Traces and the Bass Conjecture

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1. Introduction

Let A be an arbitrary ring with unit. If P is a finitely generated projective A-module then one would like to associate to P a rank function generalizing the function which assigns to the free A-module A^n the integer n. Since in the commutative case n is the trace of the identity endomorphism of A^n, one wishes to define a trace for endomorphisms of finitely generated projective A-modules in the case of noncommutative A. This was achieved independently by Hattori [7] and Stallings [12]. Unfortunately, in order for the "trace" to have the natural property of a trace function (i.e., for the trace of $a \cdot b$ to be equal to the trace of $b \cdot a$), one is forced to have the trace take values not in A but in $A/[A,A]$, where $[A,A]$ is the subgroup of A generated by all commutators $ab-ba$. The resulting trace function

$$\mathrm{tr}_P\colon \mathrm{End}_A(P) \to A/[A,A]$$

has many of the properties of the trace function in the commutative case, including additivity, commutativity, and linearity. For details, see [2]. For the results of this paper, the only two properties that will be needed are as follows.

(1) *Functoriality.* If $\alpha : A \to B$ then α induces a map $\alpha : A/[A, A] \to B/[B,B]$, and if $u \in \mathrm{End}_A(P)$ then

$$\mathrm{tr}_{P\otimes B}(u \otimes \mathrm{id}) = \alpha_*(\mathrm{tr}_P(u)).$$

(2) *Linearity.* Suppose $P = P_1 \oplus P_2$ and $u \in \mathrm{End}_A(P)$ restricts to $u_1 \in \mathrm{End}_A(P_1)$ and to $u_2 \in \mathrm{End}_A(P_2)$; then

$$\mathrm{tr}_P(u) = \mathrm{tr}_{P_1}(u_1) + \mathrm{tr}_{P_2}(u_2).$$

This last property allows one to note that, if P is a finitely generated projective A-module and one defines the rank r_P of P to be $\mathrm{tr}_P(\mathrm{id}_P)$, then if P is a direct summand of the free A-module F and $e\colon F \to F$ is the idempotent defining P (i.e., $P = e(F)$) then $r_P = \mathrm{tr}_F(e)$. Also, since $e \in M_d(A)$ for some d and the matrix defining e only involves finitely many elements of A, we see

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from Property (1) that there exists a finitely generated subring A' of A and a finitely generated projective A'-module such that $r_P = \alpha_*(r_{P'})$, where α is the natural map of A' into A.

If R is any commutative ring and G an arbitrary group, then it is easy to see, for the group ring $R(G)$, that $T(G) = RG/[RG, RG]$ is the free R-module with one generator for each conjugacy class of G. (For $g, h \in G$, $gh - hg = h^{-1}g'h - g'$, where $g = h^{-1}g'$.) One denotes the component of r_P on the conjugacy class s by $r_P(s)$. The Bass conjecture [3] is then as follows.

Let G be an arbitrary group and R any subring of the complex numbers $\mathbf{C}$ which intersects the rationals $\mathbf{Q}$ only in the ring of rational integers $\mathbf{Z}$. Then, for any finitely generated projective RG-module P, $r_P(s) = 0$ if $s \neq 1$.

The object of this paper is to describe Linnell's result [9] for the ring $\mathbf{Z}$ and our generalization to any subring of the ring of algebraic integers in $\mathbf{C}$. It will follow that the Bass conjecture is valid for locally finite and residually finite groups.

2. Statements and Proofs

We wish to give an account of Linnell's proof of the Bass conjecture for locally finite groups, and for residually finite groups in the case of integral group rings and our slight extension to group rings over rings of integers in number fields.

Let R be a ring of integers in an algebraic number field, $\mathfrak{P}$ a prime ideal of R, G an arbitrary group, and $R(G)$ the group algebra. The following lemma is implicit in Cliff [4].

