Boundary Density and the Green Function

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In this note, we generalize the following theorem on level curves of conformal mappings to domains in $\mathbf{R}^m$, $m \geq 2$.

THEOREM A. *Let Ω be a simply connected domain in $\mathbf{R}^2$ ($\Omega \neq \mathbf{R}^2$), let f be a conformal mapping from Ω onto the unit disk $|z|<1$, and let Γ be any line or circle on the plane. Then there exists an absolute constant p_0 ($1 < p_0 < 2$) such that*

$$\int_{\Gamma \cap \Omega} |f'(z)|^p |dz| \leq C(p, \Omega) < \infty \tag{0.1}$$

for $1 \leq p \leq p_0$.

For the development of the theorem, see [4], [5], [7], and [8]. Recently, Baernstein [1] constructed Ω, f, and Γ as in Theorem A, so that

$$\int_{\Gamma \cap \Omega} |f'(z)|^{2-\delta} |dz| = \infty$$

for some $\delta > 0$.

Suppose that G is the Green function on Ω with pole at $f^{-1}(0)$. It follows from (0.1) that

$$\int_{\Gamma \cap \Omega} |\nabla G(z)|^p |dz| \leq C(p, \Omega) \operatorname{dist}(0, f(\Gamma))^{-p}. \tag{0.2}$$

We extend (0.2) to the following.

THEOREM. *Suppose that Ω is a domain in $\mathbf{R}^m$ ($m \geq 2$) that satisfies the $(m-1)$-dimensional density condition ($(m-1)$DC). Let P be a fixed point in Ω, G the Green function of Ω with pole at P, and Γ an $(m-1)$-dimensional hyperplane with $P \notin \Gamma$. Then there exists a constant $p_0 > 1$ depending on the $(m-1)$DC constant, so that if $1 \leq p \leq p_0$ then*

$$\int_{\Gamma \cap \Omega} |\nabla G(x)|^p \, d\sigma(x) < B, \tag{0.3}$$

where $d\sigma$ is the $(m-1)$-dimensional measure on Γ and B is a constant depending on p, the $(m-1)$DC constant, $\operatorname{dist}(P, \partial\Omega)$, *and* $\operatorname{dist}(P, \Gamma)$.

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Let Ω be a domain in $\mathbf{R}^m$ ($m \geq 2$) and $E = \mathbf{R}^m \setminus \Omega$. We say that Ω satisfies the *capacity density condition* (CDC) if there is a fixed $\eta > 0$ such that, whenever $x \in \Omega$ and $d(x) \equiv \operatorname{dist}(x, \partial\Omega)$,

$$\text{capacity}\left(\frac{E \cap B(x, 2d(x))}{10d(x)}\right) > \eta. \tag{0.4}$$

Here, for a set S and $a > 0$, S/a denotes $\{(1/a)x : x \in S\}$ and

$$\text{capacity}(S) = \sup\{\mu(S) : \mu \text{ is a positive measure on } S \text{ satisfying } \textstyle\int_S K(x,y)\, d\mu(y) \leq 1\},$$

with

$$K(x,y) = \begin{cases} -\log|x-y| & \text{when } m = 2, \\ |x-y|^{-m+2} & \text{when } m \geq 3. \end{cases}$$

This type of complementary thickness condition has been used by many authors before (see, e.g., [11]).

Denote by Λ^α the *α-dimensional content* of a set, that is,

$$\Lambda^\alpha(S) = \inf \sum_n r_n^\alpha,$$

where the infimum is taken over all coverings of S with countably many balls of radii r_n. We say that Ω has the *α-dimensional density condition* (αDC) for some $\alpha \leq m-1$ if there is a fixed $\zeta > 0$ such that, for any $x \in \Omega$,

$$\Lambda^\alpha\left(\frac{E \cap B(x, 2d(x))}{d(x)}\right) > \zeta. \tag{0.5}$$

We note that simply connected planar domains satisfy 1DC, and that αDC for some $\alpha > m-2$ implies CDC, because of the Frostman theorem [2].

In Section 3 and Section 4, we show by examples that the integral (0.3) indeed depends on the thickness of $\mathbf{R}^m \setminus \Omega$ near the boundary, and that $(m-1)$DC is essential and cannot be replaced by αDC for any $\alpha < m-1$.

In the following, we use $c, C, c_1, c_2, \ldots$ to denote positive constants that depend at most on m; we use $C(x, y, z, \ldots)$ to denote constants depending also on x, y, z, etc.

We denote by $\omega(z, E, D)$ or $\omega_D^z(E)$ the harmonic measure of $E \subseteq \partial D$ with respect to D at z, and by $B(x, r)$ the ball $\{y : |y-x| < r\}$ in $\mathbf{R}^m$.

1. Preliminary Lemmas

First, we state a theorem which is essential in the proof.

THEOREM B. *Let Ω be a domain in $\mathbf{R}^m$ ($m \geq 2$), and let $\{D_j\}$ be a sequence of closed sets contained in Ω with* $\operatorname{dist}(D_i, D_j) > 0$ *whenever $i \neq j$. Set $\Omega_j = \Omega \setminus \bigcup_{k \neq j} D_k$. If*

$$\inf_j \inf_{z \in D_j} \omega(z, \partial\Omega, \Omega_j) = a > 0 \tag{1.1}$$

then, for any $x \in \Omega \setminus \bigcup_j D_j$,

$$\sum_j \omega(x, D_j, \Omega \setminus D_j) < \frac{1}{a} \omega(x, \bigcup D_j, \Omega \setminus \bigcup D_j).$$

When Ω and D_j's are all disks in $\mathbf{R}^2$, this theorem is due to Garnett, Gehring, and Jones [7] and is used in their proof of Theorem A $(p=1)$. A Brownian motion proof by Davis [6] gives Theorem B. Other related results can be found in [10] and [13].

