Boundary Behavior of Certain Holomorphic Maps

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1. Introduction

Our point of departure is the recent work of Alinhac, Baouendi, and Rothschild [3] and of Bell and Lempert [4] on the boundary behavior of holomorphic maps from $\mathbf{C}$ to $\mathbf{C}^n$. In the scalar case $n=1$, the results may be formulated as follows. Let H_r denote the intersection of the open disk of radius r centered at the origin with the upper half-plane, and let σ_r denote the closed semi-circle in its boundary.

THEOREM. *Let Γ be a smooth Jordan arc in $\mathbf{C}$. Let f be a holomorphic function on H_r such that the cluster set of f along $[-r,r]$ is contained in Γ. Then*

(a1) *f extends to be continuous on $(-r,r)$,*
(a2) *f is smooth on $(-r,r)\cup H_r$, and*
(b) *f has finite order at each point of $(-r,r)$ unless f is constant.*

Part (a1) is not explicitly stated in [4] but follows from the argument there because the classical reflection principle yields such continuity. The meaning of "smoothness" is $\mathcal{C}^\infty$ for f and Γ in [4], while [3] treats f in Lipschitz spaces and Γ being $\mathcal{C}^k$ with $k\geq 2$. In the former case, "finite order" at x simply means that some derivative $f^{(N)}(x)\neq 0$; in the latter case it means not of infinite order, that is, $f(z)-f(x)=O((z-x)^N)$ does not hold for every N.

The main objective in the cited work is to handle higher-dimensional mappings where Γ is replaced by a totally real manifold. The first results of this type were due to Chirka [6]; previous work has also been done by Rosay [12] and Pinchuk and Khasanov [10]. However, according to [3], the unique continuation property (b) is new even in the scalar case. It is proved in [3] and [4] by PDE methods. We first consider the case when Γ is not assumed to be smooth but is just a (continuous) Jordan arc. It turns out that a sort of finiteness (b1) still holds, with no assumption of smoothness.

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THEOREM 1. *Let Γ be a Jordan arc (or Jordan curve) in* $\mathbf{C}$. *Let f be a holomorphic function on H_r such that the cluster set of f on $[-r,r]$ is contained in Γ. Then*

(a) *f extends to be continuous in $(-r,r)\cup H_r$.*

(b1) *For each point x in $(-r,r)$ there is, assuming f to be nonconstant, a positive integer N such that f has finite multiplicity N near x in the following sense: There exists a Jordan arc γ containing $f(x)$ in its interior and a one-to-one conformal map g of a small semi-disk centered at x and contained in H_r to a one-sided neighborhood of γ at $f(x)$, with the real axis near x being mapped by g to γ, $g(x)=f(x)$, and such that $f=f(x)+(g-g(x))^N$ near x. There are $N-1$ Jordan arcs each of which has x as one endpoint and all of whose other points lie in H_r and which are otherwise disjoint. These arcs together with the real axis split H_r locally at x into N sectors, each of which is mapped conformally by f onto a one-sided neighborhood of Γ at $f(x)$; the derivative of f is nonzero at all points of H_r sufficiently close to x.*

(b2) *Suppose that Γ has a tangent line at $f(x)$. Then, for all $\epsilon>0$,*

(i) $$\limsup_{\substack{z\to x\\ \operatorname{Im} z\geq 0}} \frac{|f(z)-f(x)|}{|z-x|^{N+\epsilon}}=\infty,$$

(ii) $$\liminf_{\substack{z\to x\\ \operatorname{Im} z\geq 0}} \frac{|f(z)-f(x)|}{|z-x|^{N-\epsilon}}=0,$$

(iii) *if f vanishes to infinite order at x, then $f\equiv f(x)$ in H_r.*

The hypothesis that Γ have a tangent line at $f(x)$ means by definition that the real tangent cone to the set Γ at the point $f(x)$ is a real line. Part (b1) shows that near a point $x\in(-r,r)$, f behaves geometrically much like a function which is analytic in a full neighborhood of a point; that is, after a (one-sided) change of coordinate, f looks like a power of z. In general (b2) does not hold for $\epsilon=0$, even if Γ is a $\mathcal{C}^1$ curve. Rosay [13] discusses the example $f(z)=z\log(1/z)$ at $x=0$ where $N=1$ and the image of the real axis is $\mathcal{C}^1$. Clearly in this case (ii) fails with $\epsilon=0$. Similarly the function $f(z)=z/\log(1/z)$ is $\mathcal{C}^1$ on H_r and maps the real axis to a $\mathcal{C}^1$ curve through the origin. It satisfies (i) and (ii) for $N=1$, but $f'(0)=0$ and (i) fails for $\epsilon=0$. If, on the other hand, an f is known to be $\mathcal{C}^{N+\alpha}$ (where $\alpha>0$) on $[-r,r]$, then (i) and (ii) imply that $f^{(k)}(x)=0$ for $0<k<N$ and $f^{(N)}(x)\neq 0$. The proof of (b2) involves very simple estimates of extremal length in certain quadrilaterals.

