Remarks Concerning Cyclic Vectors in Hardy and Bergman Spaces

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To the memory of Allen L. Shields

1. Introduction

Let G be a bounded, simply connected region in the complex plane $\mathbf{C}$. For $0<p<\infty$, let $L_a^p(G)=\{f: f$ is analytic in G and $\int|f|^p dA<\infty$ (dA is area measure)$\}$ and $H^p(G)=\{f: f$ is analytic in G and $|f|^p$ has a harmonic majorant in $G\}$ denote (respectively) the Bergman and Hardy spaces in G. Note that for $1\le p<\infty$, $L_a^p(G)$ is a Banach space with respect to the norm $\|f\|_{L_a^p(G)}:=\{\int|f|^p dA\}^{1/p}$ and, when $p=2$, is a Hilbert space with respect to the inner product $\langle f,g\rangle_{L_a^2(G)}:=\int_G f\bar{g}\,dA$. As is known, any function h in $H^p(G)$ has well-defined boundary values $\tilde{h}$ belonging to $L^p(\omega_\alpha)$, where $\omega_\alpha=\omega(\cdot,G,\alpha)$ denotes harmonic measure on ∂G (for G) evaluated at a fixed point α in G. For $1\le p<\infty$, this allows us to define a norm on $H^p(G)$ by

$$\|f\|_{H^p(G)}:=\left\{\int_{\partial G}|\tilde{f}|^p\,d\omega_\alpha\right\}^{1/p}$$

and an inner product on $H^2(G)$ by

$$\langle f,g\rangle_{H^2(G)}:=\int_{\partial G}\tilde{f}(\bar{\tilde{g}})\,d\omega_\alpha.$$

$H^p(G)$ is a Banach space with respect to $\|\cdot\|_{H^p(G)}$ and $H^2(G)$ is a Hilbert space with respect to $\langle\cdot,\cdot\rangle_{H^2(G)}$. By Harnack's inequality, replacing α with another point β in G leads to an equivalent norm on $H^p(G)$. An alternate perspective on $H^p(G)$ is as follows. Let φ be a conformal map from the unit disk $\mathbf{D}:=\{z:|z|<1\}$ onto G such that $\varphi(0)=\alpha$. For $0<\rho<1$, define $G_\rho=\varphi(\{z:|z|<\rho\})$ and let $\omega_\rho:=\omega(\cdot,G_\rho,\alpha)$ be harmonic measure on ∂G_ρ evaluated at α. Then an analytic function f defined in G belongs to $H^p(G)$ if and only if $\lim_{\rho\to 1}\int_{\partial G_\rho}|f|^p\,d\omega_\rho<\infty$, that is, if and only if $f\circ\varphi$ belongs to the Hardy class $H^p(\mathbf{D})$. For more detail on Hardy spaces in general domains, we refer the reader to [Du], [Fi], and [Go]; see also [Kh] and references cited there.

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Define the shift operator M_z^G on each of the spaces $L_a^2(G)$ (the Bergman space) and $H^2(G)$ (the Hardy space) by $M_z^G(f) = zf$. In this paper we study the following question: For which G is the Bergman (resp., Hardy) shift M_z^G cyclic?

Recall that an operator T on a (complex) Hilbert space $\mathcal{H}$ is said to be *cyclic* if there exists h in $\mathcal{H}$ such that $\{p(T)h : p \text{ is a polynomial}\}$ is dense in $\mathcal{H}$. In that case h is called a *cyclic vector* for T in $\mathcal{H}$. A basic account on cyclic vectors for shifts on Hardy and Bergman spaces can be found in [Shi]. A good general reference for this material is [Co]. We remark that the question of whether 1 is a cyclic vector for M_z^G on $L_a^2(G)$, that is, whether the polynomials are dense in $L_a^2(G)$, has been thoroughly studied by a number of authors (see, e.g., [Br], [CM], [Ke1], [Ke2], [Me], [Si]; see also [MS], [He], and [Shi] for further references). The counterpart of this problem for $H^2(G)$ was treated in [Ak] and [Ro].

The rest of this paper is organized as follows. Section 2 contains some basic facts about Hardy and Bergman spaces and some auxiliary estimates which will prove useful later in the paper. In Section 3 we discuss various examples of simply connected non-Carathéodory domains G (e.g., crescents), some of which have the property that the shift M_z^G on $H^2(G)$ is cyclic and yet is without 1 as a cyclic vector, and others such that M_z^G on $H^2(G)$ is not cyclic. In Section 4 we treat the Bergman shift. We note that, surprisingly, for crescents G there is little difference between when the shift M_z^G on $L_a^2(G)$ and the shift M_z^G on $H^2(G)$ are cylic (cf. Theorems 4.1 and 4.7). Section 5 contains some further remarks on the problems discussed here and some open questions.

ACKNOWLEDGMENT. At the early stages of working on these problems the authors communicated with the late Professor Allen Shields, whose death in September of 1989 tragically interrupted our collaboration. We want to dedicate this article in remembrance of him as a small token of our love and respect for a close friend, an excellent mathematician, and a kind man.

2. Preliminaries

Some of the results in this section are well known and so we shall state them without proofs.

2.1. PROPOSITION. *Let G be a bounded, simply connected region with rectifiable boundary. Choose α in G and let $\omega_\alpha := \omega(\cdot, G, \alpha)$ be harmonic measure on ∂G evaluated at α. Let ψ be a conformal map from G onto $\mathbf{D} := \{z : |z| < 1\}$ such that $\psi(\alpha) = 0$. Then $d\omega = |\psi'|\,ds$, where ds is arclength measure on ∂G and ψ' is understood in the sense of nontangential boundary values. In particular, if ∂G is smooth then ω_α is boundedly equivalent to arclength measure.*

For a simple proof of Proposition 2.1, see for example [Go].

2.2. EXAMPLE. Consider the slit disk $\mathbf{D}\setminus[0,1]$. In this case, $d\omega|_{[0,1]} = |\psi'_+|ds + |\psi'_-|ds$, where ψ'_+ and ψ'_- are the boundary values of ψ' on $[0,1]$ from above and below, respectively.

The next result is due to L. I. Hedberg (unpublished). For an indication of the proof see [Shi].

2.3. THEOREM. *Let G be a bounded region such that ∂G is a finite union of continua. Then the bounded analytic functions in G are dense in $L_a^2(G)$.*

2.4. REMARK. The corresponding statement for $H^2(G)$, if G is simply connected, is obvious in view of the conformal invariance of $H^2(G)$ (e.g., cf. [Fi]).