We also need an extension of Hall's lemma [14].

THEOREM C. *Suppose that D is a $C^{1+\epsilon}$ domain in $\mathbf{R}^m$ for some $\epsilon > 0$, and that P is a fixed point in D. Then, for any closed set $E \subseteq \bar{D}$,*

$$\omega(P, E, D \setminus E) \geq c(P, D) \Lambda^{m-1}(E).$$

Finally, we require a theorem of Baernstein [4] on comparison of Green functions. Let $x = (x_1, \ldots, x_m) \in \mathbf{R}^m$ and $S \subseteq \mathbf{R}^m$, denote by $\bar{x} = (x_1, \ldots, x_{m-1}, -x_m)$, $\tilde{x} = (x_1, \ldots, x_{m-1}, -|x_m|)$, $\bar{S} = \{\bar{x} : x \in S\}$, and $\tilde{S} = \{\tilde{x} : x \in S\}$. Let Ω be a domain in $\mathbf{R}^m$, $E = \mathbf{R}^m \setminus \Omega$, and let G and $\tilde{G}$ be the Green functions for the domains $\hat{\mathbf{R}}^m \setminus E$ and $\hat{\mathbf{R}}^m \setminus \tilde{E}$ (respectively), with value 0 outside.

THEOREM D. *Let $x \in \mathbf{R}^m$ with $x_m = 0$ and $y \in \mathbf{R}^m$. Then*

$$G(x, y) \leq \tilde{G}(x, \bar{y}).$$

Originally the theorem is proved for $m = 2$; the proof for $m \geq 3$ is very similar.

Assume from now on that Ω is a domain in $\mathbf{R}^m$ whose complement $\mathbf{R}^m \setminus \Omega$ has positive capacity, P is a fixed point in Ω, and G is the Green function of Ω with pole P. Denote by $d = \operatorname{dist}(P, \partial\Omega)$.

LEMMA 1. *Let Y be a point in Ω with $\operatorname{dist}(Y, \partial\Omega) < |P - Y|$, let $r > 0$ with $\operatorname{dist}(Y, \partial\Omega)/4 < r < \operatorname{dist}(Y, \partial\Omega)/2$, and let $B = B(Y, r)$. When $m \geq 3$, we have*

$$G(X) \approx |P - X|^{-m+2} \quad \text{for } |X - P| \leq \frac{d}{2}$$

and

$$\omega(P, B, \Omega \setminus B) \approx r^{m-2} G(Y).$$

When $m = 2$, assume also that Ω satisfies CDC; *then*

$$\text{(1.2)} \qquad \log \frac{d}{|X - P|} \leq G(X) \leq C(\eta) + \log \frac{d}{|X - P|} \quad \text{for } |X - P| \leq \frac{d}{2}$$

and

$$\text{(1.3)} \qquad \omega(P, B, \Omega \setminus B) \leq G(Y) \leq C(\eta) \omega(P, B, \Omega \setminus B).$$

By $\approx$, we mean that the ratio of both sides is bounded above and below by constants C and c.

Proof. When $m=2$, we assume first that $\operatorname{diam}\Omega<+\infty$. Let $g(X)$ be the Green function for $|X-P|<d$ with pole P. Let $S=\{3d/4\le|X-P|\le 2d\}\backslash\Omega$, let $K=10\operatorname{diam}\Omega$, and let g^* be the Green function for $\{|X-P|<K\}\backslash S$ with pole P. Then

$$g(X)\le G(X)\le g^*(X) \quad \text{in } |X-P|<d.$$

We note that $g(X)=\log(d/|X-P|)$. From the CDC for Ω we deduce that

$$C\ge \operatorname{capacity}\left(\frac{S}{K}\right)\ge\left(\log\frac{K}{c(\eta)d}\right)^{-1}.$$

We recall that $g^*(X)=\log(K/|X-P|)-h(X)$, where $h(X)$ is harmonic in $\{|X-P|<K\}\backslash S$ with boundary values

$$h(x)=\begin{cases}\log(K/|X-P|) & \text{on } S,\\ 0 & \text{on } |X-P|=K.\end{cases}$$

Let μ be the capacitary measure of S/K. Thus

$$f(X)=\int_S \log\frac{K}{|X-Z|}\,d\mu\left(\frac{Z}{K}\right)$$

has value 1 nearly everywhere on S, and

$$C\ge\mu\left(\frac{S}{K}\right)\ge\left(\log\frac{K}{c(\eta)d}\right)^{-1}.$$

Note that

$$f(X)\le\frac{cd}{K} \quad \text{for } |X-P|=K,$$

and that

$$f(X)\ge\frac{\log(K/4d)}{\log(K/c(\eta)d)} \quad \text{for } |X-P|\le\frac{d}{2}.$$

It follows from the maximum principle that, on $|X-P|\le d/2$,

$$h(X)\ge\log\frac{K}{2d}\left(f(X)-\frac{cd}{K}\right)\ge\log\frac{K}{d}-c(\eta).$$

Therefore

$$G(X)\le g^*(X)\le C(\eta)+\log\frac{d}{|X-P|} \quad \text{for } |X-P|\le\frac{d}{2}.$$

Because the above estimates are independent of the diameter, (1.2) holds for $\operatorname{diam}\Omega=\infty$ also. The estimate (1.3) follows from (1.2), the symmetry property of the Green function, and the maximum principle.

The case $m\ge 3$ is much simpler, and CDC is not required. We omit the proof. □

The next lemma follows from the Poisson integral formula.

