition (9) of δ, we have in Case 1 that

$$\log M\left(\rho,\frac{U}{V}\right)<\delta+2 \tag{11}$$

for all $U, V\in S$.

Case 2. If Case 1 does not hold, then, with one possible exception, each of the two-term derived fractions is at least equal to ϵ in modulus at some point in $B(0,\rho)$. This allows us to estimate the inverse of a two-term derived fraction using Lemma 4. Without loss of generality, we assume that the exceptional two-term derived fraction is $|ZT|/ZT$. We now choose three of the variables. To avoid using $|ZT|/ZT$, we must choose either X, Y, Z or X, Y, T; we choose X, Y, Z. With these variables, we may form three three-term derived fractions, not distinguishing expressions that differ only in sign, namely,

$$\frac{X|XYZ|}{|XY||XZ|},\quad \frac{Y|YZX|}{|YZ||YX|},\quad \frac{Z|ZXY|}{|ZX||ZY|}.$$

We let

$$r_1=\frac{3+\rho}{4},\quad r_2=\frac{1+\rho}{2},\quad r_3=\frac{1+3\rho}{4}, \tag{12}$$

so that $\rho<r_3<r_2<r_1<1$. (The radii r_1, r_2, r_3 will not play precisely the same roles as R_1, R_2, R_3 in Bloch's paper.)

Case 2.1. Let

$$\beta=\frac{1-\rho}{48}. \tag{13}$$

Suppose that, on $B(0,r_2)$, each of the three-term derived fractions formed with X, Y, Z is less than 1 in modulus, except possibly on a finite union of discs such that the sum of the radii is less than β. Then the set of points where at least one of the three-term derived fractions formed with X, Y, Z is greater than or equal to 1 is contained in a set E which is a finite union of discs such that the sum of the radii is less than 3β. The circles centred at 0 that meet E correspond to a set of radii in $[0, 1)$ which is a finite union of intervals of total length at most 6β. The circles that meet E thus form a union of annuli, possibly with an innermost disc, and the circles that do not meet E form a union of annuli, possibly with an innermost disc.

Let $A(0, a, b)$, where $0 < a < b \le r_2$, be an annulus that does not meet E. Any two points $x, y \in A(0, a, b)$ are connected in $A(0, a, b)$ by a path of length less than $\pi + 1$. We integrate along such a path to find that

$$\frac{1}{63}\left|\frac{|XY|}{|XZ|}(x)\right| < \left|\frac{|XY|}{|XZ|}(y)\right| < 63\left|\frac{|XY|}{|XZ|}(x)\right|,$$

using Lemma 1 and the inequality $\exp(\pi + 1) < 63$. It follows that one of the inequalities

$$\left|\frac{|XY|}{|XZ|}\right| < 63 \quad \text{and} \quad \left|\frac{|XZ|}{|XY|}\right| < 63$$

holds for all points in $A(0, a, b)$. We obtain similar statements from the derived fractions $Y|YZX|/|YZ||YX|$ and $Z|ZXY|/|ZX||ZY|$.

Consider the two "cyclic" sets of ratios

$$\text{(14)} \qquad \left\{\frac{|XY|}{|XZ|}, \frac{|XZ|}{|YZ|}, \frac{|YZ|}{|XY|}\right\} \quad \text{and} \quad \left\{\frac{|XZ|}{|XY|}, \frac{|YZ|}{|XZ|}, \frac{|XY|}{|YZ|}\right\}.$$

In one of these sets, at least two of the ratios are bounded above on $A(0, a, b)$ by 63. Without loss of generality, we may assume that the two ratios in question are $|XY|/|XZ|$ and $|XZ|/|YZ|$.

We define

$$m(r) = \max\left(m\left(r, \frac{X}{Y}\right), m\left(r, \frac{Y}{Z}\right), m\left(r, \frac{X}{Z}\right)\right)$$

and proceed to estimate $m(r)$. We have

$$\frac{Y}{Z} = \frac{|XY|}{|XZ|} \cdot \frac{XY}{|XY|} \cdot \frac{|XZ|}{XZ},$$

and so, by Lemma 2,

$$m\left(r, \frac{Y}{Z}\right) \le 63 + m\left(r, \frac{XY}{|XY|}\right) + m\left(r, \frac{|XZ|}{XZ}\right).$$