The following lemma is from [Sha]. For the sake of completeness we shall give a proof.

2.5. LEMMA. *If G is a bounded, simply connected region and p is a polynomial such that $p(z) \neq 0$ for all z in G, then $pL_a^2(G) := \{pg : g \in L_a^2(G)\}$ is dense in $L_a^2(G)$.*

Proof. It will suffice to show that $(z-\beta)L_a^2(G)$ is dense in $L_a^2(G)$ whenever $\beta \notin G$. Fix β in $\mathbf{C}\setminus G$. Since G is simply connected, there is a single-valued analytic branch of $\log(z-\beta)$ in G and hence $(z-\beta)^{1/2}, (z-\beta)^{-1/2} \in L_a^2(G)$. So, if g is a bounded, analytic function in G, then $(z-\beta)^{-1/2}g \in L_a^2(G)$. Consequently, $g \in (z-\beta)^{1/2}L_a^2(G)$. By Theorem 2.3, $(z-\beta)^{1/2}L_a^2(G)$ is dense in $L_a^2(G)$. Therefore, $(z-\beta)L_a^2(G)$ is dense in $L_a^2(G)$. □

The same holds for $H^2(G)$.

2.6. LEMMA. *If G is a bounded, simply connected region and p is a polynomial such that $p(z) \neq 0$ for all z in G, then $pH^2(G)$ is dense in $H^2(G)$.*

Proof. Fix β in $\mathbf{C}\setminus G$. Now $(z-\beta)$ and $(z-\beta)^{-1}$ are both univalent in G and hence (see [Du, p. 50]) are both in $H^p(G)$ whenever $0 < p < \frac{1}{2}$. Therefore, $(z-\beta)^{1/5}, (z-\beta)^{-1/5} \in H^2(G)$. Arguing as in Lemma 2.5, we complete the proof. □

The following lemma relates the Bergman space of a given region G to the Hardy space of certain subregions in G.

2.7. LEMMA. *Let G be a bounded, simply connected region and, for any z in G, define $r_z := \operatorname{dist}(z, \partial G)$. Let D be a simply connected region contained in G and $\omega_\alpha := \omega(\cdot, D, \alpha)$ be harmonic measure on ∂D evaluated at some fixed point α in D. If $\int_{\partial D}(1/r_z^2)\,d\omega_\alpha(z) < \infty$ then, given any f in $L_a^2(G)$, $f \in H^2(D)$ and*

$$\int_{\partial D}|f|^2\,d\omega_\alpha := \|f\|^2_{H^2(D)} \le c\|f\|^2_{L^2_a(G)}, \quad \textit{where } c=\frac{1}{\pi}\int_{\partial D}\frac{1}{r_z^2}\,d\omega_\alpha(z).$$

Proof. Let φ be a conformal map from $\mathbf{D}=\{z:|z|<1\}$ onto D such that $\varphi(0)=\alpha$. For $0<\rho<1$, let $D_\rho=\varphi(\{z:|z|<\rho\})$ and let $\omega_\rho:=\omega_\rho(\cdot,D_\rho,\alpha)$ be harmonic measure on ∂D_ρ evaluated at α. For z in G, let $\Delta_z=\{\zeta:|\zeta-z|<r_z\}$. If $f\in L^2_a(G)$, then

$$\begin{aligned}\int_{D_\rho}|f|^2\,d\omega_\rho &= \int_{\partial D_\rho}\left|\frac{1}{\pi r_z^2}\iint_{\Delta_z} f\,dA\right|^2 d\omega_\rho(z)\\ &\le \frac{1}{\pi}\int_{\partial D_\rho}\frac{1}{r_z^2}\left(\iint_{\Delta_z}|f|^2\,dA\right)d\omega_\rho(z)\\ &\le \left(\frac{1}{\pi}\int_{\partial D}\frac{1}{r_z^2}\,d\omega_\rho\right)\|f\|^2_{L^2_a(G)}.\end{aligned}$$

Now it is well known that $z\mapsto 1/r_z^2$ is subharmonic in G. Hence,

$$\begin{aligned}\int_{D_\rho}|f|^2\,d\omega_\rho &\le \left(\frac{1}{\pi}\int_{\partial D_\rho}\frac{1}{r_z^2}\,d\omega_\rho\right)\|f\|^2_{L^2_a(G)}\\ &\le \left(\frac{1}{\pi}\int_{\partial D}\frac{1}{r_z^2}\,d\omega_\alpha\right)\|f\|_{L^2_a(G)}.\end{aligned}$$

If $\int_{\partial D}(1/r_z^2)\,d\omega_\alpha(z)<\infty$ then, by letting $\rho\to 1$, we see that $f\in H^2(D)$ and furthermore that

$$\int_{\partial D}|f|^2\,d\omega_\alpha := \|f\|^2_{H^2(G)} \le c\|f\|^2_{L^2_a(G)}, \quad \text{where } c=\frac{1}{\pi}\int\frac{1}{r_z^2}\,d\omega_\alpha. \qquad \square$$

The next lemma could be called the "edge-of-the-wedge estimate" for harmonic measure.

2.8. LEMMA. *Let G be a Jordan domain with rectifiable boundary such that ∂G is smooth, except at one point z_0 where ∂G forms an angle θ, $0<\theta<\pi$. Let $m=\pi/\theta$ and fix α in G. Then, near z_0, $d\omega_\alpha:=d\omega(\cdot,G,\alpha)$ is boundedly equivalent to $|z-z_0|^{m-1}\,ds$, where ds is arclength measure on ∂G.*

Proof. Let $\varphi(z)=(z-z_0)^m$. Now φ maps a portion of G near z_0 onto a smoothly bounded domain G'. By Proposition 2.1 and conformal invariance of harmonic measure,

$$\begin{aligned}d\omega_\alpha(z) &:= d\omega(z,G,\alpha)=d\omega(\varphi(z),G',\varphi(\alpha))\\ &\sim d|\varphi(z)|=|z-z_0|^{m-1}\,d|z|\end{aligned}$$

near z_0. $\square$

3. The Shift Operator on the Hardy Space

3.1. DEFINITION. A region G in $\mathbf{C}$ is called a *crescent* if $G=D\backslash\bar{\Omega}$, where D and Ω are Jordan regions such that $\Omega\subseteq D$ and $\bar{\Omega}\cap\partial D$ is a single point (the multiple boundary point of G).