LEMMA 2. *Let u be a bounded harmonic function in a ball $B(X,r)$. Then*

$$|\nabla u(X)|\le\frac{C}{r}\left(\sup_B u-\inf_B u\right).$$

2. Proof of Theorem

In this section, we assume that $\Gamma=\{x_m=0\}$ and that $P=(0,0,\ldots,0,a)$. We partition $\Gamma\cap\Omega$ into $(m-1)$-dimensional closed dyadic squares $\{Q_j\}$ with mutually disjoint interiors so that

$$c_1<\frac{\text{side length of } Q_j}{\operatorname{dist}(Q_j,\partial\Omega)}\leq\frac{1}{2}. \tag{2.1}$$

Let P_j be the center of Q_j, $B_j=B(P_j,c_2\operatorname{dist}(P_j,\partial\Omega))$, and $D_j=B_j\cap\Gamma$. Choose $c_2>0$ small enough so that $2B_j\subseteq\Omega$ and that $\{2B_j\}$ are mutually disjoint. There are at most c_3 squares Q_j satisfying

$$\operatorname{dist}(P,Q_j)\leq\frac{\operatorname{dist}(P,\partial\Omega)}{100}.$$

Denote by J the collection of the indices of the remaining Q_j's, and note from Lemma 2 that

$$\sum_{j\notin J}\int_{Q_j}|\nabla G|^p\,d\sigma(x)\leq C(p,\operatorname{dist}(P,\Gamma),\operatorname{dist}(P,\partial\Omega)). \tag{2.2}$$

We deduce from Lemmas 1 and 2 and the Harnack inequality that, for $j\in J$ and $x\in Q_j$,

$$\begin{aligned}|\nabla G(x)|&\leq CG(P_j)\operatorname{dist}(P_j,\partial\Omega)^{-1}\\&\leq C(\zeta)\omega(P,B_j,\Omega\backslash B_j)\operatorname{dist}(P_j,\partial\Omega)^{-m+1}\\&\leq C(\zeta)\omega(P,D_j,\Omega\backslash D_j)\operatorname{dist}(P_j,\partial\Omega)^{-m+1}.\end{aligned} \tag{2.3}$$

We now verify the condition (1.1). Fix $j\in J$, and let $\rho=\operatorname{dist}(P_j,\partial\Omega)$, $S=E\cap B(P_j,2\rho)$, and $\Omega_j'=\Omega\backslash\bigcup_{\substack{k\neq j\\k\in J}}D_k$. In view of $(m-1)$DC, we may assume that

$$\Lambda^{m-1}\left(\frac{S\cap\{x_m\geq 0\}}{\rho}\right)\geq\frac{\zeta}{2}.$$

Let $Y=P_j+(0,\ldots,0,\rho/2)$ and $D=\{x_m\geq 0\}\backslash S$. It follows from $(m-1)$DC, Theorem C, the maximum principle, and the Harnack inequality that

$$\omega(P_j,\partial\Omega,\Omega_j')\geq C\omega(Y,\partial\Omega,\Omega_j')\geq C\omega(Y,S,D)\geq C(\zeta).$$

Again by the Harnack inequality,

$$\inf_J\inf_{x\in D_j}\omega(x,\partial\Omega,\Omega_j')>C(\zeta)>0. \tag{2.4}$$

From (2.2), (2.3), (2.4), and Theorem B, it follows that

$$\int_{\Gamma\cap\Omega}|\nabla G|\,d\sigma(x)\leq C(\zeta,\operatorname{dist}(P,\Gamma),\operatorname{dist}(P,\partial\Omega)). \tag{2.5}$$

To prove (0.3) for $1<p\leq p_0$, we first impose the extra condition $\mathbf{R}^m\backslash\Omega\subseteq\{x_m\leq 0\}$, and we define a measure μ on Γ as follows:

$$d\mu(x) = \begin{cases} d\omega_\Omega^P(x) & \text{for } x \in \Gamma \cap \partial\Omega, \\ \omega(P, Q_j, \Omega \setminus Q_j)/\sigma(Q_j)\, d\sigma(x) & \text{for } x \in Q_j. \end{cases} \tag{2.6}$$

Recall that $P = (0, 0, \dots, 0, a)$ with $a > 0$, and that σ is the $(m-1)$-dimensional measure on Γ.

PROPOSITION. *Under the extra assumption that $\mathbf{R}^m \setminus \Omega \subseteq \{x_m \leq 0\}$, the measures μ and σ are mutually absolutely continuous on Γ, and*

$$\frac{\mu(F)}{\mu(I)} \geq C(\zeta) \frac{\sigma(F)}{\sigma(I)} \tag{2.7}$$

for any square $I \subseteq \Gamma$ of side length $\leq a/10$ and $F \subseteq I$. Consequently, there exists $p_0 > 1$ depending on ζ such that, for $1 \leq p \leq p_0$,

$$\left\| \frac{d\omega_\Omega^P}{d\sigma} \right\|_{L^p(\sigma)} \leq C(\zeta, p, \operatorname{dist}(P, \Gamma)) \tag{2.8}$$

and

$$\| |\nabla G| \|_{L^p(\sigma)} \leq C(\zeta, p, \operatorname{dist}(P, \Gamma), \operatorname{dist}(P, \partial\Omega)). \tag{2.9}$$

Jones and Marshall [10] have proved (2.8) for domains in $\mathbf{R}^2$ with complements in $\{x_2 = 0\}$ and satisfying the 1-dimensional density condition.

Proof. To prove $\mu \ll \sigma$, we assume that $F \subseteq \Gamma \cap \partial\Omega$ with $\sigma(F) = 0$ and claim that $\omega(P, F, \Omega) = 0$.