When $r_3 \le r$, we can apply Lemma 4 to obtain

$$m\left(r, \frac{XY}{|XY|}\right) \le \left(\frac{8}{1-\rho}\right)\log\frac{1}{\epsilon} + \left(\frac{8}{1-\rho}\right)^2 m\left(r, \frac{|XY|}{XY}\right).$$

Choose R to satisfy $r < R$ and apply Lemma 5 to obtain

$$\begin{aligned} m\left(r, \frac{Y}{Z}\right) < 63 + \left(\frac{8}{1-\rho}\right)\log\frac{1}{\epsilon} \\ + \left(1 + \left(\frac{8}{1-\rho}\right)^2\right)\left(3 + \overset{+}{\log}\,\delta + 2\log\frac{1}{R-r} + \overset{+}{\log}\, m(R)\right). \end{aligned}$$

We obtain the same bound for $m(r, X/Y)$.

By Lemma 2,

$$m\left(r, \frac{X}{Z}\right) \leq m\left(r, \frac{X}{Y}\right) + m\left(r, \frac{Y}{Z}\right),$$

and so, after making some approximations, we have

$$(15) \qquad m(r) < \left(\frac{8}{1-\rho}\right)^2 \left(9 + \log\frac{1}{\epsilon} + 3\delta + 5\log\frac{1}{R-r} + 3\overset{+}{\log}\, m(R)\right)$$

We have derived the inequality (15) under the assumption that a particular pair of ratios $|XY|/|XZ|$ and $|XZ|/|YZ|$ is bounded above by 63 on $A(0, a, b)$. On another annulus, it may be a different pair of ratios that is bounded.

We now consider the set of numbers r in $[r_3, r_2)$ such that the circle of radius r and centre 0 does not meet E. This is a finite union of intervals, each interval corresponding to an annulus that can play the role of $A(0, a, b)$ in the previous argument. On each of these annuli, some pair of ratios from (14) is bounded above by 63. Since there are six possible pairs of ratios, at least one pair is bounded above by 63 for 1/6 of the possible radii. The total measure of radii intersecting E is at most 6β, and so, by (12) and (13), we have a pair of ratios bounded above by 63 on a set of measure

$$\frac{1}{6}(r_2 - r_3 - 6\beta) = \frac{1-\rho}{48}.$$

Without loss of generality, we may assume that we have the inequality (15) on a set of measure $(1-\rho)/48$ in $[r_3, r_2)$. We apply Lemma 8, and after some approximation obtain the estimate

$$m(r_3) < \max\left(144\left(\frac{8}{1-\rho}\right)^4, 4\left(\frac{8}{1-\rho}\right)^2 \left(33 + \log\frac{1}{\epsilon} + 3\delta + 6\log\frac{1}{1-\rho}\right)\right).$$

This is an estimate for the three quantities $X/Y, Y/Z, X/Z$. We use Lemma 4 to obtain estimates for $Y/X, Z/Y, Z/X$. The identity

$$\frac{T}{X} = -1 - \frac{Y}{X} - \frac{Z}{X}$$

then yields an estimate for T/X, and there are corresponding estimates for T/Y and T/Z. Lemma 4 again gives estimates for $X/T, Y/T$, and Z/T.

For any $U, V \in S$ we have

$$m\left(r_3, \frac{U}{V}\right) < \max\left(289\left(\frac{8}{1-\rho}\right)^4 + 3\delta,\right.$$
$$\left. 8\left(\frac{8}{1-\rho}\right)^2 \left(34 + \log\frac{1}{\epsilon} + 4\delta + 6\log\frac{1}{1-\rho}\right)\right).$$

Lemma 3 gives

$$\log M\left(\rho, \frac{U}{V}\right) < \max\left(289\left(\frac{8}{1-\rho}\right)^5 + 3\left(\frac{8}{1-\rho}\right)\delta,\right. \tag{16}$$
$$\left. 8\left(\frac{8}{1-\rho}\right)^3\left(34+\log\frac{1}{\epsilon}+4\delta+6\log\frac{1}{1-\rho}\right)\right).$$

Case 2.2. If Case 2.1 does not hold, then at least one of the three-term derived fractions formed with X, Y, Z has the property that the set of points in $B(0, r_2)$ at which it is greater than or equal to 1 is not contained in any union of discs such that the sum of the radii is less than β. Without loss of generality, we may take $X|XYZ|/|XY||XZ|$ to be the derived fraction with this property.

