An Explicit Plancherel Formula for Completely Solvable Lie Groups

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Introduction

Let G be a connected, simply connected and completely solvable Lie group with Lie algebra $\mathfrak{g}$. The dual $G^\wedge$ of G is parameterized by the space $\mathfrak{g}^*/G$ of coadjoint orbits in the dual $\mathfrak{g}^*$ of $\mathfrak{g}$ via the canonical Kirillov bijection K: $G^\wedge \to \mathfrak{g}^*/G$. If $\xi \in G^\wedge$ and $l \in K(\xi)$, then there is an analytic subgroup H of G and a unitary character χ of H such that $l_{|\mathfrak{h}} = \mathrm{id}\,\chi$ ($\mathfrak{h}$ is the Lie algebra of H) and such that the representation $\rho(l, \mathfrak{h})$ of G induced by χ belongs to the equivalence class ξ. Moreover, K is a Borel isomorphism, and the measure class on $G^\wedge$ which gives the canonical decomposition of the regular representation corresponds to the pushforward to $\mathfrak{g}^*/G$ of the Lebesgue measure class on $\mathfrak{g}^*$. On the other hand the space $\mathfrak{g}^*/G$ can be parameterized (generically) by an explicit algebraic submanifold Σ of $\mathfrak{g}^*$. In this work we describe the Plancherel measure for G as a measure on Σ and show that as such it is a "rational" measure. The resulting Plancherel formula generalizes the well-known formula of L. Pukanszky for nilpotent Lie groups.

Recall the Plancherel formula obtained by Pukanszky for nilpotent Lie groups in [6]. It is shown that there is a G-invariant Zariski open set $W \subset \mathfrak{g}^*$, and a subspace V_0 of $\mathfrak{g}^*$, such that the orbits in W are of maximal dimension, and such that $\Lambda = V_0 \cap W$ is a cross-section for the coadjoint orbits in W. The Plancherel measure is realized as a measure on Λ. There is a rational function $r(\lambda)$ on V_0 which is nonsingular on Λ and such that the Plancherel formula is given by

$$\phi(e) = \int_\Lambda \mathrm{Tr}(\rho_\lambda(\phi))|r(\lambda)|\, d\lambda,$$

where $[\rho_\lambda]$ is the equivalence class in $G^\wedge$ corresponding to $\lambda \in \Lambda$, and $d\lambda$ is a Euclidean measure on V_0.

The Plancherel formula for exponential Lie groups was obtained by Duflo and Rais [4]. Of course G is not necessarily unimodular and points in $G^\wedge$ are not necessarily closed. For each orbit O in $\mathfrak{g}^*/G$, denote the corresponding

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element of $G^\wedge$ by $[\rho_O]$, and the canonical measure on O by β_O. The formula here takes the form

$$\phi(e)=\int_{\mathfrak{g}^*/G}\operatorname{Tr}(A_{\psi,O}^{-1/2}\rho_O(\phi)A_{\psi,O}^{-1/2})\,dm_\psi(O).$$

Here ψ is a rational function on $\mathfrak{g}^*$ such that $\psi(sl)=\Delta(s)^{-1}\psi(l)$, $A_{\psi,O}$ is the operator semi-invariant for ρ_O associated with the restriction of $|\psi|$ to O, and m_ψ is the (unique) measure on $\mathfrak{g}^*/G$ such that $\int_{\mathfrak{g}^*}f(l)|\psi(l)|\,dl=\int_{\mathfrak{g}^*/G}(\int_O f\,d\beta_O)\,dm_\psi(O)$. We shall give an explicit description of the measure m_ψ by means of a cross-section for (generic orbits in) $\mathfrak{g}^*/G$.

In the solvable case the set of orbits of maximal dimension, and even the set W as defined by Pukanszky, are too large to admit a smooth cross-section. However it is shown in [2] that if G is completely solvable then there is a G-invariant Zariski open subset $\Omega\subset W$, and an algebraic subset $\Sigma\subset\Omega$, such that Σ is a cross-section for the coadjoint orbits in Ω that is homeomorphic to Ω/G. The set Ω is a finite union of disjoint open subsets Ω_ϵ, and each of the subsets $\Sigma_\epsilon=\Sigma\cap\Omega_\epsilon$ of Σ can be described as follows. There are subspaces V_0 and V_1 of $\mathfrak{g}^*$ with $\mathfrak{g}^*=V_0\oplus V_1$, and for each Ω_ϵ there is a Zariski open subset Λ_ϵ of V_0 and a rational function $p_\epsilon\colon\Lambda_\epsilon\to V_1$ such that $\Sigma_\epsilon=\operatorname{graph}(p_\epsilon)$. Thus, projection onto V_0 parallel to V_1 defines a rational diffeomorphism $\pi_\epsilon\colon\Sigma_\epsilon\to\Lambda_\epsilon$. The restriction of the Plancherel measure m_ψ to Ω_ϵ/G is then realized as a measure on Λ_ϵ. Let $d\lambda$ be a Euclidean measure on V_0. The main result of this paper is the following.