We note by the maximum principle that, for any $y \in \Omega$,

$$\omega(y, F, \Omega) \leq \sup_{x \in \Omega \cap \Gamma} \omega(x, F, \Omega) \equiv b.$$

Fix a point $x \in \Omega \cap \Gamma$, let $d(x) = \operatorname{dist}(x, \partial\Omega)$, and let Λ be the spherical cap on $\partial B(x, d(x)/2)$ defined by $\partial B(x, d(x)/2) \cap \{x_m \leq -d(x)/4\}$. Let $\alpha = \omega(x, \Lambda, B(x, d(x)/2))$, a number between 0 and 1 depending only on m. We note by the Markov property that

$$\omega(x, F, \Omega) \leq (1-\alpha)b + \alpha \sup_{y \in \Lambda} \omega(y, F, \Omega).$$

For $y \in \Lambda$, we note that

$$\omega(y, F, \Omega) \leq b\omega(y, \Gamma \cap \Omega, \Omega \cap \{x_m < 0\}) = b(1 - \omega(y, \partial\Omega, \Omega \cap \{x_m < 0\}),$$

and also from $(m-1)$DC for Ω and Theorem C that

$$\begin{aligned} \omega(y, \partial\Omega, \Omega \cap \{x_m < 0\}) &\geq \omega(y, \partial\Omega \cap B(x, 2d(x)), \{x_m < 0\}) \\ &> c(\zeta) > 0. \end{aligned}$$

Combining the above estimates, we obtain that $b \leq (1-\alpha)b + \alpha b(1 - c(\zeta))$. This is possible only when $b = 0$. This proves the claim and thus $\mu \ll \sigma$.

To show (2.7), we may assume that I is a dyadic square on Γ with side length $\leq a/10$. Then either $I \subseteq Q_{j_0}$ for some integer j_0, or there exists a collection K of natural numbers such that

(2.10) $$I=(I\cap\partial\Omega)\cup\bigcup_{j\in K}Q_j.$$

In the first case, (2.7) follows from the definition of μ. Thus, we proceed with the assumption (2.10). An inequality similar to (2.4) still holds for Ω and $\{D_j\}_{j\in K}$. Thus it follows from Theorem B and Harnack inequality that

$$\begin{aligned}\sum_{j\in K}\omega(P,Q_j,\Omega\backslash Q_j)&\le C\sum_K\omega(P,D_j,\Omega\backslash D_j)\\&\le C(\zeta)\omega\Big(P,\bigcup_K D_j,\Omega\backslash\bigcup D_j\Big)\\&\le C(\zeta)\omega\Big(P,\bigcup_K Q_j,\Omega\backslash\bigcup_K Q_j\Big)=C(\zeta)\omega(P,I,\Omega\backslash I).\end{aligned}$$

Hence

$$\mu(I)\le C(\zeta)\omega(P,I,\Omega\backslash I).$$

Let $F\subseteq I$, and write $F=(F\cap\partial\Omega)\cup\bigcup_{j\in K}F_j$ with $F_j\subseteq Q_j$. Thus,

$$\mu\Big(\bigcup_K F_j\Big)=\sum_K\frac{\sigma(F_j)}{\sigma(Q_j)}\omega(P,Q_j,\Omega\backslash Q_j).$$

Therefore, in order to prove (2.7), it is enough to show that

(2.11) $$\frac{\omega(P,Q_j,\Omega\backslash Q_j)}{\omega(P,I,\Omega\backslash I)}\ge c(\zeta)\frac{\sigma(Q_j)}{\sigma(I)}\quad\text{for } j\in K,$$

and that

(2.12) $$\frac{\omega(P,F\cap\partial\Omega,\Omega)}{\omega(P,I,\Omega\backslash I)}\ge c(\zeta)\frac{\sigma(F\cap\partial\Omega)}{\sigma(I)}.$$

Let $I^*=2I$ and let U be the rectangular cylinder $\{x:(x_1,\dots,x_{m-1},0)\in I^*$ and $-l(I)\le x_m\le 3l(I)\}$. Then, for $x\in\{x_m=l(I)\}\backslash\bar U$,

(2.13) $$\omega(x,\partial U\cap\{x_m\ge l(I)\},\Omega\backslash\bar U)\ge\omega(x,(\partial U\cap\Omega)\cap\{x_m<l(I)\},\Omega\backslash\bar U).$$

Proof of (2.13) shall be given in the next paragraph. Inequality (2.13) holds also for $x=P$. From the Markov property and the maximum principle it follows that, for $j\in K$,

$$\begin{aligned}&\omega(P,Q_j,\Omega\backslash Q_j)\\&\quad\ge\omega(P,\partial U\cap\{x_m\ge l(I)\},\Omega\backslash\bar U)\inf_{x\in\partial U\cap\{x_m\ge l(I)\}}\omega(x,Q_j,\Omega\backslash Q_j)\\&\quad\ge\omega(P,\partial U\cap\{x_m\ge l(I)\},\Omega\backslash\bar U)\inf_{x\in\partial U\cap\{x_m\ge l(I)\}}\omega(x,Q_j,\{x_m>0\})\\&\quad\ge c\frac{\sigma(Q_j)}{\sigma(I)}\omega(P,\partial U\cap\{x_m\ge l(I)\},\Omega\backslash\bar U).\end{aligned}$$

On the other hand,

$$\omega(P,I,\Omega\backslash I)\le\omega(P,\partial U\cap\Omega,\Omega\backslash\bar U)\le 2\omega(P,\partial U\cap\{x_m\ge l(I)\},\Omega\backslash\bar U).$$

Thus (2.11) follows from the above estimates; the proof of (2.12) is similar. The absolute continuity $\sigma\ll\mu$ and the doubling property $\mu(2I)\le c(\zeta)\mu(I)$ for squares with $l(I)\le a/10$ follow from (2.7).