LEMMA 1. *Let A be a ring of characteristic p^n and $e \in M_d(A(G))$ an idempotent. Then $\exists N > 0$ and $a_1, \ldots, a_N \in A$, $g_1, \ldots, g_N \in G$ such that, for any $k \geq 1$,*

$$\operatorname{tr}(e) = \sum a_i^{p^k} \cdot [g_i^{p^k}] \in \frac{A(G)}{[A(G), A(G)]}.$$

Now suppose that p is a rational prime in R, $(p) = \mathfrak{P}^e \ldots$; that is, $\mathfrak{P}$ lies over p and has exact power e dividing (p). Then using valuations it is easy to see $\mathfrak{P}^{(s-1)e+j} \cap \mathbf{Z} = (p^s)$ with $s \geq 1$ and $1 \leq j \leq e$. It follows that the ring $R/\mathfrak{P}^n$ contains the ring $\mathbf{Z}/p^k$ where $n = (k-1)e + j$, $1 \leq j \leq e$.

LEMMA 2. *The characteristic of $(R/\mathfrak{P}^n) = p^k$, $n = (k-1)e + j$, $1 \leq j \leq e$.*

Proof. In light of the above discussion we see that $p^k \cdot 1 = 0$; thus p^k is a multiple of the characteristic of $R/\mathfrak{P}^n$ and therefore $\operatorname{char}(R/\mathfrak{P}^n) = p^r$ with $1 \leq r \leq k$.

If $p^r \cdot 1 = 0$ then there exists a map $\mathbf{Z}/p^r \to R/\mathfrak{P}^n$ induced by the inclusion of $\mathbf{Z}$ in R, and therefore a map $(p)^r \to \mathfrak{P}^n$ which must be a monomorphism since it is the restriction of one. That is, $(p)^r \subseteq \mathfrak{P}^n \cap \mathbf{Z} = (p^k)$. Hence $r \geq k$. □

We will also have need of the following.

LEMMA 3. *Let f = residue degree = $|U(R/\mathfrak{P}^n)|$. Then for all $a \in R$ and any $n \geq 1$, $(a^{p^{n-1}})^{p^f} = a^{p^{n-1}} \bmod \mathfrak{P}^n$.*

Proof. If $q = p^f$ then $R/\mathfrak{P} \cong \mathbf{F}_q$ and we must show, for any $a \in R$, that

$$(a^{p^{n-1}})^q - a^{p^{n-1}} = a^{p^{n-1}}((a^{p^{n-1}})^{q-1} - 1) \in \mathfrak{P}^n.$$

If $a \in \mathfrak{P}$ then $a^n \in \mathfrak{P}^n$ and $p^{n-1} \geq n$ for all primes p and all $n \geq 1$. If $a \notin \mathfrak{P}$ then a is a unit in the local ring $R/\mathfrak{P}^n$ and maps to a unit in $R/\mathfrak{P} \cong \mathbf{F}_q$ under the natural map. Hence $a^{q-1} \in \operatorname{kernel}(U(R/\mathfrak{P}^n) \to U(R/\mathfrak{P})) = 1 + J$, where $J = \mathfrak{P}/\mathfrak{P}^n$. Filtering $1+J$ by $1+J^i$ with $J^i = \mathfrak{P}^i/\mathfrak{P}^n$ and using the fact that the multiplicative group $1+J^i/1+J^{i+1}$ is isomorphic to the additive group $\mathfrak{P}^i/\mathfrak{P}^{i+1}$ (which in turn is isomorphic to $R/\mathfrak{P}$), we see that if $b \in 1+J$ then $b^{p^{n-1}} = 1$. Therefore $(a^{q-1})^{p^{n-1}}$ is congruent to 1 modulo $\mathfrak{P}^n$. □

By raising both sides of the above congruence to the $p^{(n-1)(q-1)}$-power we have the following.

COROLLARY. *If $a \in R$ and $n \geq 1$, then $a^{p^{nf}} \equiv a^{p^{(n-1)f}} \bmod \mathfrak{P}^n$.*

LEMMA 4. *Let $x \in G$ and suppose that x is conjugate to x^{p^u} for some $u \sim 0$ and that x is of finite order. Then* order(x) *is relatively prime to p.*

Proof. Let $x^s = 1$ and suppose $s = p \cdot m$. Then

$$\mathrm{id} = \mathrm{id}^{p^{u-1}} = (x^s)^{p^{u-1}} = x^{p^u m} \sim x^m$$

since $x^{p^u} \sim x$. Hence $x^m = \mathrm{id}$. □

We are now in a position to state and prove the generalization of Linnell's result.