If G is a crescent then $\mathbf{C}\setminus\bar{G}$ has two components, one of which is bounded. Denote the bounded component of $\mathbf{C}\setminus\bar{G}$ by Ω_G.

3.2. THEOREM. *Let G be a simply connected region and suppose there is a crescent G' such that $G'\subseteq G$ and $(\mathbf{C}\setminus G)\cap\Omega_{G'}\neq\varnothing$. Choose α in G' and β in $\Omega_{G'}$ and let $\omega_\alpha=\omega(\cdot,G',\alpha)$ and $\nu_\beta=\omega(\cdot,\Omega_{G'},\beta)$ be harmonic measures for G' and $\Omega_{G'}$, respectively. If there exists a positive constant c such that $\nu_\beta\le c\omega_\alpha$ on $\partial\Omega_{G'}$, then the shift M_z^G on $H^2(G)$ is not cyclic.*

Proof. Suppose to the contrary that M_z^G on $H^2(G)$ is cyclic and that f in $H^2(G)$ is a cyclic vector for M_z^G. Clearly $f\not\equiv 0$, and since $f\in H^2(G)$, $\log|\tilde{f}|\in L^1(\omega_\alpha)$, where $\tilde{f}$ denotes the boundary values of f. Since $\nu_\beta\le c\omega_\alpha$ on $\partial\Omega_{G'}$, $\tilde{f}$ is defined ν_β-a.e. on $\partial\Omega_{G'}$ and $\log|\tilde{f}|\in L^1(\nu_\beta)$.

By our assumption that f is a cyclic vector for M_z^G on $H^2(G)$, $\{pf\colon p$ is a polynomial$\}$ is dense in $H^2(G)$. Now we may assume that in fact $\beta\in(\mathbf{C}\setminus G)\cap\Omega_{G'}$. So, by Lemma 2.6, $\{pf\colon p$ is a polynomial and $p(\beta)=0\}$ is dense in $H^2(G)$. In particular, there exists a sequence of polynomials $\{p_n\}$ such that $p_n(\beta)=0$ for all n and $\|p_nf-f\|_{H^2(G)}\to 0$ as $n\to\infty$. Since $G'\subseteq G$, it follows that $f\in H^2(G')$, and also that $\|p_nf-f\|_{H^2(G')}\le\|p_nf-f\|_{H^2(G)}$, where both norms are taken with respect to harmonic measure evaluated at α. Therefore,

$$\begin{aligned}\int_{\partial\Omega_{G'}}|p_n-1|^2|\tilde{f}|^2\,d\nu_\beta&=\int_{\partial\Omega_{G'}}|p_n\tilde{f}-\tilde{f}|^2\,d\nu_\beta\\&\le c\int_{\partial\Omega_{G'}}|p_n\tilde{f}-\tilde{f}|^2\,d\omega_\alpha\le c\int_{\partial G'}|p_n\tilde{f}-\tilde{f}|^2\,d\omega_\alpha\\&=c\|p_nf-f\|^2_{H^2(G')}\to 0\quad\text{as } n\to\infty.\end{aligned}$$

Applying Szegö's theorem [Ga, p. 136], we conclude that $2\log|\tilde{f}|\notin L^1(\nu_\beta)$, which is a contradiction. So, M_z^G on $H^2(G)$ is not cyclic. □

Let us now introduce some more notation. For $0\le\theta<\pi$, let $V(\theta)=\mathrm{co}(\{z\colon|z|\le\sin(\theta/2)\}\cup\{1\})$, that is, the closed convex hull of $\{z\colon|z|\le\sin(\theta/2)\}\cup\{1\}$; see Figure 1. Notice that $\partial V(\theta)$ is smooth except at $z=1$, and there, $\partial V(\theta)$ forms an angle θ. Let $G(\theta)=\{z\colon|z|<1\}\setminus V(\theta)$; see Figure 2.

3.3. COROLLARY. *If $0\le\theta\le\pi/3$, then the shift $M_z^{G(\theta)}$ on $H^2(G(\theta))$ is not cyclic.*

Proof. If $0\le\theta\le\pi/3$, then $G(\pi/3)\subseteq G(\theta)$ and $(\mathbf{C}\setminus G(\theta))\cap\Omega_{G(\pi/3)}\neq\emptyset$. Choose α in $G(\pi/3)$ and β in $\mathrm{int}(V(\pi/3))=\Omega_{G(\pi/3)}$, and let

$$\omega_\alpha=\omega(\cdot,G(\pi/3),\alpha)\quad\text{and}\quad\nu_\beta=\omega(\cdot,\mathrm{int}(V(\pi/3)),\beta).$$

According to Lemma 2.8, $d\omega_\alpha$ restricted to $\partial V(\pi/3)$ and $d\nu_\beta$ are both boundedly equivalent to $|z-1|^2\,ds$, where ds is arclength measure on $\partial V(\pi/3)$. Applying Theorem 3.2, we complete the proof. □