In considering (a) of Theorem 1 it is natural to ask, more generally, what properties of a set E, assumed to be a continuum (i.e., compact, connected, and not reducing to a single point), ensure that a holomorphic function f on H_r with cluster set on $[-r,r]$ contained in E always extends continuously to $(-r, r)$. One such property, that E have finite 1-dimensional Hausdorff measure, was conjectured by Globevnik and Stout [9] and verified independently using different methods by Pommerenke [11] and in [2]. More general results involving local connectivity were subsequently obtained by Carmona and Cufí [5]; their hypotheses are directly on the cluster set of f, rather than on a containing set E, and have a global character as they take f to be holomorphic on the unit disk with hypotheses on its global cluster set. The following gives a simple necessary condition for E which can often be easily checked; for example, (a) of Theorem 1 is an obvious consequence. The method of proof uses ideas from [2] which will also be useful for considering mappings into $\mathbf{C}^n$ in Theorem 3 below.

THEOREM 2. *Let E be a plane continuum with empty interior with the following property: For every continuum Q contained in E and every point p in Q there exists a Jordan curve J such that*

(i) $p \notin J$,
(ii) *$E \cap J$ is finite, and*
(iii) *Q meets both components of the complement of J.*

Let f be a holomorphic function on H_r whose cluster set along $[-r, r]$ is contained in E. Then f extends to be continuous along $(-r, r)$.

The previously mentioned result of the case when E has finite linear measure is a simple corollary; indeed, using the fact that almost every vertical line hits E finitely often, one can easily produce J satisfying (i), (ii), and (iii).

Our last result involves extending Theorem 1 to the case when the Jordan curve Γ lies in $\mathbf{C}^n$. It is not known whether this result holds for an arbitrary continuous Jordan curve Γ. Globevnik and Stout [7] have shown, among other things, that if Γ is a rectifiable Jordan curve in $\mathbf{C}^n$ and if f is a bounded proper holomorphic mapping of the open unit disk into $\mathbf{C}^n \setminus \Gamma$ (this amounts to saying that f has cluster set contained in Γ), then f extends to be continuous on the unit circle. In fact, since coordinate projection decreases linear measure, the continuity of the component functions of f follows immediately from the result (mentioned above) on functions with cluster sets of finite linear measure. In this connection, an example of Globevnik and Stout [8, Ex. 8] is perhaps relevant. They showed that f can be one-to-one and regular on the open unit disk, continuous on the closed disk, and map the circle in a two-to-one fashion onto a Jordan curve Γ.

In order to formulate a result which includes both the case of a Jordan curve in $\mathbf{C}$ where no smoothness is required as well as the higher-dimensional case where the assumption that Γ be $\mathcal{C}^1$ is sufficient, we utilize a notion of complex tangent cone to Γ. We define the complex tangent cone $TC(p)$ of Γ at $p \in \Gamma$ as follows: $v \in \mathbf{C}^n$ is in the tangent cone $TC(p)$ if there exist $q_n \in \Gamma$ and $p_n \in \Gamma$ converging to p and $z_n \in \mathbf{C}$ such that $z_n(q_n - p_n)$ converges to v. Whitney ([14], [15]) discusses a number of different complex tangent cones, of which this is of his type C_5. In general, $TC(p)$ is a union of complex lines.

For our purposes, to define the real tangent cone at p (a notion which we have already used above), we take the z_n to be real and positive and $p_n = p$ for all n; this is the real analogue of Whitney's C_3 type tangent cone.

THEOREM 3. *Let Γ be a Jordan curve in $\mathbf{C}^n$ such that, at a dense set of points p in Γ, $TC(p)$ is a complex line. Let f be a holomorphic map from H_r such that the cluster set of f along $[-r, r]$ is contained in Γ. Then*

(a) *f extends to be continuous on $(-r, r) \cup H_r$.*
(b) *Moreover, if $TC(f(x))$ is a complex line for $x \in (-r, r)$, and if the real tangent cone to Γ at $f(x)$ is a real line, then $f - f(x)$ does not vanish to infinite order at x unless f is constant.*

Two cases where the complex tangent cone hypothesis is clearly satisfied at a point p are: (a) locally, near p, when Γ is contained in a holomorphic 1-variety which is nonsingular at p; and (b) when Γ is a $\mathcal{C}^1$ manifold near p.

2. Proof of Theorem 1(b1)

As noted above, (a) follows from Theorem 2. We shall thus assume that f is continuous up to the real axis and, in proving (b1) and (b2), we may assume that $x = 0$ and that $f(x) = 0$.