THEOREM. *Let R be the ring of integers in a number field and let G be an arbitrary group.*

(a) *Let $\mathfrak{P}$ be a prime in R with $\mathfrak{P} \cap \mathbf{Z} = (p)$ and let Q be a finitely generated projective $R_{\mathfrak{P}}(G)$-module. Let $f = [R/\mathfrak{P} : \mathbf{F}_p]$ and suppose that $x \in G$ with $r_Q(x) \neq 0$. Then x is conjugate to x^{p^u} for some $u > 0$ and $r_Q(x) = r_Q(x^{p^f})$.*

(b) *Suppose P is a finitely generated projective $R(G)$-module and $x \in G \setminus 1$ with $r_P(x) \neq 0$. Then there exist subgroups $C, H \subseteq G$ with $x \in C \subseteq H$ and $C \cong \mathbf{Q}^+$, H is finitely generated, and the elements of C lie in finitely many H-conjugacy classes.*

Proof. (a) Let Q correspond to the idempotent $e \in M_d(R_{\mathfrak{P}}G)$. Since $r_Q(x) = \operatorname{tr}(e)_x \neq 0 \in R_{\mathfrak{P}}$, $\exists n \geq 2$ such that $r_Q(x) \notin \mathfrak{P}^n$. Let $S = R_{\mathfrak{P}}/\mathfrak{P}^n$ and let α be the image of e in $M_d(SG)$. By Lemma 2, char $S = p^m$ where $n = (m+1)e + j$, $1 \leq j \leq e$. Cliff's result, together with Lemma 3, implies that by choosing $k = n, n+1$ there exist $N > 0$ and $a_1, \ldots, a_N \in S$, $g_1, \ldots, g_N \in G$ such that

$$\operatorname{tr}(\alpha) = \sum a_i \cdot [g_i] = \sum a_i \cdot [g_i^{p^f}].$$

Let $\mathcal{C}(G)$ denote the conjugacy classes of G and $T\colon \mathcal{C}(G) \to C(G)$ the map sending $[g]$ to $[g^{p^f}]$. Let $b_i = \sum a_j$, where $[g_j] \sim [g_i]$; let $\tilde{b}_i = \sum b_j$, where $[g_j^{p^f}] \sim [g_i^{p^f}]$. Then

$$\operatorname{tr}(\alpha) = \sum b_i [g_i] = \sum \tilde{b}_i [g_i^{p^f}] \in T(SG).$$

If $\Lambda = \operatorname{supp}(\alpha) \subseteq \mathcal{C}(G)$ and $[g_i] \in \Lambda$ (i.e., $b_i \neq 0$), then the above show that there exists j such that $[g_j^{p^f}] = [g_i]$ and $\tilde{b}_j = b_i \neq 0$. Since $\tilde{b}_j = \sum b_i$, $\exists g_k \in \Lambda$ with $g_i \sim g_k^{p^f}$; that is, $\Lambda \subseteq T(\Lambda)$ and hence equal since Λ is finite. Since $x \in \Lambda$, $u = |\Lambda| \geq 1$ and $\lambda \sim \lambda^{p^u}$ for all $\lambda \in \Lambda$; therefore $x \sim x^{p^u}$. Moreover, $T|_\Lambda$ a bijection implies that $\tilde{g}_i \sim x$ if and only if $\tilde{g}_i^{p^f} \sim x^{p^f}$, and so

$$rk_Q(x) \equiv rk_Q(x^{p^f}) \bmod \mathfrak{P}^n.$$

Since n can be arbitrarily large, rk_Q must have the same value on x and x^{p^f}.

(b) Let P be a finitely generated projective RG-module. By considering the idempotent determining P we can find a finitely generated subgroup H of G and a finitely generated projective module Q over RH such that r_Q maps to r_P under the natural map $TH \to TG$. Since $r_P(x) \neq 0$ we may (by taking a conjugate in G, if necessary) assume $x \in H$.