To prove (2.13) we note that, on $\Omega\setminus\bar{U}$,

$$\omega(x, \partial U\cap\{x_m \geq l(I)\}, \Omega\setminus\bar{U}) = u(x) - \tilde{u}(x) \tag{2.14}$$

and

$$\omega(x, (\partial U\cap\Omega)\cap\{x_m < l(I)\}, \Omega\setminus\bar{U}) = v(x) - \tilde{v}(x), \tag{2.15}$$

where u and v are bounded harmonic functions in $\mathbf{R}^m\setminus U$ with boundary values $u=1$ on $\partial U\cap\{x_m > l(I)\}$, $u=0$ on $\partial U\cap\{x_m < l(I)\}$, and $v\equiv 1-u$; and where $\tilde{u}$ and $\tilde{v}$ are bounded harmonic functions in $\Omega\setminus\bar{U}$ with boundary values $\tilde{u}(x)=u(x)$ and $\tilde{v}(x)=v(x)$ on $\partial\Omega\setminus U$, $\tilde{u}=\tilde{v}=0$ on $\partial U\cap\Omega$. By symmetry, $u(x)=v(x)=\frac{1}{2}$ on $\{x_m = l(I)\}\setminus\bar{U}$, and by the maximum principle, $\tilde{u}(x)\leq\frac{1}{2}\leq\tilde{v}(x)$ on $\partial\Omega\setminus U$. By the maximum principle again, $\tilde{u}(x)\leq\tilde{v}(x)$ in $\Omega\setminus\bar{U}$. Thus (2.13) follows from (2.14) and (2.15).

In view of theorems of Coifman and Fefferman [3], on each square I with $l(I)\leq a/10$, the measure $d\mu$ belongs to the Muckenhoupt class $A_\infty(d\sigma)$ and vice versa; moreover, there exists $p_0 > 1$ depending on ζ such that

$$\left(\frac{1}{\sigma(I)}\int_I \left|\frac{d\mu}{d\sigma}\right|^p d\sigma\right)^{1/p} \leq C(\zeta, p)\frac{\mu(I)}{\sigma(I)} \tag{2.16}$$

for $1\leq p\leq p_0$. Covering Γ by squares $\{I_j\}$ with mutually disjoint interiors and of side lengths between $a/100$ and $a/10$, we may deduce (2.8), (2.9), and

$$\int_\Gamma \left|\frac{d\mu}{d\sigma}\right|^p d\sigma \leq C(p, \zeta, d(P,\Gamma), d(P,\partial\Omega))$$

from (2.3), (2.5), and (2.16). This proves the proposition. □

Finally, we remove the restriction $\mathbf{R}^m\setminus\Omega\subseteq\{x_m\leq 0\}$, and define $\tilde{\Omega}$ and $\tilde{G}$ as in Theorem D. From (2.3) and Theorem D, it follows that

$$\begin{aligned}|\nabla G(x)| &\leq CG(P_j)\operatorname{dist}(P_j,\partial\Omega)^{-1}\\ &= C\tilde{G}(P_j)\operatorname{dist}(P_j,\partial\tilde{\Omega})^{-1}\\ &\leq C(\zeta)\omega(P, D_j, \tilde{\Omega}\setminus D_j)\operatorname{dist}(P_j,\partial\tilde{\Omega})^{-m+1}.\end{aligned}$$

Applying the proposition to $\tilde{\Omega}$, we obtain (0.3) for $1 < p < p_0$. This proves the theorem. □

3. Example 1

In $\mathbf{R}^3$, let N be a large integer, and let $P_1, P_2, \ldots, P_N$ be N points on $\{|x_1|\leq\frac{1}{2}, |x_2|\leq\frac{1}{2}, x_3=\frac{1}{2}\}$ satisfying $\operatorname{dist}(P_i,P_j)\geq N^{-1/2}$ for all $i\neq j$. Let $0=(0,0,0)$, $B_j = B(P_j, c_4N^{-1})$, $\Omega = B(0,1)\setminus\bigcup_1^N B_j$, and G be the Green function with pole at 0. Then

$$\int_{\{x_3=1/2\}\cap\Omega} |\nabla G(x)|\,d\sigma(x) \geq C\log N$$

if c_4 is sufficiently small.

We need the following lemma.

LEMMA 3. *Let $0<\rho<1/10$, and let w $(0\le w\le 1)$ be a function continuous in $\rho\le|x|\le 1$ and harmonic in $\rho<|x|<1$, with boundary value 0 on $|x|=\rho$. Then*

$$|\nabla w|\le \frac{C\rho}{|x|^2} \quad \textit{for } 2\rho<|x|<\sqrt{\rho}.$$

Proof. Write $w=w_1-w_2$, where w_1 is harmonic in $|x|<1$ with boundary values $w_1=w$ on $|x|=1$, and where w_2 is harmonic in $\rho<|x|<1$ with boundary values $w_2=w_1$ on $|x|=\rho$ and $w_2=0$ on $|x|=1$. From the Poisson integral formula, we deduce that

$$|w_1(x)-w_1(y)|\le C\rho \quad \text{for } |x|,|y|\le\rho, \tag{3.1}$$

and that

$$|\nabla w_1(x)|\le C \quad \text{on } |x|\le \tfrac{1}{2}.$$

Let w_3 be a harmonic function in $\rho<|x|<1$ with boundary values $w_3=w_1(0)$ on $|x|=\rho$ and $w_3=0$ on $|x|=1$, and let $w_4=w_2-w_3$. Because of (3.1), $|w_4(x)|\le C\rho$ on $\rho<|x|<1$. In view of Lemma 2,

$$|\nabla w_4(x)|\le \frac{C\rho}{|x|} \quad \text{for } 2\rho<|x|<\frac{1}{2}.$$