We next define two multiplicities $N_1(f)$ and $N_2(f)$ for f at $x = 0$. By decreasing r, we may assume that f is continuous on $\bar{H}_r$ and that $f \neq 0$ on the closed circular part σ_r of ∂H_r, since, taking f nonconstant, $f \neq 0$ on a dense subset of $[-r, r]$. Then $f(\sigma_r)$ is a curve with endpoints on Γ and $f(\sigma_r)$ is bounded away from 0. Now Γ subdivides a small neighborhood of 0 into two components, Ω_1 and Ω_2, which are disjoint from $f(\sigma_r)$ (and Γ) if the neighborhood is sufficiently small. Then $f(\partial H_r)$ is the union of $f(\sigma_r)$ and the subset $f[-r, r]$ of Γ. The winding number of $f(\partial H_r)$ about all points of Ω_1 is the same, call it $N_1(f)$; likewise define $N_2(f)$. These numbers cannot both be zero, and moreover they decrease as r decreases. We may assume that r is chosen so small that $N_j(f)$ are minimal. We claim that this implies that $f \neq 0$ on $\bar{H}_r \backslash \{0\}$. In fact, if $f(z) = 0$ for some $z \in \bar{H}_r$ $(z \neq 0)$ then, for some $z' \in H_r$ near z, $f(z')$ is in Ω_1 or Ω_2. But by decreasing r so that z' is not $\bar{H}_r$ we could then, by the argument principle, decrease $N_1(f)$ or $N_2(f)$, a contradiction of minimality.

The origin subdivides Γ into two closed "legs" meeting only at the origin. We consider two cases. In the first case, the endpoints of $f(\sigma_r)$ lie on the same leg (and, as we know, are nonzero). We can deform $f([-r, r])$ in Γ to the arc in Γ joining $f(-r)$ and $f(r)$. This does not change the winding numbers $N_1(f)$ and $N_2(f)$. Since the deformed closed curve (we have $f(\sigma_r)$ unchanged during the deformation) is bounded away from 0, we conclude that $N_1(f) = N_2(f)$.

In the second case, the endpoints of $f(\sigma_r)$ lie on different legs of Γ. We again deform $f(\partial H_r)$ to a curve τ by deforming $f([-r, r])$ to the arc in Γ

joining $f(-r)$ to $f(r)$. Since τ agrees with Γ locally at 0, we conclude that the winding numbers $N_1(f)$ and $N_2(f)$ differ by 1. By relabelling, we may assume that $N_1(f)=N_2(f)+1$. We define $N=N(f)\equiv N_1(f)+N_2(f)$; $N\geq 1$.

Set $g=f^{1/N}$ (choose a branch). Then g is continuous on $\bar{H}_r$, $g=0$ only at $z=0$, and g is holomorphic on H_r. To define γ, consider the images of the two legs Γ_+ and Γ_- of Γ under the maps $w\mapsto w^{1/N}$. We get a set of $2N$ Jordan arcs through 0 which are otherwise disjoint. Since $g=0$ only at 0, g maps $\mathbf{R}^+$ and $\mathbf{R}^-$ each into one of the $2N$ arcs. If we get two such arcs, let γ be their union. If they should coincide (we shall see below that in fact they are different), choose γ to be the union of the image arc and any other of the $2N-1$ remaining arcs.

Now g becomes a mapping satisfying the hypotheses that f does, except that Γ is replaced by γ; in particular, we have that $N_1(g)$ and $N_2(g)$ are well defined.

LEMMA 1. $N_1(g)=1$ *and* $N_2(g)=0$.

Proof. Choose $q\in\Omega_1(\Gamma)$ close to 0 such that q is a regular value of f and such that the N roots $q^{1/N}=b_1,b_2,\ldots,b_N$ are regular values of g and such that none of the b_k's are contained in γ. By reordering, we may assume that there is an s $(0\leq s\leq N)$ such that $b_k\in\Omega_1(\gamma)$ for $k\leq s$ and $b_k\in\Omega_2(\gamma)$ for $s<k\leq N$. Set $h(\zeta)=\zeta^N$. Since $h\circ g=f$,

$$f^{-1}\{q\}=g^{-1}\{b_1,b_2,\ldots,b_N\}.$$

Counting these sets in two ways we get, by the argument principle,

$$N_1(f)=s\cdot N_1(g)+(N-s)N_2(g)\equiv A.$$

Hence, since $N_1(g)\geq N_2(g)$,

$$N\geq N_1(f)=A\geq N_2(g)\cdot N.$$

This gives $1\geq N_2(g)$.

If $N_2(g)=1$ then all of the above inequalities are really equalities. Hence $N=N_1(f)$ and $N_1(g)=N_2(g)=1$. Therefore $N_2(f)=0$ and so $N_1(f)=1$ and $N=1$. Hence $g=f$ and $N_2(g)=N_2(f)=0$, a contradiction.

Therefore, $N_2(g)=0$ and so $N_1(g)=1$ and we are through. □

Continuing with the proof of (b1), we have $f=g^N$, where g is a conformal map of a one-sided neighborhood of 0 in H_r to a one-sided neighborhood $\Omega_1(\gamma)$ of γ at 0.