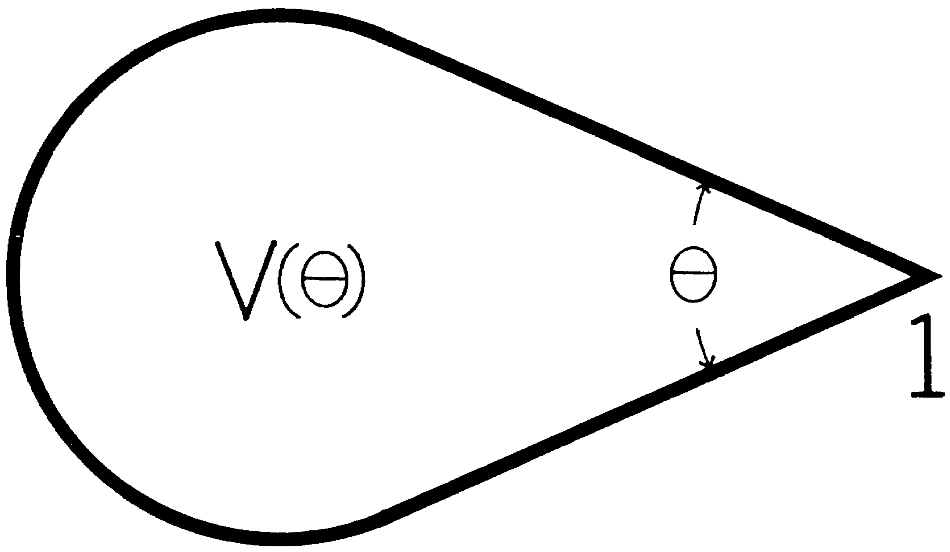

Figure 1

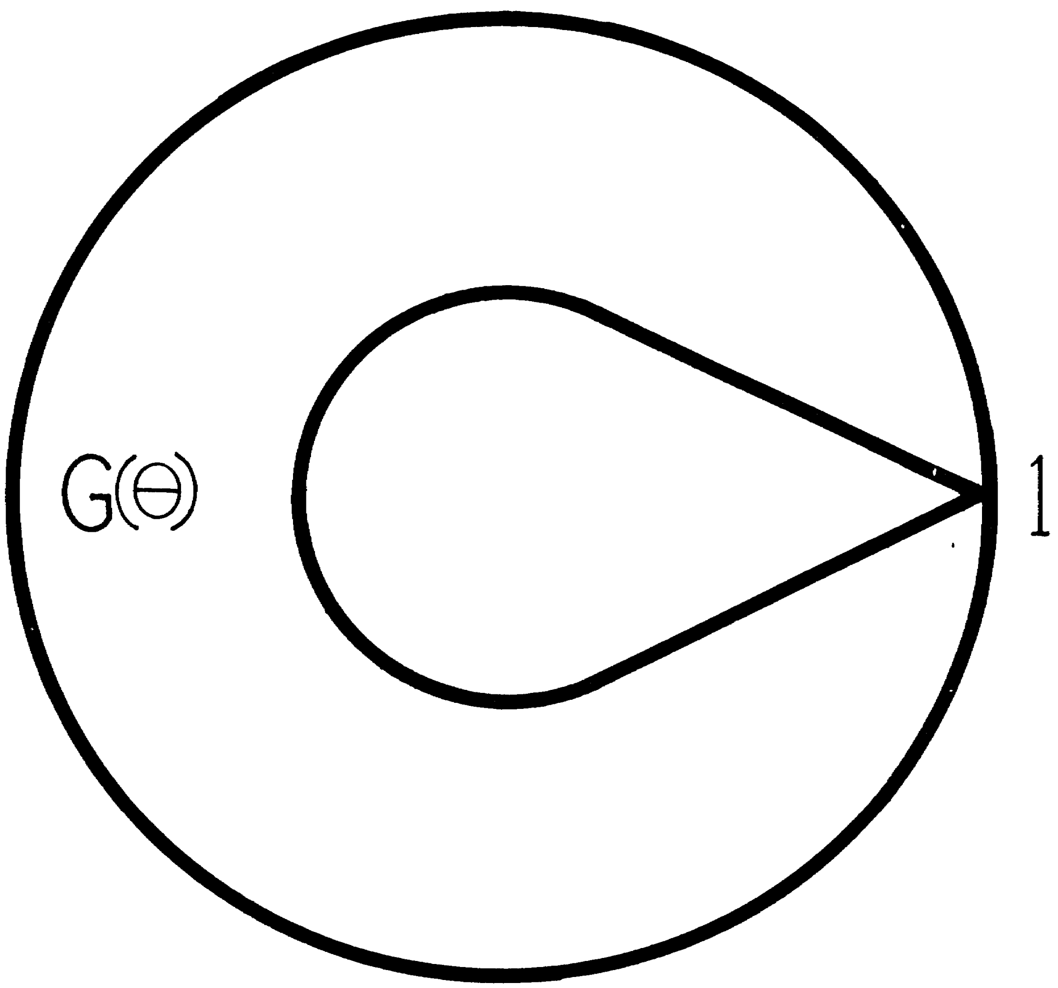

Figure 2

3.4. REMARK. Notice that $G(0)$ is the slit disk $\mathbf{D}\setminus[0,1]$. So, in particular, Corollary 3.3 implies that the shift on the Hardy space of the slit disk is not cyclic.

3.5. LEMMA. *Let G and G' be bounded, simply connected regions and let φ be a conformal map from G onto G'. If φ is uniformly approximable by polynomials on $\bar{G}$ and φ^{-1} is uniformly approximable by polynomials on $\overline{G'}$, then M_z^G on $H^2(G)$ is cyclic if and only if $M_z^{G'}$ on $H^2(G')$ is cyclic.*

Proof. Suppose M_z^G on $H^2(G)$ is cyclic with cyclic vector f; we may assume that $\|f\|_{H^2(G)}=1$. It suffices to show that $f\circ\varphi^{-1}$ is cyclic for $M_z^{G'}$ on $H^2(G')$. Now choose g in $H^2(G')$. Then $g\circ\varphi\in H^2(G)$. Fix $\epsilon>0$. Since f is a cyclic vector for M_z^G on $H^2(G)$, there exists a polynomial p such that $\|pf-g\circ\varphi\|_{H^2(G)}<\epsilon$. By the uniform continuity of p on $\bar{G}$ and our hypothesis, we can find a polynomial q such that

$$\|p\circ\varphi^{-1}-p\circ q\|_\infty:=\max_{w\in G'}|p(\varphi^{-1}(w))-p(q(w))|<\epsilon,$$

where $Q:=p\circ q$ is a polynomial. A simple application of the triangle inequality then yields ($\|f\|_{H^2(G)}=\|f\circ\varphi^{-1}\|_{H^2(G')}=1$):

$$\begin{aligned}\|Q\cdot(f\circ\varphi^{-1})-g\|_{H^2(G')}&\le\|(p\circ\varphi^{-1})\cdot(f\circ\varphi^{-1})-g\|_{H^2(G')}+\epsilon\|f\circ\varphi^{-1}\|_{H^2(G')}\\&=\|pf-g\circ\varphi\|_{H^2(G)}+\epsilon<2\epsilon,\end{aligned}$$

and the result follows. □

3.6. THEOREM. *If $\pi/2\le\theta<\pi$, then $M_z^{G(\theta)}$ on $H^2(G(\theta))$ is cyclic and yet 1 is not a cyclic vector for $M_z^{G(\theta)}$.*

Proof. In [Ak] it was shown that the polynomials are not dense in $H^2(G(\theta))$ and so 1 is not a cyclic vector for $M_z^{G(\theta)}$ on $H^2(G(\theta))$. Note that among those crescents that narrow "tangentially" at their multiple boundary points, there are some for which the polynomials are dense in their respective Hardy spaces and others for which they are not. If G is a crescent bounded by two internally tangent circles, then the polynomials are dense in $H^2(G)$ (see [Ak]).