It is clear that

$$|\nabla w_3(x)|\le 2w_1(0)\frac{\rho}{|x|^2} \quad \text{for } \rho<|x|<\frac{1}{2}.$$

By combining the above estimates, we establish the lemma. □

Denote by $a=1/\sqrt{N}$ and $r=c_4N^{-1}$. We note that

$$G(x)=\frac{1}{|x|}-1-h(x),$$

where $h(x)$ is harmonic in Ω with boundary values

$$h(x)=\begin{cases} 1/|x|-1 & \text{on } \bigcup \partial B_j, \\ 0 & \text{on } |x|=1. \end{cases}$$

Let $u(x)$ be harmonic in $B(0,1)\backslash B_1$ with boundary values

$$u(x)=\begin{cases} 1/|x|-1 & \text{on } \partial B_1, \\ 0 & \text{on } |x|=1; \end{cases}$$

and let $v(x)$ be harmonic in Ω with boundary values

$$v(x)=\begin{cases} 1/|x|-1-u(x) & \text{on } \bigcup_2^N \partial B_j, \\ 0 & \text{on } \partial B_1\cup\{|x|=1\}. \end{cases}$$

Thus $h(x)=u(x)+v(x)$, and

$$|\nabla G(x)|\ge|\nabla u(x)|-|\nabla v(x)|-\frac{1}{|x|^2} \quad \text{in } \Omega. \tag{3.2}$$

To estimate $|\nabla v(x)|$ we let $V(x)=\sum_2^N(r/|x-P_j|)$, and note by the maximum principle that $|v(x)|\le 4V(x)$ in Ω. Because $\operatorname{dist}(P_i,P_j)\ge a$, we have $V(x)\le (C\sqrt{N}/a)r\le CrN$ in $|x-P_1|\le a/2$. Hence $-CrN\le v(x)\le CrN$ on $|x-P_1|=a/2$. We deduce from Lemma 3 that

$$(3.3)\qquad |\nabla v(x)|\le \frac{Cr^2N}{|x-P_1|^2}\quad \text{on } 2r\le|x-P_1|\le\sqrt{ra}.$$

To estimate $|\nabla u|$, we let $u_1(x)$ be harmonic in $B(0,1)\backslash B_1$ with boundary values

$$u_1(x)=\begin{cases}1/|P_1|-1 & \text{on } \partial B_1,\\ 0 & \text{on } |x|=1,\end{cases}$$

and $u_2(x)=u(x)-u_1(x)$. Therefore

$$|u_2|\le \sup_{x\in\partial B_1}\left|\frac{1}{|P_1|}-\frac{1}{|x|}\right|\le 16r\quad\text{in } B(0,1)\backslash B_1.$$

In view of Lemma 2,

$$(3.4)\qquad |\nabla u_2(x)|\le C\quad\text{on } 2r\le|x-P_1|\le\frac{a}{2}.$$

To estimate $|\nabla u_1|$, we let $b=(1/|P_1|-1)^{-1}$, let u_3 be harmonic in $r<|x-P_1|<1/10$ with boundary values

$$u_3(x)=\begin{cases}1 & \text{on } \partial B_1,\\ 0 & \text{on } |x-P_1|=1/10,\end{cases}$$

and let $u_4(x)=bu_1(x)-u_3(x)$. Note that $u_4(x)=0$ on $|x-P_1|=r$ and that

$$u_4(x)=bu(x)\le 20r\quad\text{on } |x-P_1|=\frac{1}{10}.$$

Therefore, by Lemma 2,

$$(3.5)\qquad |\nabla u_4|\le C\quad\text{on } 2r\le|x-P_1|\le\frac{a}{2}.$$

Note also that

$$(3.6)\qquad |\nabla u_3(x)|\ge\frac{r}{|x-P_1|^2}\quad\text{on } 2r\le|x-P_1|\le\frac{a}{2}.$$

Combining (3.2)-(3.6) we conclude that, in $2r\le|x-P_1|\le\sqrt{ra}$,

$$\begin{aligned}|\nabla G(x)|&\ge\left(\frac{1}{|P_1|}-1\right)\left(\frac{r}{|x-P_1|^2}-C\right)-C-\frac{1}{|x|^2}-\frac{Cr^2N}{|x-P_1|^2}\\ &\ge\frac{1}{10}\frac{r}{|x-P_1|^2}-\frac{cr^2N}{|x-P_1|^2}-C\\ &\ge C\frac{r}{|x-P_1|^2},\end{aligned}$$

provided that c_4 is small and N is large. Thus

$$\int_{2r\le|x-P_1|\le\sqrt{ra}}|\nabla G(x)|\,d\sigma(x)\ge cr\log\frac{a}{r}\ge C\frac{1}{N}\log N$$

and

$$\int_{\{x_3=1/2\}\cap\Omega}|\nabla G(x)|\,d\sigma(x)\ge C\log N.$$

REMARK 1. In $\mathbf{R}^3$, line segments have zero capacity, so we may obtain a simply connected domain in Example 1 by deleting from Ω N very narrow cylinders joining $\{B_j\}$ to $|x|=1$.

REMARK 2. In $\mathbf{R}^2$, given a positive integer N, let $r=e^{-CN}$, let $\{B_j\}$ be N disks with centers equally spaced on $|x|=\frac{1}{2}$ of radii r, and let $\Omega=B(0,1)\setminus\bigcup_1^N B_j$. If G is the Green function with pole at 0, and if N and C are sufficiently large, then

$$\int_{\{|x|=1/2\}\cap\Omega}|\nabla G(x)|\,ds(x)\ge cN.$$

Detail is similar to Example 1.