Consider the $2N$ arcs $\Gamma_+^{1/N}$ and $\Gamma_-^{1/N}$ discussed above. Some of these lie in $\Omega_1(\gamma)$ and subdivide $\Omega_1(\gamma)$ into sectors each of which is mapped by $\zeta\mapsto\zeta^N\equiv h(\zeta)$ to a one-sided neighborhood of Γ at 0. Since $f=h\circ g$ and $N_2(g)=0$, we conclude that in $\Omega_1(\gamma)$ there are $N_1(f)$ sectors which are mapped to $\Omega_1(\Gamma)$ and $N_2(f)$ sectors which are mapped to $\Omega_2(\Gamma)$. Thus there are N sectors in $\Omega_1(\gamma)$. Pulling these sectors back to H_r by g^{-1} gives sectors claimed for H_r. Finally, since g is conformal and one-to-one, we have $g'\neq 0$ and hence $f'\neq 0$ in H_r. □

3. Proof of Theorem 1(b2)

The main step will be the following lemma. We shall consider quadrilaterals of the form

$$S(r_1, r_2, \theta_1, \theta_2) = \{re^{i\theta} : r_1 < r < r_2 \text{ and } \theta_1 < \theta < \theta_2\},$$

where $\theta_2 < \theta_1 + 2\pi$; the four distinguished points are the intersections of the boundary circular arcs with the boundary radial arcs. Consider the family C of curves joining the two circular arcs. Then the extremal length $\lambda(C) = \log(r_2/r_1)/(\theta_2 - \theta_1)$. In fact, $\log z$ maps the quadrilateral to the rectangle $\{z : \log r_1 < \operatorname{Re} z < \log r_2,\ \theta_1 < \operatorname{Im} z < \theta_2\}$ conformally. We just employ the known extremal length for the rectangle; see Ahlfors [1, p. 53].

LEMMA 2. *Let γ_1 and γ_2 be Jordan arcs containing the origin as an interior point and suppose that γ_1 and γ_2 each have a tangent line at 0. Let φ be a one-to-one conformal map of a one-sided neighborhood W_1 of γ_1 at 0 to a one-sided neighborhood W_2 of γ_2 at 0 such that $\varphi(\gamma_1) \subseteq \gamma_2$ and $\varphi(0) = 0$. Then, for all $\epsilon > 0$,*

(i)
$$\limsup_{\substack{z \to 0 \\ z \in W_1}} \frac{|\varphi(z)|}{|z|^{1+\epsilon}} = \infty$$

and

(ii)
$$\liminf_{\substack{z \to 0 \\ z \in W_1}} \frac{|\varphi(z)|}{|z|^{1-\epsilon}} = 0.$$

Proof. For (i), it suffices to show that the lim sup is positive for all $\epsilon > 0$. Suppose not. Then $|\varphi(z)| \le |z|^{1+\epsilon}$ for all $z \in W_1$ with $|z|$ sufficiently small, say for $|z| \le \rho$.

We may assume that the tangent line to γ_1 and to γ_2 at $z = 0$ is the real axis. Choose $\eta > 0$. We may also suppose that ρ is sufficiently small so that, if $w \in \partial W_j$ $(0 < |w| < \rho)$, then $w \in \gamma_j$ and thus either $|\arg w| < \eta$ or $|\arg w - \pi| < \eta$. Take $0 < \delta < \rho$. It follows that $S_1 \equiv S(\delta, \rho, \eta, \pi - \eta)$ is contained in W_1. Choose $R > 0$ such that $|\varphi(z)| \ge R$ if $z \in W_1$ and $|z| = \rho$. We have that if $z \in W_1$ and $|z| = \delta$ then $|\varphi(z)| \le \delta^{1+\epsilon}$. Then $\varphi(S_1) \subseteq W_2$, and the images of the circular arcs of S_1 which constitute two edges of the quadrilateral $\varphi(S_1)$ are separated by the circular edges of $S_2 \equiv S(\delta^{1+\epsilon}, R, -\eta, \pi + \eta)$. Since the radial edges of S_2 lie outside W_2, they are separated by the corresponding edges of $\varphi(S_1)$. By the fundamental comparison principle on extremal length ([1, p. 54], cf. Fig. 4-1 there), we have $\lambda(\varphi(C_1)) \ge \lambda(C_2)$, where C_j is the family of curves joining the circular boundary curves of S_j. By conformal invariance, $\lambda(C_1) \ge \lambda(C_2)$. Hence

$$\frac{\log(\rho/\delta)}{\pi - 2\eta} \ge \frac{\log(R/\delta^{1+\epsilon})}{\pi + 2\eta}.$$

Dividing by $\log(1/\delta)$ and letting $\delta \to 0$, we have

$$\frac{1}{\pi-2\eta} \geq \frac{1+\epsilon}{\pi+2\eta}.$$

Letting $\eta \to 0$ gives $1/\pi \geq (1+\epsilon)/\pi$, a contradiction. We conclude that (i) holds.