Now let $S(z)=(1-z)^2$ and $E(\theta)=S(G(\theta))$. $E(\theta)$ is a crescent with multiple boundary point 0. Clearly S is a polynomial and $S^{-1}(w)=1-\sqrt{w}$ on $\overline{E(\theta)}$ is a uniform limit of polynomials. Therefore, by Lemma 3.5, $M_z^{G(\theta)}$ on $H^2(G(\theta))$ is cyclic if and only if $M_z^{E(\theta)}$ is cyclic.

Let $\mu:=\omega(\cdot,E(\theta),\alpha)$ be harmonic measure on $\partial E(\theta)$ evaluated at some point α in $E(\theta)$. Applying Lemma 2.8 we can find a positive constant c such that in a neighborhood of 0, $d\mu\le c|z|d|z|$. Therefore $1/z\in L^1(\mu)$; in fact, $1/z\in L^p(\mu)$ whenever $0<p<2$. So, if φ is a conformal map from $\mathbf{D}:=\{\xi: |\xi|<1\}$ onto $E(\theta)$ and dm denotes normalized arclength measure on $\partial\mathbf{D}$, then $h:=\exp(-1/|\varphi|)$ is in $L^\infty(dm)$ and $\log(h)\in L^1(dm)$. Consequently,

$$F(\xi):=\exp\left(\frac{1}{2\pi}\int_0^{2\pi}\frac{e^{it}+\xi}{e^{it}-\xi}\log|h(e^{it})|\,dt\right)$$

is an outer function in $H^\infty(\mathbf{D})$, and if $\tilde{F}$ denotes the nontangential boundary values of F then $|\tilde{F}(e^{it})|=|h(e^{it})|$ a.e. Let $f=F\circ\varphi^{-1}$ and $\tilde{f}=\tilde{F}\circ\varphi^{-1}$. Since F is an outer function, $FH^2(\mathbf{D})$ is dense in $H^2(\mathbf{D})$ and so $fH^2(E(\theta))$ is dense in $H^2(E(\theta))$. Hence, in order to show that f is a cyclic vector for

$M_z^{E(\theta)}$ on $H^2(E(\theta))$, it suffices to show that $\{pf : p \text{ is a polynomial}\}$ is dense in $fH^2(E(\theta))$.

To this end, choose g in $L^2(|\tilde{f}|^2\,d\mu)$ such that $\int_{\partial E(\theta)} pg|\tilde{f}|^2\,d\mu = 0$ for all polynomials p. For ζ in $\mathbf{C}\backslash\partial E(\theta)$, consider the Cauchy transform

$$\hat{g}(\zeta) := \int_{\partial E(\theta)} \frac{g(z)|\tilde{f}(z)|^2}{z-\zeta}\,d\mu(z).$$

Notice that there exists $\lambda > 0$ such that $\Delta := \{\zeta : |\zeta-\lambda| < \lambda\} \subseteq S(\mathrm{int}(V(\theta))) = \Omega_{E(\theta)}$; this is why we are considering $E(\theta)$ rather than $G(\theta)$.

Now $|\tilde{f}(z)|^2 = \exp(-2/|z|)\,d\mu$-a.e. on $\partial E(\theta)$. Thus, it is easily seen that $1/z^n \in L^2(|\tilde{f}|^2\,d\mu)$ for all positive integers n, and in fact $1/z^n \in P^2(|\tilde{f}|^2\,d\mu)$ (= the closure of the polynomials in $L^2(|\tilde{f}|^2\,d\mu)$) since

$$\frac{1}{(z+1/k)^n} \to \frac{1}{z^n} \quad \text{in } L^2(|\tilde{f}|^2\,d\mu) \quad \text{as } k\to\infty$$

and, by Runge's theorem, $1/(z+1/k)^n$ is uniformly approximable by polynomials in $\overline{E(\theta)}$. Moreover, since harmonic measure has total mass 1 and the maximum of $x^{-2n}\exp(-2/x)$ on $(0,\infty)$ is $(n/e)^{2n}$,

$$\|z^{-n}\|_2 := \left\{\int_{\partial E(\theta)} |z^{-n}|^2|\tilde{f}|^2\,d\mu\right\}^{1/2} \le n^n$$

for all positive integers n. Therefore, $\sum_{n=1}^{\infty}\|z^{-n}\|_2^{-1/n} = \infty$.

For z in $\overline{E(\theta)}\backslash\{0\}$, ζ in Δ, and any positive integer n,

$$\frac{1}{z-\zeta} = \frac{1}{z} + \cdots + \frac{\zeta^{n-1}}{z^n} + \frac{\zeta^n}{z^n(z-\zeta)}.$$

As $z \mapsto \zeta^{k-1}/z^k \in P^2(|\tilde{f}|^2\,d\mu)$ for any positive integer k, and $g \perp P^2(|\tilde{f}|^2\,d\mu)$, it follows that

$$\hat{g}(\zeta) = \zeta^n \cdot \int_{\partial E(\theta)} \frac{g(z)|\tilde{f}(z)|^2}{z^n(z-\zeta)}\,d\mu(z).$$

Now there exists a positive constant c such that $|z-\zeta| \ge c|z|^2$ for any z in $\overline{E(\theta)}$ and ζ in Δ. Applying the Cauchy-Schwarz inequality, we obtain that, for all ζ in Δ,

$$\begin{aligned}|\hat{g}(\zeta)| &\le \mathrm{const}|\zeta|^n\|g\|_2\|z^{-n-2}\|_2\\ &\le \mathrm{const}|\zeta|^n M_n,\end{aligned}$$

where $\|\cdot\|_2$ denotes the norm in $L^2(|\tilde{f}|^2\,d\mu)$ and $\sum_{n=1}^{\infty} M_n^{-1/n} = \infty$. Moreover, applying the Cauchy-Schwarz inequality and rescaling if necessary, we may assume that the sequence $\{M_n\}$ is log-convex (cf. [CM]). From a classical result of Carleman [Ca1], it follows that $\hat{g}|_\Delta \equiv 0$. Thus, the polynomials are dense in $R^2(E(\theta), |\tilde{f}|^2\,d\mu)$ (= the closure in $L^2(|\tilde{f}|^2\,d\mu)$ of the rational functions with poles off $\overline{E(\theta)}$). Since the rational functions with poles off $\overline{E(\theta)}$ are dense in $H^2(E(\theta))$, we have that, given any function k in $H^2(E(\theta))$, there exists a sequence of polynomials $\{p_n\}$ such that

$$\|p_n f - kf\|^2_{H^2(E(\theta))} = \int_{\partial E(\theta)} |p_n\tilde{f} - \tilde{k}\tilde{f}|^2\,d\mu = \int_{\partial E(\theta)} |p_n - \tilde{k}|^2|\tilde{f}|^2\,d\mu \to 0$$

as $n\to\infty$. Therefore, $\{pf: p$ is a polynomial$\}$ is dense in $fH^2(E(\theta))$ and so, by previous observations, $M_z^{G(\theta)}$ on $H^2(G(\theta))$ is cyclic. □

The following corollary is just a restatement of Theorem 3.6.