For each rational prime p let

$$f(p) = \min\{[R/\mathfrak{P}\colon \mathbf{F}_p] \mid \mathfrak{P} \mid p\}.$$

By (a), $r_Q(x) = r_Q(x^{p^{f(p)}})$ for all p. This implies that, if n is any rational integer with $f(p) \mid \nu_p(n)$ $\forall p \mid n$, then $r_Q(x) = r_Q(x^n)$. Also from (a) we have that, for all p, $x \sim x^{p^{u(p)}}$ for some $u(p) > 0$. Note that $f(p) \mid u(p)$. Let p_i be the ith prime. By the above there exists $g_i \in H$ with $g_i \cdot x \cdot g_i^{-1} = x^{n(i)}$ where $n(i) = p_i^{u(p_i)}$. Let r_i be any sequence of integers > 0, $m_i = n(r_i) \cdots n(r_1)$, and $h_i = g_{r_i} \cdots g_{r_1}$. If we define $x_k = h_k^{-1} \cdot x \cdot h_k$ and $x_0 = x$, then it is clear that $x_{k-1} = x_k^{n(r_k)}$. Hence if C_k is the cyclic subgroup generated by x_k then we have $x \in C_1 \subseteq \cdots \subseteq C_k \subseteq \cdots \subseteq H$. Let $C = \bigcup C_k \subseteq H$. To show $C \cong Q$ we first note that x is of infinite order by Lemma 4, since x is conjugate to a pth power of x for all primes p. We obtain the following exact ladder:

$$\begin{array}{ccccccc}
\mathbf{Z} & \xrightarrow{n(r_1)} & \mathbf{Z} & \xrightarrow{n(r_2)} & \mathbf{Z} & \xrightarrow{n(r_3)} & \mathbf{Z} \cdots \\
\downarrow 1 & & \downarrow m_1^{-1} & & \downarrow m_2^{-1} & & \downarrow m_3^{-1} \\
\mathbf{Q} & \xrightarrow{1} & \mathbf{Q} & \xrightarrow{1} & \mathbf{Q} & \xrightarrow{1} & \mathbf{Q} \cdots.
\end{array}$$

We thus obtain a monomorphism $C \to \mathbf{Q}$. In order to see that this map is onto, we first choose the sequence r_i so that every positive integer appears infinitely often. This guarantees that, by going far enough out in the sequence, any arbitrarily high power of p will be inverted and hence any $a/b \in \mathbf{Q}$ will be in the image. To see that C is contained in only finitely many H-conjugacy classes, we note that any element of C equals x_k^s for some k and s and that $x_k^s = (x_{k+1})^{n(k)s}$. Since $n(i) = p_i^{u(p_i)}$ and $f(p_i) \mid u(p_i)$, we see that by going far enough out in the sequence we can represent any element of C by $(x_k)^n$

where, for all $p|n$, $f(p)|\nu_p(n)$. Since x_k is H-conjugate to x,

$$r_Q(x_k^n) = r_Q(x^n) = r_Q(x)$$

by the divisibility condition on n. Since $r_Q(x) \neq 0$ and $\operatorname{supp} rk_Q$ is finite, we see that the elements of C must be contained in finitely many H-conjugacy classes. □

COROLLARY 1. *Let R be any subring of the ring of algebraic integers* $\mathbf{A}$ *in* $\mathbf{C}$. *Then the preceding theorem is valid for R.*

Proof. If P is a finitely generated projective $R(G)$-module and $e \in M_d(RG)$ the corresponding idempotent, then the entries of e contain only a finite number of elements from R (and from G). So if S is the subring of R generated by these elements then P is induced from a projective $S(G)$-module and $S \subseteq K \subseteq \mathbf{C}$ (K a number field), and so $S \subseteq \mathbf{A}_K$ the integers in K. Hence the validity of the theorem for $\mathbf{A}_K$ implies its validity for R by the functoriality of the rank. □

COROLLARY 2. *Let $R \subseteq \mathbf{A}$, and let G be an arbitrary group. If P is any finitely generated projective RG-module, then r_P vanishes on any conjugacy class of finite order not equal to the identity.*