4. Example 2

Given $1<\alpha<2$, there exists a domain Ω in $\mathbf{R}^3$ satisfying αDC and CDC, a 2-dimensional plane Γ and a point $P\in\Omega\setminus\Gamma$ such that

$$\int_{\Gamma\cap\Omega}|\nabla G(P,x)|\,d\sigma(x)=\infty.$$

First we construct a domain when $\alpha=\log 4/\log 3$.

Let D be the snowflake domain in $\mathbf{R}^2$ constructed as follows: Let T_0 be a closed equilateral triangle with side length 1 and center $(0,0)$. After the polygon T_n is constructed, we subdivide each side of T_n into three equal subintervals and build an equilateral triangle over each middle subinterval, exterior to T_n, and with one side on that subinterval. The polygon so obtained is called T_{n+1}, which has $3\cdot 4^{n+1}$ sides of side length 3^{-n-1} each. Let D be the interior of $\bigcup T_n$ and $\gamma=\partial D$.

Corresponding to each side $I_{n,k}$ $(1\le k\le 4\cdot 3^n)$ of T_n, let $P_{n,k}$ be the center of the equilateral triangle built over the middle third of $I_{n,k}$ and let $Y_{n,k}$ be the vertex of that triangle exterior to T_n; let $Q_{n,k}=B(P_{n,k},3^{-n-5})\subseteq\mathbf{R}^2$.

We identify sets just constructed in $\mathbf{R}^2$ with sets in $\mathbf{R}^2\times\{x_3=0\}\subseteq\mathbf{R}^3$ and keep the same notations. Define in $\mathbf{R}^3$ a domain $\Omega=\{|x|<10\}\setminus\gamma$. Because of the self-similarity of γ, Ω satisfies the CDC and αDC for $\alpha=\log 4/\log 3$.

Denote by P the point $(0,0,1/20)$ and by G the Green function for Ω with pole at P. We claim that

$$\int_{\{x_3=0\}\cap\Omega}|\nabla G(x)|\,d\sigma(x)=\infty. \tag{4.1}$$

We shall actually prove that

(4.2) $$\sum_n \sum_{k=1}^{4\cdot 3^n} \int_{Q_{n,k}} |\nabla G(x)|\, d\sigma(x) = \infty.$$

Since $\{Q_{n,k}\}_{n,k}$ are mutually disjoint, (4.1) follows.

The domain Ω is not a nontangentially accessible (NTA) domain in the sense of Jerison and Kenig [9]. However, it satisfies the interior corkscrew condition and the Harnack chain condition; the CDC of Ω is a proper substitute for the exterior corkscrew condition in obtaining the estimates of harmonic functions needed. We have the following.

LEMMA 4. *There exists* $\beta > 0$ *such that for all* $Y \in \gamma \subseteq \partial\Omega$, $0 < r < 1/10$, *and every positive harmonic function* u *in* $\Omega \backslash B(0, 1/10)$: *if* u *vanishes continuously on* $\gamma \cap B(Y, r)$ *then, for* $X \in \Omega \cap B(Y, r)$,

$$u(X) \le C(|X-Y| r^{-1})^{\beta} M(u),$$

where $M(u) = \sup\{u(Z) \colon Z \in \partial B(Y, r) \cap \Omega\}$.

LEMMA 5. *If* u *is positive harmonic in* $\Omega \backslash \{|x| \le 1/10\}$ *and* u *vanishes continuously on some* $B(Y_{n,k}, 3^{-n}) \cap \gamma$, *then*

$$u(X) \le C u(P_{n,k})$$

for all $X \in B(Y_{n,k}, 3^{-n-1}) \cap \Omega$.

Lemmas 4 and 5 and their proof are analogous to Lemmas (4.1) and (4.4) in [9]. In their proofs, instead of the exterior corkscrew condition, the following simple consequence of CDC is used. We omit the proofs.

LEMMA 6. *There exists* λ $(0 < \lambda < 1)$ *such that*

$$\omega(X, \partial B(Y, r) \cap \Omega, B(Y, r) \cap \Omega) < \lambda$$

whenever $X \in \Omega$, $Y \in \partial\Omega$, $0 < r < 1/10$, *and* $|X - Y| = r/2$.

From Lemma 5 and the Harnack principle it follows that $G(X) \le CG(P_{n,k})$ in $B(Y_{n,k}, 3^{-n-1}) \cap \Omega$, and that $G(X) \approx G(P_{n,k})$ on $Q_{n,k}$. Since $G \equiv 0$ on γ, we deduce by normal family argument that there exists $c > 0$ such that

$$\int_{Q_{n,k}} |\nabla G(x)|\, d\sigma(x) \ge c G(P_{n,k}) 3^{-n}$$

for all (n, k). In view of Lemma 1 and the Harnack inequality, we have

(4.3) $$\int_{Q_{n,k}} |\nabla G(x)|\, d\sigma(x) \ge c\,\omega(0, Q_{n,k}, \Omega \backslash Q_{n,k}).$$

We need the following property of Ω.

LEMMA 7. *Each* X *in* Ω *with* $\operatorname{dist}(X, \gamma) = 3^{-n-5}$ *can be joined to some* $P_{n,k}$ *by a curve* τ *in* Ω *of length less than* $C3^{-n}$ *with* $\operatorname{dist}(\gamma, \tau) > c3^{-n}$.