We define

$$m(r) = \max\left(m\left(r, \frac{X}{T}\right), m\left(r, \frac{Y}{T}\right), m\left(r, \frac{Z}{T}\right)\right)$$

and proceed to estimate $m(r)$. From

$$X+Y+Z+T=0 \tag{7}$$

we have

$$|TYZ| = -|XYZ|,$$

and hence

$$\frac{X}{T} = -\frac{|TYZ|}{TYZ}\cdot\frac{|XY||XZ|}{X|XYZ|}\cdot\frac{XY}{|XY|}\cdot\frac{XZ}{|XZ|}. \tag{17}$$

The main problem in estimating this expression is to replace $|XY||XZ|/X|XYZ|$ by its inverse using Lemma 4, because it is necessary to account for the zeros. We proceed as follows.

Every zero of $|XY||XZ|/X|XYZ|$ is a zero of $(|XY|/XY)\cdot(|XZ|/XZ)$ with at least the same multiplicity. Because we are in Case 2, there is some $y \in B(0, \rho)$ such that

$$\left|\frac{|XY|}{XY}(y)\right| \geq \epsilon.$$

By Lemma 4 we obtain, for $r_1 < r < 1$,

$$\sum_{\lambda} \log\left|\frac{r^2-\bar{a}_\lambda y}{r(y-a_\lambda)}\right| \leq \log\frac{1}{\epsilon} + \frac{1}{3}\left(\frac{8}{1-\rho}\right)m\left(r, \frac{|XY|}{XY}\right) \tag{18}$$
$$-3\left(\frac{1-\rho}{8}\right)m\left(r, \frac{XY}{|XY|}\right),$$

where a_λ runs over the zeros of $|XY|/XY$. Similarly, there is some $z \in B(0, \rho)$ for which

$$\sum_{\mu} \log\left|\frac{r^2-\bar{a}_\mu z}{r(z-a_\mu)}\right| \leq \log\frac{1}{\epsilon} + \frac{1}{3}\left(\frac{8}{1-\rho}\right)m\left(r, \frac{|XZ|}{XZ}\right) \tag{19}$$
$$-3\left(\frac{1-\rho}{8}\right)m\left(r, \frac{XZ}{|XZ|}\right),$$

where a_μ runs over the zeros of $|XZ|/XZ$.

Neither y nor z need be a point at which

$$\left|\frac{X|XYZ|}{|XY||XZ|}\right| \geq 1.$$

This is where we apply Lemma 7, which (roughly speaking) allows us to move the point at which an estimate of the form (18) or (19) is made. We take $\gamma = \beta/2$ and obtain, instead of (18) and (19),

$$\begin{aligned}&\sum_{\lambda} \log\left|\frac{r^2 - \bar{a}_{\lambda} x}{r(x - a_{\lambda})}\right| \\ (20)\qquad &\leq \left(\frac{8}{1-\rho}\right)^2 \left(7 + \log\frac{1}{1-\rho}\right) \\ &\quad \times\left(\log\frac{1}{\epsilon} + \frac{1}{3}\left(\frac{8}{1-\rho}\right) m\left(r, \frac{|XY|}{XY}\right) - 3\left(\frac{1-\rho}{8}\right) m\left(r, \frac{XY}{|XY|}\right)\right),\end{aligned}$$

and the corresponding modification of (19), both of which hold for x outside a certain subset E of $B(0, r_2)$ which is a finite union of discs such that the sum of the radii is less than β. By assumption, there is some point $x \in B(0, r_2) - E$ such that

$$\left|\frac{X|XYZ|}{|XY||XZ|}(x)\right| \geq 1.$$

We take this value of x in Lemma 4 and use (20) and the corresponding expression for $|XZ|/XZ$ to obtain the estimate

$$\begin{aligned}m\left(r, \frac{|XY||XZ|}{X|XYZ|}\right) &\leq \left(\frac{8}{1-\rho}\right)^3 \left(7 + \log\frac{1}{1-\rho}\right) \\ &\quad \times\left(2\log\frac{1}{\epsilon} + \frac{1}{3}\left(\frac{8}{1-\rho}\right)\left(m\left(r, \frac{|XY|}{XY}\right) + m\left(r, \frac{|XZ|}{XZ}\right)\right)\right. \\ (21)\qquad &\qquad \left. - 3\left(\frac{1-\rho}{8}\right)\left(m\left(r, \frac{XY}{|XY|}\right) + m\left(r, \frac{XZ}{|XZ|}\right)\right)\right) \\ &\quad + \left(\frac{8}{1-\rho}\right)^2 m\left(r, \frac{X|XYZ|}{|XY||XZ|}\right).\end{aligned}$$