3.7. COROLLARY. *If $\pi/2\le\theta<\pi$ and φ is a conformal map from $\mathbf{D}:=\{z:|z|<1\}$ onto $G(\theta)$, then the analytic Toeplitz operator T_φ on $H^2(\mathbf{D})$ defined by $T_\varphi(f)=\varphi f$ is cyclic and yet 1 is not a cyclic vector for T_φ.*

3.8. REMARK. For any crescent G, the geometry of G near its multiple boundary point is primarily what determines whether or not M_z^G on $H^2(G)$ is cyclic. One can therefore easily generalize Theorem 3.6 by replacing $G(\theta)$ with any crescent which has the same geometry as $G(\theta)$ near $z=1$.

4. The Shift Operator on the Bergman Space

For which G does M_z^G on $L_a^2(G)$ (resp., on $H^2(G)$) have 1 as a cyclic vector? The answer to this question in the context of Bergman spaces differs considerably from that of Hardy spaces (cf. [Br], [CM], [Ke1], [Ke2], and [Ak]). However, at least for crescents, the answer to the related question—For which G is M_z^G on $L_a^2(G)$ (resp., on $H^2(G)$) cyclic?—turns out to be surprisingly similar in both cases.

Given $0\le\theta<\pi$, let $V(\theta)$ and $G(\theta)$ be as in Section 3.

4.1. THEOREM. *If G is a bounded, simply connected domain and there exists θ $(0\le\theta\le\frac{\pi}{3})$ such that $G(\theta)\subseteq G$ and $(\mathbf{C}\setminus G)\cap\Omega_{G(\theta)}\neq\emptyset$, then M_z^G on $L_a^2(G)$ is not cyclic.*

Proof. Let $\sigma=\frac{1}{2}(\frac{\pi}{3}-\theta)$, define $D(\theta)=(\operatorname{int}V(\pi-\sigma))\setminus V(\theta+\sigma)$, choose α in $D(\theta)$, and let $\omega_\alpha:=\omega(\cdot,D(\theta),\alpha)$ be the harmonic measure on $\partial D(\theta)$ evaluated at α. Since the angle between the outer and inner boundaries of $D(\theta)$ at the multiple boundary point 1 is precisely $\frac{\pi}{3}$, we can apply Lemma 2.8 and conclude that $d\omega$ is boundedly equivalent to $|z-1|^2d|z|$ in a neighborhood of 1. Therefore, if $r_z=\operatorname{dist}(z,\partial G)$ for z in G (as in §2), then

$$c=\frac{1}{\pi}\int_{\partial D(\theta)}\frac{1}{r_z^2}\,d\omega(z)<\infty$$

($r_z=|z-1|\sin\sigma$ for z in $\partial D(\theta)$). So, by Lemma 2.7, if $f\in L_a^2(G)$ then $f\in H^2(D(\theta))$ and

$$\int_{\partial D(\theta)}|f|^2\,d\omega=\|f\|_{H^2(D(\theta))}^2\le c\|f\|_{L_a^2(G)}^2.$$

Suppose that M_z^G on $L_a^2(G)$ is cyclic and that f is a cyclic vector for M_z^G. Fix β in $(\mathbf{C}\setminus G)\cap\Omega_{G(\theta)}$. It follows from Lemma 2.5 that $(z-\beta)^kf$ is also a cyclic vector for M_z^G on $L_a^2(G)$ for any positive integer k. Therefore, for any positive integer k, there exists a sequence of polynomials $\{p_n\}$ such that

$$\int_{\partial D(\theta)} |(z-\beta)^k f p_n - 1|^2 \, d\omega_\alpha = \|(z-\beta)^k f p_n - 1\|^2_{H^2(D(\theta))} \le c \|(z-\beta)^k f p_n - 1\|^2_{L^2_a(G)} \to 0$$

as $n \to \infty$. Since $(z-\beta)^{-k}$ is bounded on $D(\theta)$, evidently

$$\int_{\partial D(\theta)} |f p_n - (z-\beta)^{-k}|^2 \, d\omega_\alpha \to 0$$

as $n \to \infty$. Because the rational functions with poles off $\overline{D(\theta)}$ are dense in $H^2(D(\theta))$, Runge's theorem implies that f is a cyclic vector for $M_z^{D(\theta)}$ on $H^2(D(\theta))$. However, $\theta + \sigma = \frac{\theta}{2} + \frac{\pi}{6} < \frac{\pi}{3}$ and so, if $\nu_\beta := \omega(\cdot, \Omega_{D(\theta)}, \beta)$ is harmonic measure on $\partial\Omega_{D(\theta)} = \partial V(\theta+\sigma)$ evaluated at β, then (by Lemma 2.8) $d\nu_\beta \le \text{const}|z-1|^{2+\epsilon} d|z|$ for some $\epsilon > 0$ in a neighborhood of $z = 1$. Since $d\omega_\alpha$ is boundedly equivalent to $|z-1|^2 d|z|$ in a neighborhood of $z=1$, it follows from Theorem 3.2 that $M_z^{D(\theta)}$ on $H^2(D(\theta))$ cannot be cyclic. Therefore, we conclude that M_z^G on $L^2_a(G)$ is not cyclic. □

A slight modification of the above argument yields the following (cf. Corollary 3.3 and Remark 3.4).

4.2. COROLLARY. *If G is a disk with finitely many radial slits, then M_z^G on $L^2_a(G)$ is not cyclic.*

As in the proof of Theorem 4.1, one can use Lemma 2.7 to show that, if M_z^G on $L^2_a(G)$ is cyclic, then there exist subregions D of G such that $(\partial D) \cap (\partial G) \ne \emptyset$ for which M_z^D on $H^2(D)$ is cyclic. Conversely, it turns out that if the boundary of G is rectifiable then M_z^G being cyclic on $H^2(G)$ implies that it is cyclic on $L^2_a(G)$. In order to show this, we must recall the concept of Smirnov classes.