To obtain (ii) we need only apply (i) to $\psi = \varphi^{-1}$. □

To complete the proof of (b2), consider the $2N$ arcs, $\Gamma_+^{1/N}$ and $\Gamma_-^{1/N}$, discussed above. Since the tangent cone to Γ at 0 is a real line, it follows that the tangent cones at 0 to any two consecutive arcs of the $2N$ arcs are rays through the origin making an angle of π/N. Hence the curve γ constructed above and associated to the mapping $g = f^{1/N}$ is such that its two legs γ_+ and γ_- make an angle of π at 0. Thus we can apply Lemma 2 with $\varphi = g$, $\gamma_1 = \mathbf{R}$, and $\gamma_2 = \gamma$ to conclude that

$$\limsup_{\substack{z \to 0 \\ \operatorname{Im} z \geq 0}} \frac{|g(z)|}{|z|^{1+\epsilon}} = \infty,$$

and a corresponding statement for the lim inf. Then (i) and (ii) of the theorem follow from $|g(z)|^N \equiv |f(z)|$. Part (iii) follows since (i) and (ii) hold unless f is constant. □

4. Proof of Theorem 2

It suffices to show that f is continuous at $x=0$. By decreasing r, we may suppose that f is bounded on H_r and holomorphic on $\bar{H}_r \cap \{\operatorname{Im} z > 0\}$, that $f' \neq 0$ on σ_r^0, and that f has nontangential limits a.e. on $(-r, r)$. We argue by contradiction and suppose that the cluster set Q of f at 0 is a set of more than one point; that is, Q is a continuum contained in E.

Our hypothesis gives, for an arbitrary choice of p in Q, a Jordan curve J which meets E in a finite set $\{p_1, p_2, \dots, p_N\}$. Then $J \backslash E$ is a finite union of open Jordan arcs $\gamma_1, \gamma_2, \dots, \gamma_N$, where the endpoints of γ_k are p_k and p_{k+1} ($p_{N+1} = p_1$). By modifying the γ_k we may assume that each γ_k contains none of the singular values of f, the latter set being countable. Then all components of $f^{-1}(\gamma_k)$ in H_r are open arcs and these are mutually disjoint.

Consider a component τ of $f^{-1}(\gamma_k)$ for any fixed k. Consider the cluster set in $\bar{H}_r$ of one of the two "ends" of τ. If this cluster set meets σ_r^0 at some point b, then $f(b) \in \bar{\gamma}_k$ and f is locally one-to-one at b, and so τ continuously approaches b. Otherwise, the cluster set of the end of τ is a connected subset of $A \cup [-r, r]$, where A is the countable set $f^{-1}\{p_1, p_2, \dots, p_N\}$. If this (connected!) cluster set meets A then again it is a single point in A, by connectedness. Finally the cluster set could be a subinterval of $[-r, r]$. We claim this subinterval K reduces to a point. Suppose not. Since f maps τ to γ_k, the cluster set of f along the end of τ is a single point, either p_k or p_{k+1}. This implies that the nontangential limit of f is a.e. on K equal to p_k or p_{k+1}. This implies that f is constant, which we are assuming is not the case. We conclude that K reduces to a single point. We have shown then that τ has a limit in $\bar{H}_r$ along both its "ends."

LEMMA 3. *For each k $(1 \leq k \leq N)$ there exists ρ $(0 < \rho < r)$ such that only finitely many components of $f^{-1}(\gamma_k)$ meet H_ρ.*

Proof. Fix $q \in \gamma_k$. Choose ρ $(0 < \rho < r)$ such that

(a) $f^{-1}(q) \cap H_\rho$ is empty, and

(b) the limit of f along σ_ρ at $\pm\rho$ exists and is not equal to p_j for $1 \leq j \leq N$.

Clearly (a) is true for all ρ sufficiently small since $q \notin E$, and (b) holds for all ρ except a set of measure zero.

From the above discussion of the components of $f^{-1}(\gamma_k)$, it follows that each component τ is one of two types:

(1) $f(\tau) = \gamma_k$, or

(2) $f(\tau) \subsetneqq \gamma_k$.

In type (2), at least one endpoint b of τ lies on σ_r^0 and satisfies $f(b) \in \bar{\gamma}_k$.

We argue by contradiction and suppose that $f^{-1}(\gamma_k)$ contains an infinite set of distinct components $\{\tau_j\}$ such that each τ_j meets H_ρ. We may assume that either all τ_j are of type 1 or all are of type 2.

Suppose that all τ_j are of type 1. Then there is a $q_j \in \tau_j$ such that $f(q_j) = q$. By (a), q_j is not in H_ρ. Hence τ connects q_j to a point in H_ρ and so meets σ_ρ^0 at a point z_j. Let $z \in \sigma_\rho$ be a limit point of the z_j. If $z \in \sigma_\rho^0$ then $f(z) \in J$ and $f^{-1}(J)$ is, locally at z, just a finite set of arcs through z which are otherwise disjoint, as f' vanishes at most to finite order at z. This contradicts the fact that $f^{-1}(J) \supseteq \{\tau_k\}$ which cluster at z. The alternative is that z be an endpoint of σ_ρ. Then $f^*(z) \in E \cap J$. But $f^*(z) \neq p_j$ by (b) for all j, another contradiction.