We have

$$\frac{X|XYZ|}{|XY||XZ|} = \frac{|XYZ|}{XYZ} \cdot \frac{XY}{|XY|} \cdot \frac{XZ}{|XZ|},$$

and so, by Lemma 2,

$$(22)\quad m\left(r, \frac{X|XYZ|}{|XY||XZ|}\right) \leq m\left(r, \frac{|XYZ|}{XYZ}\right) + m\left(r, \frac{XY}{|XY|}\right) + m\left(r, \frac{XZ}{|XZ|}\right).$$

From (17) we have

$$m\left(r, \frac{X}{T}\right) \leq m\left(r, \frac{|TYZ|}{TYZ}\right) + m\left(r, \frac{|XY||XZ|}{X|XYZ|}\right) + m\left(r, \frac{XY}{|XY|}\right) + m\left(r, \frac{XZ}{|XZ|}\right).$$

We combine this with (21) and (22) to obtain

$$m\left(r,\frac{X}{T}\right)\le m\left(r,\frac{|TYZ|}{TYZ}\right)+\left(\frac{8}{1-\rho}\right)^2 m\left(r,\frac{|XYZ|}{XYZ}\right)$$
$$+\left(\frac{8}{1-\rho}\right)^3\left(7+\log\frac{1}{1-\rho}\right)\left(2\log\frac{1}{\epsilon}+\frac{1}{3}\left(\frac{8}{1-\rho}\right)\right.$$
$$\left.\times\left(m\left(r,\frac{|XY|}{XY}\right)+m\left(r,\frac{|XZ|}{XZ}\right)\right)\right).$$

The corresponding estimates for Y/T and Z/T are obtained in the same way. Now

$$|XY|=T^2\left|\frac{X}{T}\,\frac{Y}{T}\right|,\qquad |XYZ|=T^3\left|\frac{X}{T}\,\frac{Y}{T}\,\frac{Z}{T}\right|,$$

and so on, so that Lemmas 5 and 6 give

$$m(r)<2\left(\frac{8}{1-\rho}\right)^3\left(7+\log\frac{1}{1-\rho}\right)$$
$$\times\left(\log\frac{1}{\epsilon}+\frac{1}{3}\left(\frac{8}{1-\rho}\right)\left(7+2\delta+4\log\frac{1}{R-r}+2\overset{+}{\log}\, m(R)\right)\right)$$

whenever $r_1\le r<R<1$. We apply Lemma 8 to obtain

$$m(r_1)\le\max\left\{\frac{256}{9}\left(\frac{8}{1-\rho}\right)^8\left(8+\log\frac{1}{1-\rho}\right)^2,\right.$$
$$8\left(\frac{8}{1-\rho}\right)^3\left(8+\log\frac{1}{1-\rho}\right)\left(\log\frac{1}{\epsilon}+\frac{1}{3}\left(\frac{8}{1-\rho}\right)\right. \tag{23}$$
$$\left.\left.\times\left(36+2\delta+10\log\frac{1}{1-\rho}\right)\right)\right\}.$$

To estimate T/X, T/Y, T/Z we use Lemma 4, which gives δ plus the previous estimate (23). The other six ratios may now be estimated by Lemma 2.

This estimates $m(r, U/V)$ for any ratio U/V with $U, V\in S$, and we now apply Lemma 3 to estimate $\log M(\rho, U/V)$. We obtain

$$\log M\left(\rho,\frac{U}{V}\right)<\max\left\{\frac{512}{27}\left(\frac{8}{1-\rho}\right)^9\left(8+\log\frac{1}{1-\rho}\right)^2+\frac{1}{3}\left(\frac{8}{1-\rho}\right)\delta,\right.$$
$$\frac{16}{3}\left(\frac{8}{1-\rho}\right)^4\left(8+\log\frac{1}{1-\rho}\right) \tag{24}$$
$$\left.\times\left(\log\frac{1}{\epsilon}+\left(\frac{8}{1-\rho}\right)\left(12+\delta+4\log\frac{1}{1-\rho}\right)\right)\right\}.$$

This expression is larger in almost every way than the bounds (11) and (16) obtained in the other cases. The only exception is that the multiple of δ in the first term of the maximum of (24) is smaller than the multiple of δ in the first term of the maximum of (16). Therefore we have the expression (8) of the theorem.