Denote by $S_n = \{X: \operatorname{dist}(X, \Gamma) = 3^{-n-5}\}$. We obtain, by Lemma 7 and the Harnack principle, that

$$\sum_{k=1}^{4\cdot 3^n} \omega(X, Q_{n,k}, \Omega \backslash Q_{n,k}) > c > 0$$

for all $X \in S_n$. Since capacity$(S_n) >$ capacity$(\Gamma) > 0$,

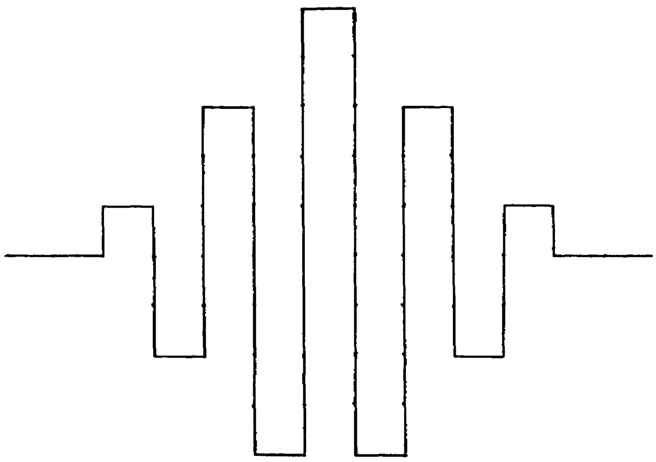
$$\sum_{k=1}^{4\cdot 3^n} \omega(0, Q_{n,k}, \Omega \backslash Q_{n,k}) > c > 0 \quad \text{for each } n. \tag{4.4}$$

Combining (4.3) and (4.4), we conclude (4.2), and thus Example 2, when $\alpha = \log 4/\log 3$.

For an arbitrary α $(1 < \alpha < 2)$, we choose a positive integer $N \equiv 1 \pmod 4$ with $\log(N^2/2 - 2N + 9/2)/\log N > \alpha$. Let I be the interval $\{0 \le x_1 \le N,\ x_2 = 0\} \subseteq \mathbf{R}^2$ and J be the polygonal path with sides parallel to the axes, symmetric about the line $x_1 = N/2$ and joining the points

$$(0,0), (1,0), (2,0), (2,1), (3,1), (3,-2), (4,-2), (4,3), (5,3), (5,-4), \ldots,$$
$$\left(\frac{N-3}{2}, -\frac{N-5}{2}\right), \left(\frac{N-1}{2}, -\frac{N-5}{2}\right), \left(\frac{N-1}{2}, \frac{N-3}{2}\right), \left(\frac{N+1}{2}, \frac{N-3}{2}\right)$$

in succession. Consider J as a polygonal path with side length 1, with vertices at all its lattice points; the path J has total length $N^2/2 - 2N + 9/2$ (see Figure 1).



J (N = 13)

Figure 1

Let S be the square with I as a diagonal, and let Q be the disk centered at $(N/2, -(N-3)/2)$ of radius $\frac{1}{2}$. We note that $\operatorname{dist}(Q, J) > \sqrt{2}/2$, and that

$$\operatorname{dist}(Q \cup J \backslash (I_1 \cup I_2), \partial S) = \frac{\sqrt{2}}{2},$$

where $I_1 = \{0 \le x_1 \le 1,\ x_2 = 0\}$ and $I_2 = \{N-1 \le x_1 \le N,\ x_2 = 0\}$.

Let T_0 be a closed unit square centered at $(0,0)$. After a polygon T_{n-1} is constructed, we shall replace each side $\tilde{I}$ of T_{n-1} by a polygonal path $F(J)$, where F is the linear transformation on $\mathbf{R}^2$ that maps I onto $\tilde{I}$ and Q into the interior of T_{n-1}. The union of these polygonal paths form the boundary of a new polygon T_n, which has $4(N^2/2-2N+9/2)^n$ sides of length N^{-n} each. Let $D=\lim T_n$ and $\gamma=\partial D$. The construction of γ is adapted from [12].

Let Ω be the domain in $\mathbf{R}^3$ defined by

$$\Omega=\{|x|<10\}\setminus(\gamma\times\{x_3=0\}).$$

Clearly Ω satisfies α_0DC and CDC for $\alpha_0=\log(N^2/2-2N+9/2)/\log N$, hence αDC since $\alpha<\alpha_0$. There exists a sequence $\{Q_{n,k}\}$ of disks on $x_3=0$ (namely, the images of Q in T_n while constructing T_{n+1}) such that each point X in Ω with $\operatorname{dist}(X,\Gamma)=N^{-n-5}$ can be joined to some $Q_{n,k}$ by a curve γ in Ω of length less than CN^{-n} with $\operatorname{dist}(\gamma,\Gamma)>cN^{-n}$.

Following the proof above, we obtain

$$\int_{\Omega\cap\{x_3=0\}}|\nabla G(x)|\,d\sigma(x)=\infty.$$

REMARK 3. Given $0<\alpha<1$, let S be the Cantor set on the interval $\{0\le x_1\le 1, x_2=0\}$ obtained by successively deleting the middle β portions of the intervals, where $\beta=1-2^{1-1/\alpha}$. Then S has dimension α, and the domain Ω in $\mathbf{R}^2$ defined by $\Omega=\{|x|<10\}\setminus S$ satisfies the αDC. Let G be the Green function on Ω with pole at any point in $\Omega\setminus\{x_2=0\}$. We may deduce as in Example 2 that

$$\int_{\{x_2=0\}\cap\Omega}|\nabla G(x)|\,dx_1=\infty.$$

Added in proof: Since the submission of this manuscript, a related paper by J. Fernández has appeared in Revista Math. Iberoamericana 5 (1989).

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