4.3. DEFINITION. An analytic function f in a simply connected domain G with rectifiable boundary is said to belong to the *Smirnov class* $E_p(G)$ if there exists a system of rectifiable Jordan curves $\{\Gamma_i\}$ in G such that $\Gamma_i \to \partial G$ and $\sup_i \int_{\Gamma_i} |f(z)|^p d|z| < \infty$, $0 < p < \infty$.

Let $\varphi : \mathbf{D} \to G$ be a conformal map of the unit disk onto G and let $\psi = \varphi^{-1}$ be its inverse. For $0 < \rho < 1$, let $\Gamma_\rho = \varphi(\{z : |z| < \rho\})$. The following result of Keldyš and Lavrent'ev (see [Du, Cor. to Thm. 10.1]) provides a link between Smirnov and Hardy classes.

4.4. THEOREM. *For $0 < p < \infty$, the following are equivalent:*

(i) $f \in E_p(G)$;
(ii) $\sup_{0<\rho<1} \int_{\Gamma_\rho} |f(z)|^p d|z| < \infty$;
(iii) $f \cdot [\psi']^{-1/p} \in H^p(G)$.

4.5. COROLLARY. *A function f in $E_p(G)$ has well-defined nontangential boundary values $\tilde{f}$ a.e. on ∂G, and*

$$\|f\|_{E_p(G)} := \left\{\int_{\partial G} |\tilde{f}(\zeta)|^p d|\zeta|\right\}^{1/p} < \infty.$$

Moreover, for $1 \le p < \infty$, $E_p(G)$ *is a Banach space with respect to the norm* $\|\cdot\|_{E_p(G)}$, *and the mapping* $f \mapsto f \cdot [\psi']^{-1/p}$ *is an isometry of* $E_p(G)$ *onto* $H^p(G)$.

The following "isoperimetric inequality" is due to Carleman [Ca2]. For a different proof see [Ar].

4.6. THEOREM. *Let G be a simply connected region with rectifiable boundary. Then*

$$\iint_G |f|^2 dA \le \frac{1}{4\pi}\left(\int_{\partial G} |\tilde{f}(z)|\, d|z|\right)^2$$

for all f in $E_1(G)$.

With these preliminaries we can prove the following.

4.7. THEOREM. *Let G be a simply connected region with rectifiable boundary. If* M_z^G *is cyclic on* $H^2(G)$, *then it is also cyclic on* $L_a^2(G)$.

For the proof we need two simple lemmas.

4.8. LEMMA. *Let G be as above and let* ψ *be a conformal map of G onto the unit disk. Then* $\{f \cdot \sqrt{\psi'} : f \in H^\infty(G)\}$ *is dense in* $E_2(G)$.

Proof. It suffices to observe that $H^\infty(G)$ is dense in $H^2(G)$ and then apply Corollary 4.5. □

4.9. LEMMA. *If f is a cyclic vector for* M_z^G *on* $H^2(G)$, *then* $f \cdot \sqrt{\psi'}$ *is a cyclic vector for* M_z^G *on* $E_2(G)$ *(G and* ψ *are as in Lemma 4.8).*

Proof. Choose g in $H^\infty(G)$. By our hypothesis there exists a sequence of polynomials $\{p_n\}$ such that $\|p_n f - g\|_{H^2(G)} \to 0$ as $n \to \infty$. By Corollary 4.5,

$$\|g\sqrt{\psi'} - p_n f \sqrt{\psi'}\|_{E_2(G)} = \|g - p_n f\|_{H^2(G)} \to 0$$

as $n \to \infty$. Applying Lemma 4.8, we complete the proof. □

Proof of Theorem 4.7. Let f be a cyclic vector for M_z^G on $H^2(G)$ and choose g in $H^\infty(G)$. By Lemma 4.9, there is a sequence of polynomials $\{p_n\}$ such that $\|p_n f\sqrt{\psi'} - g\|_{E_2(G)}^2 \to 0$ as $n \to \infty$. Then, by applying Theorem 4.6 and the Cauchy-Schwarz inequality, we obtain

$$\|p_n f\sqrt{\psi'} - g\|_{L_a^2(G)}^2 \le \frac{1}{4\pi}\left(\int_{\partial G} |p_n \tilde{f}\sqrt{\tilde{\psi}'} - \tilde{g}|\, d|z|\right)^2$$

$$\le \frac{1}{4\pi}\|p_n f\sqrt{\psi'} - g\|_{E_2(G)}^2 [\text{Perimeter}(G)]^2 \to 0$$

as $n \to \infty$. By Theorem 2.3 the proof is now complete. □

4.10. COROLLARY. *If $\pi/2 \le \theta < \pi$, then $M_z^{G(\theta)}$ on $L_a^2(G(\theta))$ is cyclic and yet 1 is not a cyclic vector for $M_z^{G(\theta)}$.*

Proof. By Theorems 4.7 and 3.6, $M_z^{G(\theta)}$ on $L_a^2(G(\theta))$ is cyclic. To see that 1 is not a cyclic vector for M_z^G on $L_a^2(G)$ (i.e., that the polynomials are not dense in $L_a^2(G)$), one may combine Corollary 3.7 with [Bo, Cor. 3.4]. □

4.11. REMARK. If G is a crescent bounded by two internally tangent circles, then 1 is a cyclic vector for M_z^G on $H^2(G)$ (see [Ak]). On the other hand, polynomials are not dense in $L_a^2(G)$ (see, e.g., [CM], [Ke2], or [Si]) and therefore 1 is not a cyclic vector for M_z^G on $L_a^2(G)$. However, by Corollary 4.10, M_z^G on $L_a^2(G)$ is cyclic.

5. Concluding Remarks

1. Corollary 4.2 has an independent proof based on a totally different idea which we think is worth sketching here. To fix the ideas, we assume that $G = \mathbf{D}\setminus[0,1]$. The following statement is obvious.

