A New Proof of the Bott–Samelson Theorem

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Let (M, g) be a compact connected Riemannian manifold with $\dim M = n \geq 2$. Let i_M and d_M denote the injectivity radius and the diameter of M. Given any point p in M, let i_p and $C(p)$ denote the injectivity radius at p and the cut locus of p. We shall call M an S_l-*manifold* if, for some point $p \in M$, $C(p)$ is an l-dimensional submanifold of M. If, for every point $p \in M$, $C(p)$ is an l-dimensional submanifold, then we shall call M an ES_l-*manifold*. For example, according to the Allamigeon–Warner theorem [B], all Blaschke manifolds are ES_l-manifolds. In particular, all compact symmetric spaces of rank one (CROSSes) are ES_l-manifolds for some l. The Bott–Samelson theorem [B] can be stated in the following way.

THEOREM. *The integral cohomology ring of a Blaschke manifold is the same as that of a CROSS.*

The purpose of this note is to give a new proof of this theorem by using the Thom isomorphism theorem. More precisely, we shall prove the following two theorems.

THEOREM A. *If M is an ES_l-manifold with $l = 0$, then M is isometric to the standard unit sphere S^n up to a constant factor.*

THEOREM B. *The integral cohomology ring of an S_l-manifold M is the same as that of a CROSS. More precisely, $\pi_1(M) = 0$ or $\mathbf{Z}_2$.*

(1) *$\pi_1(M) = \mathbf{Z}_2$ if and only if $l = n-1$. In this case, M has the homotopy type of $\mathbf{R}P^n$.*
(2) *If $\pi_1(M)$ is trivial, one has only the following possibilities:*
  (a) *$l = 0$, and M is homeomorphic to S^n;*
  (b) *$n = 2m$, $l = n-2$, and M has the homotopy type of $\mathbf{C}P^m$;*
  (c) *$n = 4m$, $l = n-4$, and M has the integral cohomology ring of $\mathbf{H}P^m$;*
  (d) *$n = 16$, $l = 8$, and M has the integral cohomology ring of $\mathbf{Ca}P^2$.*

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REMARKS.

(1) Theorem B(1) is actually due to Gómez and Muñoz [GM].
(2) For any point p in an exotic sphere, one can find a metric such that $C(p)$ contains a single point.
(3) Theorems A and B support the following topological Blaschke conjectures.

CONJECTURE I. If M is an ES_l-manifold, then M is isometric to a CROSS up to a constant factor.

CONJECTURE II. If M is an S_l-manifold, then M is homeomorphic to a CROSS.

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2. Proof of Theorem A

By assumption, $l=0$ and we know that every cut locus $C(p)$ contains a single point q. Hence $C(q)=\{p\}$. Choose a point p in M with $i_p=i_M$. Without loss of generality, we can assume that $i_M=\pi$. Consider any normal geodesic γ emanating from p. γ hits q at the time π. Since $\gamma|_{[\pi,2\pi]}$ is a minimal geodesic starting from q, γ comes back to p at the time 2π. Hence γ is a geodesic loop about p. For any $\alpha\in(0,\pi)$, $\gamma|_{[\alpha,\alpha+\pi]}$ is a minimal geodesic from $\gamma(\alpha)$ to $\gamma(\alpha+\pi)$, since $i_x\geq i_p=\pi$ where $x=\gamma(\alpha)$. The curve $(-\gamma|_{[0,\alpha]})\cup(-\gamma|_{[\alpha+\pi,2\pi]})$ is from x, through p, to $\gamma(\alpha+\pi)$ and has length π. Thus it must be a minimal geodesic. Therefore γ is a closed geodesic with period 2π, and every point x on γ has injectivity radius $i_x=\pi$. This implies that $i_M=d_M=\pi$ and that M is a Wiedersehen manifold. Since the only Wiedersehen manifold is the standard unit sphere ([Be], [W], [Y]), M is isometric to the standard unit sphere up to a constant factor. □

3. Proof of Theorem B

First, if $\pi_1(M)$ is nontrivial, by the transversality theorem it is easy to see that $\dim C(p)=l=n-1$. Hence, according to the result in [GM], we have that $\pi_1(M)=\mathbf{Z}_2$ and that M has the homotopy type of $\mathbf{R}P^n$. Second, if $l=0$ then it is obvious that M is homeomorphic to S^n. Hence we can assume that $n\geq 3$ and $\pi_1(M)=0$. Let $N=C(p)$ and $i:N\to M$ be the inclusion map. Because M can be viewed as a space obtained by attaching an n-cell to N, one has [K1]:

(1) $$i_*: H_j(N)\to H_j(M) \text{ is an isomorphism for } 0\leq j\leq n-1,$$

and

(2) $$i_*: \pi_j(N)\to \pi_j(M) \text{ is an isomorphism for } 1\leq j\leq n-2.$$

In particular, $\pi_1(N)=0$, and

(3) $$H_j(M)=0 \quad \text{for } l<j<n \text{ and } H_n(M)=\mathbf{Z}.$$

Consider the ϵ-neighborhood $N(\epsilon)$ of N in M, that is,

$$N(\epsilon)=\{y\in M \mid d(y,x)<\epsilon \text{ for some } x\in N\}.$$

It is well known that, for sufficiently small ϵ, $N(\epsilon)$ can be viewed as the normal bundle of N in M via the exponential map. Since $\pi_1(M)$ and $\pi_1(N)$ are trivial, the normal bundle $E=N(\epsilon)$ is orientable. The base space N is embedded as the zero cross-section in the space E. Let E_0 be the complement of N in E. Since for every point $x\in M-(\{p\}\cup N)$ there is a unique geodesic emanating from p, through x, to $N(\epsilon)$, an easy excision argument shows that

(4) $$H_j(M,p)\cong H_j(E,E_0) \quad \text{for all } j\geq 0.$$

The Thom isomorphism for the orientable bundle E over N [MS] gives us an isomorphism:

(5) $$H_j(E,E_0)\xrightarrow{u\cap} H_{j-k}(N),$$

where $k=n-l$ and where $u\in H^k(E,E_0)$ is the Thom class.

Combining (1), (4), and (5) with the Poincaré duality, one has the isomorphisms:

(6) $$H^j(M)\xrightarrow{\cup e} H^{j+k}(M) \quad \text{for all } j\geq 0,$$

where $e\in H^k(M)$ corresponds to the Thom class u (or the Euler class of the normal bundle E over N). By (3) and the Poincaré duality again, one has

(7) $$H^j(M)=0 \quad \text{for } 0<j<k.$$

Together with (6), we know the whole integral cohomology ring of M: one has $H^*(M)\cong\mathbf{Z}(e)/e^{m+1}$, where $m=n/k$.

Now, according to a deep result in cohomology theory ([A], [Ad], [M]), if the integral cohomology ring of a manifold has only one generator then the only possibilities are:

(a) $l=0$;
(b) $l=n-2$, $n=2m$;
(c) $l=n-4$, $n=4m$;
(d) $l=8$, $n=16$.

If $l=n-2$ and $n=2m$ (i.e., if M has the same integral cohomology ring as $\mathbf{C}P^m$), then one can construct a map $f: M\to\mathbf{C}P^m$ inducing an isomorphism of cohomology rings and hence a homotopy equivalence by Whitehead's theorem ([K2], [B]). This completes the proof of Theorem B. □

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