Now suppose that all τ_j are of type 2. Then τ_j joins some point of H_ρ to σ_r and so again meets σ_ρ^0 at some point z_j. Just as before, this leads to a contradiction. This proves the lemma. □

Applying Lemma 3 for each k, $1 \leq k \leq N$, we see that if ρ is sufficiently small then $f^{-1}(\gamma_k) \cap H_\rho$ contains a finite number of components for all k. We claim that one of these components converges to 0, at one endpoint. Suppose not. Then, since the number of these components in H_ρ is finite, there is a δ $(0 < \delta < \rho)$ such that $H_\delta \cap f^{-1}(\gamma_k)$ is empty for all k. Consider the open set $W = f(H_\delta)$. Then $W \cap J$ could contain at most the set $\{p_1, p_2, \ldots, p_N\}$. Since W is open, we conclude that W is disjoint from J. Since W is connected, then either W lies in the bounded component of the complement of J or in the unbounded component. Hence $\bar{W}$ does not meet both components of $\mathbf{C} \setminus J$. As $Q \subseteq \bar{W}$, this is a contradiction. We conclude that there is an arc τ in H_r which goes to 0 such that the limit of f along τ at 0 exists and equals some $p_k \in J \cap Q$ (since $f(\tau) \subseteq J$).

Now apply the hypothesis of the theorem again with $p = p_k$ and Q as before to get a Jordan curve J' not containing $p = p_k$. Repeating the above argument, we get another τ' which goes to 0 such that the limit of f along τ' at 0 exists and equals some $p' \in J'$. As $p' \neq p_k$, we get a contradiction to a classical theorem of Lindelöf. We conclude that f extends to be continuous at $x = 0$. □

5. Proof of Theorem 3(a)

It suffices to show that f is continuous at $x = 0$. For this we need the following lemma, which says that the image of f, in some sense, looks like the image of a continuous function near a dense set of points of Γ.

LEMMA 4. *There exists a dense set of points p in Γ such that, after possibly decreasing r, there exists an affine change of coordinates in $\mathbf{C}^n$ with the following properties: The point p has coordinates 0. There exist an arbitrarily small neighborhood W_1 of 0 in $\mathbf{C}^1$ and an arbitrarily small neighborhood W_2 of 0 in $\mathbf{C}^{n-1}$ such that, setting $W = W_1 \times W_2 \subseteq \mathbf{C}^n$, we have*

$$(\Gamma \cup \overline{f(H_r)}) \cap (\bar{W}_1 \times \partial W_2) = \emptyset \quad \text{and} \quad (\overline{f(H_r)} \backslash f(H_r)) \cap W \subseteq \Gamma \cap W.$$

Moreover, there exists an open Jordan arc γ in W_1 such that $W_1 \backslash \gamma$ is a union of two disjoint, nonempty domains Ω_1 and Ω_2. The coordinate function z_1 maps $(\Gamma \cup f(H_r)) \cap W$ homeomorphically to a subset L of W_1 and maps $\Gamma \cap W$ homeomorphically to γ. There are four possible cases:

(1) $L = \gamma \cup \Omega_1$,
(2) $L = \gamma \cup \Omega_2$,
(3) $L = W_1$,
(4) $L = \gamma$.

Assuming Lemma 4 for the present, we continue with the proof of the theorem. Arguing by contradiction, we suppose that the cluster set Q of f at $x = 0$ does not reduce to a single point. Then Q is a subarc of Γ. Choose an interior point $p \in Q$ where Lemma 4 holds. Choose W sufficiently small so that $\Gamma \cap W \subseteq Q$. Then, by Lemma 4, z_1 restricted to $W \cap (\Gamma \cup f(H_r))$ maps homeomorphically to $L \subseteq W_1$. Let ψ be the inverse map.

Choose a Jordan curve J_0 in W_1 such that $J_0 \cap \gamma$ contains two points and such that γ meets both components $\mathbf{C} \backslash J_0$. Set $J = \psi(J_0) \subseteq W$.

Then, if case (3) of the lemma holds, J is a Jordan curve such that $J \cap \Gamma$ consists of two points p_1 and p_2, and $J \backslash \Gamma$ is a union of two open Jordan arcs γ_1 and γ_2 both contained in $f(H_r)$. If cases (1) and (2) hold, then J is a Jordan arc with endpoints p_1 and p_2, $J \cap \Gamma = \{p_1, p_2\}$, and $\gamma_1 \equiv J \backslash \Gamma \subseteq f(H_r)$. We set $\gamma_2 = \emptyset$ for a uniform notation. Note that case (4) of the lemma does not hold, since Q is the cluster set of f at 0 and $Q \cap W = \Gamma \cap W \neq \emptyset$.

By choosing r and W sufficiently small we may assume that the closure of $f(\sigma_r)$ is disjoint from W. Also we may choose J_0 to be disjoint from the singular values of f_1. Now consider a component τ of $f^{-1}(\gamma_k)$ in H_r for $k = 1, 2$. Just as in the proof of Theorem 2, one shows that τ has a limit at each of its "ends"; just as before, τ is an open arc in H_r. Lemma 3 is also valid for f, the proof being the same except that, since $f(\sigma_r)$ is disjoint from W, there are no τ of type (2). Hence, if ρ is sufficiently small then $f^{-1}(\gamma_k)\cap H_\rho$ contains a finite number of components for $k=1,2$. We claim that one of these components τ has 0 as an endpoint. If not, then for some δ $(0<\delta<\rho)$, $H_\delta \cap f^{-1}(\gamma_k)$ is empty for $k=1,2$.