5.1. LEMMA. *For z in G let $r_z = \operatorname{dist}(z, \partial G)$. If $f \in L_a^2(G)$ then $|f(z)| \le \pi^{-1/2} r_z^{-1} \|f\|_{L_a^2(G)}$.*

5.2. LEMMA. *Let f be in $L_a^2(G)$ such that f has no zeros in G. Then there are positive constants c_1 and c_2 such that, near the slit* $(0,1)$,

$$|f(z)| \ge \frac{c_1}{r_z} \exp\left(\frac{-c_2}{r_z^2}\right). \tag{5.2.1}$$

Proof (sketch). First we note that if h is bounded in the disk of radius R $(0<R<1)$, that is, if $0<|h(z)|<M$ in $\mathbf{D}_R := \{z : |z| < R\}$, then

$$v(z) := \log\left(\frac{M}{|h(z)|}\right)$$

is a positive harmonic function in $\mathbf{D}_R$. Therefore, by Harnack's inequality, $v(z) \le 2Rv(0)/(R-|z|)$ and so

$$|h(z)| \ge M \exp\left(-\frac{\alpha}{R-|z|}\right), \tag{5.2.2}$$

where $\alpha = 2R\log(M/|h(0)|)$.

For h in $L_a^2(\mathbf{D}_R)$ we apply (5.2.2) to $h_r(z) := h(rz)$, where $0<r<1$ and M is estimated by Lemma 5.1. This will give 5.2.1 for $\mathbf{D}_R$. To obtain the estimate in G near $(0,1)$ it suffices to use a conformal map from $\mathbf{D}_R$ onto a half-disk whose "diameter" is a segment in $[0,1]$. □

The following lemma is one of many versions of a celebrated Beurling–Levinson theorem (see, e.g., [Le]; see also [Do] and [Sjö]).

5.3. LEMMA. *Let $F(y)$ be a monotone decreasing function on $\{y: 0<y<R\}$ with $F(y) \to \infty$ as $y \to 0$. Let $\mathfrak{F} = \{f: f$ is analytic in $\mathbf{D}_R := \{z: |z|<R\}$ such that $|f(z)| \le F(|y|)$; $z = x+iy\}$. If*

$$\int_0^R \log^+ \log^+ F(y)\, dy < \infty,$$

then $\mathfrak{F}$ is a normal family in $\mathbf{D}_R$.

Proof of Corollary 4.2. Suppose to the contrary that M_z^G on $L_a^2(G)$ has a cyclic vector f. Then $f(z) \ne 0$ for any z in G and there exists a sequence of polynomials $\{p_n\}$ such that $p_n f \to 1$ in $L_a^2(G)$. Hence, $\|p_n f\|_{L_a^2(G)} \le c < \infty$ for all n, and, by Lemma 5.1, we have

$$|p_n(z)| \le c(\pi^{1/2} r_z |f(z)|)^{-1}. \tag{5.3.1}$$

Then Lemma 5.3, together with estimates (5.2.1) and (5.3.1) on a sufficiently small disk Δ centered on $(0,1)$, imply that $\{p_n\}$ forms a normal family in Δ. Also, $p_n(z) \to 1/f(z)$ for all z in G. So, the sequence $\{p_n(z)\}$ converges for all z in Δ; therefore $1/f$ and also f extend analytically to Δ. By increasing the radius of Δ, we obtain that f must be analytic in $\mathbf{D}$. This contradicts our assumption that f is a cyclic vector for M_z^G on $L_a^2(G)$, since, for example, $\sqrt{z} \in L_a^2(G)$. The result now follows. □

2. We have not been able to produce an example of a domain G for which M_z^G on $L_a^2(G)$ is cyclic and yet, on $H^2(G)$, M_z^G is not cyclic. Lemma 2.7 and Theorem 4.7 seem to suggest a "wild" conjecture that M_z^G on $H^2(G)$ is cyclic if and only if it is cyclic on $L_a^2(G)$, but this appears highly unlikely.

Another interesting problem would be to construct a non-Carathéodory domain G for which M_z^G on $L_a^2(G)$ has a cyclic vector which is not in the Nevanlinna class in G. All cyclic vectors produced in this paper are in the Nevanlinna class.

3. It is not hard to see that, in all examples of regions G considered in this paper, the operator M_z^G is two-cyclic; that is, there exist two vectors f and g in $H^2(G)$ (resp., in $L_a^2(G)$) such that $\{pf+qg: p$ and q are polynomials$\}$ is dense in $H^2(G)$ (resp., in $L_a^2(G)$). Is this true for any simply connected domain G? The closest analogy which may be relevant here is the well-known fact that the algebra $R(K)$ on any compact set K in $\mathbf{C}$ always has two generators.

ADDED IN PROOF.

1. Professor P. Bourdon has kindly pointed out to us that the rectifiability hypothesis in Theorem 4.7 can be omitted and the proof significantly simplified by means of the following argument.

Let $\varphi: \mathbf{D} \to G$, $\psi = \varphi^{-1}$ be as in Section 4. If f is a cyclic vector in $H^2(G)$, then $\{p(z)f: p$ is a polynomial$\}$ is dense in $H^2(G)$ and hence $\{p(\varphi)f(\varphi):$

p is a polynomial} is dense in $H^2(\mathbf{D})$. Since $H^2(\mathbf{D}) \subset L^2_a(\mathbf{D})$ is dense in $L^2_a(\mathbf{D})$, $\{p(\varphi)f(\varphi): p$ is a polynomial} is dense in $L^2_a(\mathbf{D})$. Changing variables, one can readily see that $\{p(z)[f\psi']: p$ is a polynomial} is dense in $L^2_a(G)$. So, $f\psi'$ is a cyclic vector for M_z on $L^2_a(G)$.

We also note that the fact that $f\psi'$ (instead of $f\sqrt{\psi'}$) is cyclic in $L^2_a(G)$ can be extracted from the argument in the paper if we replace $f\sqrt{\psi'}$ by $f\psi'$ and $E_2(G)$ by $E_1(G)$ in Lemma 4.9, and then use the full strength of Carleman's estimate 4.6.

2. After this paper was submitted, the authors learned of the recent work by B. M. Solomyak and A. L. Vol'berg (*Multiplicity of analytic Toeplitz operators,* Oper. Theory: Adv. Appl., 42, pp. 87–192, Birkhäuser, Basel, 1989), which provides an affirmative answer to our Question 3 in Section 5 and also contains far-reaching generalizations concerning multi-cyclicity of the shift on the Hardy space.

The relation between cyclicity of the shift on the Hardy space (but not on the Bergman space) on crescents and size of angles at the double point has already been noted by N. K. Nikol'skii (*Outlines for the computation of the multiplicities of the spectra of orthogonal sums,* J. Soviet Math. 27 (1984), 2521–2526), although his arguments are fairly different from ours. The authors are grateful to Professor A. L. Vol'berg for bringing these references to our attention.

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