It remains to discuss what happens when $X_0+Y_0=0$. Suppose first that $X_0+Y_0=0$, $X_0+Z_0\neq 0$, and $X_0+T_0\neq 0$. Under this assumption, four of the terms in the last minimum in the definition (10) of ϵ are zero, and they must be omitted from the definition of ϵ in order to make ϵ a positive quantity. The only part of the argument that this affects is Case 1.2. If the two-term derived fractions that are bounded by ϵ happen to be $|XY|/XY$ and $|ZT|/ZT$, then, after obtaining bounds for X/Y, Y/X, Z/T, and T/Z, we cannot proceed to bound the other eight ratios.

Now suppose that $X_0+Y_0=X_0+Z_0=0$. Then $X_0+T_0\neq 0$, since X_0, Y_0, Z_0, T_0 are nonzero numbers. We must now omit eight terms from the definition (10) of ϵ, with a corresponding effect on the argument in Case 1.2. □

THEOREM 2 (Bloch's version of Landau's theorem [1, Thm. III]). *Assume the hypotheses of Theorem 1, and adopt the same notation.*

I. *Suppose that* $X_0+Y_0\neq 0, X_0+Z_0\neq 0,$ *and* $X_0+T_0\neq 0$. *Then, if* $U, V\in S$,

$$\left(\frac{U}{V}\right)'(0)<\max(\exp(1.7\times 10^{11}+25\delta),\ \epsilon^{-180000}\exp(1.7\times 10^{7}+1.5\times 10^{6}\times\delta)). \tag{25}$$

II. *If* $X_0+Y_0=0$, $X_0+Z_0\neq 0$, *and* $X_0+T_0\neq 0$, *then the inequality* (25) *holds for the ratios* X/Y, Y/X, Z/T, *and* T/Z.

III. *If* $X_0+Y_0=X_0+Z_0=0$ *and* $X_0+T_0\neq 0$, *then either the inequality* (25) *holds for* $X/Y, Y/X, Z/T, T/Z$ *or the inequality* (25) *holds for* X/Z, $Z/X, Y/T, T/Y$.

Proof ([12, p. 103], [17, p. 354]). Estimate $M(0.001, U/V)$ by inequality (8) of Theorem 1 and apply Cauchy's integral formula for the derivative. □

We now define Royden's infinitesimal form of the Kobayashi pseudometric ([11, formula (2.16)], [13, p. 88], [16, p. 127]). Let M be a complex manifold, let $p\in M$ be a point, and let $v\in T_pM$ be a tangent vector. We define $F_M(v)$ to be the infimum of $1/R$ over all holomorphic maps $\varphi: B(0,1)\to M$ such that $\varphi(0)=p$ and $\varphi'(0)=Rv$. The function F_M is a kind of infinitesimal metric called the *Royden function* of M; it is known to be upper semicontinuous.

THEOREM 3. *Let Q be the union of four lines in general position in* $\mathbf{CP}^2$ *and let Δ be the union of the diagonals of Q. Let* $M=\mathbf{CP}^2-Q$ *and let K be a compact subset of M. Let* $c:[0,1]\to K$ *be a curve such that* $c(0)\in\Delta$ *and* $c((0,1])$ *is disjoint from Δ. Let v be a unit vector field along c, measuring lengths in some Riemannian metric on* $\mathbf{CP}^2$. *Then, for* $0<t\leq 1$,

$$F_M(v(t))>Ar^{180000}(t),$$

where A is a constant depending only on K and $r(t)$ is the Riemannian distance from $c(t)$ to Δ.

Proof. Starting from Theorem 2, we wish to prove that ϵ decays at the same rate as the Riemannian distance from Δ. In the notation of Theorem 2, if (X_0, Y_0, Z_0) are the coordinates of a point of K then, when (X_0, Y_0, Z_0) is close enough to Δ, the smallest term in the minimum (10) defining ϵ is of the form

$$\log\left(1+\frac{1}{2}\left|\frac{U_0+V_0}{V_0}\right|\right)$$

for some $U, V \in S$. This expression is approximately $\frac{1}{2}|(U_0+V_0)/V_0|$, which is a bounded multiple of the Riemannian distance from (X_0, Y_0, Z_0) to the closest of the lines in Δ. □

Added in proof: These problems are treated by a different method in M. J. Cowen, *The method of negative curvature: the Kobayashi metric on* $\mathbf{P}_2$ *minus 4 lines,* Trans. Amer. Math. Soc. 319 (1990), 729–745.

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