Consider $V=f(H_\delta)$. Let Ω_0 be the bounded component of $\mathbf{C}\setminus J_0$. V is disjoint from γ_1 and γ_2, hence V is disjoint from J. Set $V_1=V\cap\psi(\Omega_0)$ and $V_2=V\setminus\psi(\bar{\Omega}_0)$. Then V is a disjoint union of its relatively open sets V_1 and V_2. Since V is connected, it follows that (i) $V=V_1$ or (ii) $V=V_2$. Since $Q\subseteq\bar{V}$, it follows that (i) $Q\subseteq\psi(\bar{\Omega}_0)$ or (ii) Q is disjoint from $\psi(\Omega_0)$. In view of the fact that $Q\cap W=\Gamma\cap W$, either one of these possibilities yields a contradiction. We conclude that some component τ of $f^{-1}(\gamma_k)$ for $k=1$ or 2 has 0 as one endpoint and that $f(z)$ has a limit equal to p_1 or p_2 as z approaches 0 along τ. As before, by choosing a different J_0 disjoint from the first choice, we get a different τ' with 0 as an endpoint along which f has a different limit. This contradicts Lindelöf's theorem, in its obvious vector formulation. We conclude that Q reduces to a single point; that is, f is continuous at $x=0$. □

Now we prove Lemma 4. Take any point $\tilde{p}$ in Γ where $TC(\tilde{p})$ is a complex line; such $\tilde{p}$ are dense in Γ. Without loss of generality, we may assume that $TC(\tilde{p})$ is the z_1 axis. Let π denote the coordinate projection $\pi(z)=z_1$ and write, for $z\in\mathbf{C}^n$, $z=(\pi(z),z')$ with $z'\in\mathbf{C}^{n-1}$. We claim that π is one-to-one on a neighborhood of $\tilde{p}$ in Γ. If not, there would exist $\{p_n\},\{q_n\}\subseteq\Gamma$ with $p_n\neq q_n$, $\pi(p_n)=\pi(q_n)$, $p_n\to\tilde{p}$, and $q_n\to\tilde{p}$. Then some subsequence of $(q_n-p_n)/\|q_n-p_n\|$ converges to a $v\in TC(\tilde{p})$ with $\|v\|=1$ and $\pi(v)=0$. Since π is injective on $TC(\tilde{p})$, this is a contradiction. Hence there exists an open Jordan arc γ in $\mathbf{C}$ containing $\pi(\tilde{p})$ such that γ is the homeomorphic image by π of a neighborhood of $\tilde{p}$ in Γ.

By decreasing r, we may assume that $f(\sigma_r)$ is bounded away from $\tilde{p}$. Thus, if $\tilde{W}$ is a sufficiently small neighborhood of $\tilde{p}$, so that $\tilde{W}$ is disjoint from $f(\sigma_r)$, then, by the proper mapping theorem, $(f(H_r)\cap\tilde{W})\setminus\Gamma$ is a subvariety of complex dimension 1 of $\tilde{W}\setminus\Gamma$.

Consider the set $\pi^{-1}(\pi(\tilde{p}))\cap(f(H_r)\cup\Gamma)$, viewed as a subset of $\mathbf{C}^{n-1}=\{z_1=\pi(\tilde{p})\}$. This set, near $\tilde{p}'\in\mathbf{C}^{n-1}$, consists of $\tilde{p}'$ and a countable set of points in $f(H_r)\setminus\Gamma$. Let W_2 be an arbitrarily small neighborhood of $\tilde{p}'$ in $\mathbf{C}^{n-1}$ such that ∂W_2 is disjoint from $f(H_r)\cup\Gamma$. Now, if W_1 is a sufficiently small neighborhood of $\pi(\tilde{p})$ in $\mathbf{C}$, then $W\equiv W_1\times W_2\subseteq\tilde{W}$ and $f(H_r)\cup\Gamma$ is disjoint from $\bar{W}_1\times\partial W_2$. We can choose W_1 such that π maps $\Gamma\cap W$ homeomorphically to $\gamma\subseteq W_1$ and such that γ divides W_1 into two nonempty domains Ω_1 and Ω_2, with $W_1\setminus\gamma=\Omega_1\cup\Omega_2$.

Let $V_j=f(H_r)\cap(\Omega_j\times W_2)$, $j=1,2$. Then V_j is a subvariety of $\Omega_j\times W_2$ and $\pi: V_j\to\Omega_j$ is a proper holomorphic map with some multiplicity m_j, $j=1,2$.

We claim that, by replacing W_1 and W_2 by smaller sets W_1' and W_2', we can arrange that the restriction of π to $(f(H_r)\cup\Gamma)\cap(W_1'\times W_2')$ has fibers over γ consisting of one point: the unique point of $\Gamma\cap(W_1'\times W_2')$ lying over a given point of γ. The smaller set $W_1'\times W_2'$ may not contain $\tilde{p}$, but it will contain some point $p\in\Gamma$.