Proof. This is just (a) of the Theorem and Lemma 4. □

COROLLARY 3. *Let $R \subseteq \mathbf{A}$. Let G be a group of residually bounded exponent; that is, given any $x \neq 1$ in G, there exists a homomorphism φ of G into a group H such that $\varphi(x) \neq 1$ and that $\exists n \geq 1$ so that $h^n = 1$ for all $h \in H$. Then, for any finitely generated projective RG-module P, $r_P(x) = 0$ for all $x \neq 1$.*

Proof. There exist no nontrivial homomorphisms of $\mathbf{Q}$ into such a group H, since the image must be divisible. The result follows from part (b) of the theorem. □

Note that since residually finite groups are obviously residually bounded exponent, the following classes of groups satisfy Bass's conjecture for subrings of the ring of algebraic integers in $\mathbf{C}$:

(a) groups with a faithful representation over some field (this is equivalent to "every finitely generated subgroup has a faithful representation over some field [M_1]");
(b) polycyclic groups (equivalent to solvable groups with maximum condition);
(c) extensions of abelian groups by nilpotent groups, and consequently any meta-abelian group;
(d) the fundamental group of a graph whose vertex groups are residually finite and whose edge groups are finite.

For (a), Mal'cev [10] proves that finitely generated subgroups of matrix groups over fields are residually finite. That polycyclic groups are residually finite was proven by Hirsch [8]. (d) follows since amalgamated free products of residually finite-by-finite groups are residually finite. As for (c), Hall [6] shows that finitely generated groups which are extensions of abelian groups by nilpotent groups are residually finite. But it is easy to see that that hypothesis on G passes to the finitely generated H of part (b) of the theorem.

REMARKS. (1) Polycyclic groups are contained in the class of solvable groups with finite Hirsch number. These groups cannot be of the type in part (b) of the theorem, since a finitely generated solvable group of finite Hirsch number contains a finite index subgroup which is an extension of a nilpotent group (the Hirsch–Plotkin radical) by a free abelian group of finite rank [1]. By Hall's theorem [6] this subgroup is residually finite and hence so is the original group.

(2) We have all the classes given by Eckmann [5] except for groups of $cd \leq 2$. Groups of $cd = 1$ are free and thus residually finite, so we only lack those of $cd = 2$.

3. An Example and a Question

In [6], Hall produces an example of a finitely generated solvable group G of class 3 which contains a normal subgroup isomorphic to $\mathbf{Q}^\omega$, a vector space of countable dimension over the rational field $\mathbf{Q}$. Let V be an infinite-dimensional vector space over $\mathbf{Q}$ with basis v_m, $m = 0, \pm 1, \pm 2, \ldots$. Let p_m $(m \in \mathbf{Z})$ be an enumeration of the primes with $p_m \neq p_n$ if $m \neq n$, and let H be the subgroup of $\mathrm{Aut}_{\mathbf{Q}}(V)$ generated by T and S where

$$T(v_m) = v_{m+1}, \qquad S(v_m) = p_m \cdot v_m.$$

G is the semi-direct product of V and H. It is not difficult to see that G is generated by three elements, H is meta-abelian, and V is a minimal normal subgroup of G. Moreover, the conjugacy class determined by v_0 contains the subgroups $\mathbf{Q}v_m$ for any m and hence any one of these subgroups is contained in a single conjugacy class. G therefore fulfills the conditions in part (b) of the theorem. Note also that G is of infinite cohomological dimension since it contains a copy of $\mathbf{Q}^\omega$ which is obviously of infinite cohomological dimension. Hence we have the following.

QUESTION. If G is a finitely generated group containing a copy of $\mathbf{Q}$ which is contained in only finitely many conjugacy classes of G, must G necessarily be of infinite cohomological dimension? Or better, must G contain a copy of $\mathbf{Q}^\omega$?

NOTE. If G is solvable then the answer to the first question is yes, since finite cohomological dimension is equivalent to finite Hirsch number for solvable groups, and these are residually finite by Remark (1).

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