Consider a point $q\in(f(H_r)\setminus\Gamma)\cap(W_1\times W_2)$ such that $\pi(q)\in\gamma$. Then π restricted to $f(H_r)$ is an open map near q and so maps each neighborhood of q in $f(H_r)$ to a neighborhood of $\pi(q)$ in $\mathbf{C}$ which consequently meets both Ω_1 and Ω_2. We conclude that for each $\lambda\in\gamma$ there are at most $1+\min(m_1,m_2)$ points q in $(f(H_r)\cup\Gamma)\cap(W_1\times W_2)$ such that $\pi(q)=\lambda$. For each $\lambda\in\gamma$, let $m(\lambda)$ be the (finite) number of such q. Choose λ_0 such that $m(\lambda_0)$ is maximal and let $q_1,q_2,\ldots,q_t$ $(t=m(\lambda_0))$ be the corresponding points in $W_1\times W_2$ with $q_1\in\Gamma$ and $q_j\in f(H_r)\setminus\Gamma$ for $2\le j\le t$; $\pi(q_j)=\lambda_0$ for $1\le j\le t$. For $2\le j\le t$, choose small neighborhoods N_j of q_j in $f(H_r)$ such that the N_j are mutually disjoint and bounded away from q_1. Then $\pi(N_j)$ is a neighborhood of λ_0. Choose W_1' a neighborhood of λ_0 in $\bigcap_2^t\pi(N_j)\subseteq W_1$ and W_2' a neighborhood of $q_1'\in\mathbf{C}^{n-1}$, where $W_2'\subseteq W_2$ such that $W_1'\times W_2'$ is disjoint from all N_j, $2\le j\le t$. By the maximality of $t=m(\lambda_0)$, we conclude that for $\lambda\in\gamma\cap W_1'$ the only point $q\in(W_1'\times W_2')\cap(f(H_r)\cup\Gamma)$ such that $\pi(q)=\lambda$ is the unique point $q\in\Gamma$ lying over λ. We can choose W_1' and W_2' so that π maps $\Gamma\cap(W_1'\times W_2')$ homeomorphically to $\gamma'=W_1'\cap\gamma$ and that $W_1'\setminus\gamma'=\Omega_1'\cup\Omega_2'$, two nonempty domains. By our construction, W_1' and W_2' have the desired property. We now change notation, dropping the primes, and thus may assume that π restricted to $f(H_r)\cup\Gamma$ has one-point fibers over γ. For the point p we take q_1.

We claim the multiplicities m_1 and m_2 over Ω_1 and Ω_2 now satisfy $m_j=0$ or 1, $j=1,2$. To see $m_1=0$ or 1 we argue by contradiction (the same argument for m_2). Suppose $m_1\ge 2$. We define a bounded holomorphic function $F(\lambda)$ for $\lambda\in\Omega_1$ as follows. We have $\pi: V_1\to\Omega_1$ an analytic cover of multiplicity m_1. For $\lambda\in\Omega_1$, set $\{w_1,w_2,\ldots,w_{m_1}\}=\pi^{-1}(\lambda)\subseteq V_1$, counting multiplicity. Set $F(\lambda)=(\prod_{i<j}(P(w_i)-P(w_j)))^2$, where P is a polynomial chosen so that F is not identically zero on Ω_1, but, as is well known, is holomorphic there. But if $\lambda_0\in\gamma$ and if $\lambda\in\Omega_1$ approaches λ_0, then $F(\lambda)$ approaches 0 because the fiber over λ_0 of $\pi\,|\,\overline{f(H_r)}$ has a single point. It follows, say by Rado's theorem, that $F\equiv 0$ in Ω_1, a contradiction. We conclude that $m_1=0$ or 1.

Now the four cases of the lemma follow from the four cases $m_1=0$ or 1, $m_2=0$ or 1. Since the original set of points $\{\tilde{p}\}\in\Gamma$ was dense in Γ, it is clear that the points $\{p\}$ are also dense. This completes the proof of the lemma. □

6. Proof of Theorem 3(b)

We may assume that $x=0$ and that $p=f(0)=0$. We may further assume that $TC(0)$ is the z_1 axis. As in the proof of Lemma 4, the projection π to the first coordinate is one-to-one on a neighborhood of 0 in Γ, and maps this neighborhood homeomorphically to a Jordan arc γ through 0 in the z_1-plane. Then, assuming that $f=(f_1, f_2, \dots, f_n)$ is continuous on $H_r \cup (-r, r)$ by (a) but not constant, it follows that f_1 is not constant.

Let l be the real tangent cone to Γ at 0, which, by hypothesis, is a real line. Clearly l is contained in the z_1 axis $= TC(0)$. It is easy to check that l is then also the real tangent cone to γ at 0. Hence we can apply Theorem 1(b2) to f_1 for some $N \geq 1$. Since $|f_1| \leq |f|$, we see that (b2)(i) for f_1 implies

$$\limsup_{\substack{z \to 0 \\ \operatorname{Im} z \geq 0}} \frac{|f(z)|}{|z|^{N+\epsilon}} = \infty$$

for all $\epsilon > 0$. This means that f vanishes at 0 to order at most N. □

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