THEOREM. *There is a real-valued polynomial function $P_0(l)$ on $\mathfrak{g}^*$ such that for each Ω_ϵ the restriction of m_ψ to Ω_ϵ/G is given by*

$$|\psi(\pi_\epsilon^{-1}(\lambda))|\,|P_0(\pi_\epsilon^{-1}(\lambda))|\,d\lambda.$$

This immediately yields an explicit Plancherel formula. It is

$$\phi(e)=\sum_\epsilon\int_{\Lambda_\epsilon}\operatorname{Tr}(A_{\psi,\lambda,\epsilon}^{-1/2}\rho_{\lambda,\epsilon}(\phi)A_{\psi,\lambda,\epsilon}^{-1/2})|\psi(\pi_\epsilon^{-1}(\lambda))|\,|P_0(\pi_\epsilon^{-1}(\lambda))|\,d\lambda,$$

where $\rho_{\lambda,\epsilon}$ is an irreducible representation corresponding to $\pi_\epsilon^{-1}(\lambda)$. The function ψ is as above, and $A_{\psi,\lambda,\epsilon}$ is the operator semi-invariant for $\rho_{\lambda,\epsilon}$ associated with $|\psi|$ as above. If G has irreducible representations that are square integrable (in the strict sense) then each Ω_ϵ is a G-orbit, $V_0=(0)$ and $d\lambda$ is just point mass measure.

In the nilpotent case one has $\Omega_\epsilon=\Omega$ and $p_\epsilon=0$. Thus $\Lambda_\epsilon=\Sigma=V_0\cap\Omega$ is Zariski open in $\Lambda=V_0\cap W$, and our formula reduces to the Plancherel formula for nilpotent Lie groups derived in [6].

1. Preliminaries

Fix a basis $\{Z_1,Z_2,\dots,Z_n\}$ of $\mathfrak{g}$, let $\mathfrak{g}_j=\operatorname{span}_{\mathbf{R}}\{Z_1,Z_2,\dots,Z_j\}$, $1\le j\le n$, and set $\mathfrak{g}_0=(0)$. We choose our basis so that for each j, $\mathfrak{g}_j$ is an ideal in $\mathfrak{g}$. Let $e_1,e_2,\dots,e_n$ be the dual basis in $\mathfrak{g}^*$. For each $j=1,2,\dots,n$, define

$$\mathfrak{g}_j^{\perp} = \{l \in \mathfrak{g}^* : \mathfrak{g}_j \subset \ker(l)\}$$

and let π_j denote the projection of $\mathfrak{g}^*$ onto $\operatorname{span}_{\mathbf{R}}\{e_1, e_2, \dots, e_j\}$ parallel to $\mathfrak{g}_j^{\perp}$. We denote $\operatorname{ad}^*(X)l$ by $[X, l]$, $l \in \mathfrak{g}^*$, $X \in \mathfrak{g}$, and we denote the coadjoint action of G on $\mathfrak{g}^*$ multiplicatively. Let μ_j denote the $(0, \infty)$-valued multiplicative character of G such that $se_j = \mu_j(s)e_j \operatorname{mod}(\mathfrak{g}_j^{\perp})$ for $s \in G$, and let $\gamma_j \in \mathfrak{g}^*$ be the differential of μ_j $(1 \le j \le n)$. For each l in $\mathfrak{g}^*$, set $\mathfrak{g}_j^l = \{X \in \mathfrak{g}: \text{for every } Y \in \mathfrak{g}_j,\ l([X, Y]) = 0\}$.

Fix a Euclidean measure dX on $\mathfrak{g}$, and let da be the left Haar measure on G such that $d(\exp X) = j_G(X)dX$, where

$$j_G(X) = \left| \frac{\det(1 - e^{-\operatorname{ad} X})}{\operatorname{ad} X} \right|.$$

Let Δ be the modular function so that for each b in G, $d(ab) = \Delta(b)da$. For each coadjoint orbit O, let β_O be the canonical measure on O (cf. [7, Lemma 6 and Prop. 4]).

In Proposition IV.4.1 of [8], it is shown that there is a nonzero rational function ψ on $\mathfrak{g}^*$ such that $\psi(sl) = \Delta(s)^{-1}\psi(l)$, $s \in G$, $l \in \mathfrak{g}^*$. Fix one such function ψ. There is a unique measure m_ψ on $\mathfrak{g}^*/G$ such that

$$\int_{\mathfrak{g}^*} \phi(l)|\psi(l)|\, dl = \int_{\mathfrak{g}^*/G} \left(\int_O \phi(l)\, d\beta_O(l) \right) dm_\psi(O)$$

holds for any nonnegative Borel function ϕ on $\mathfrak{g}^*$ [4, Lemme 5.1.7]. For each O in $\mathfrak{g}^*/G$, let $[\pi_O]$ be the corresponding element of $G^\wedge$ and let ψ_O be the restriction of ψ to O. Let U be the set of all coadjoint orbits O such that

(i) $|\psi(l)|^{-1} d\beta_O(l)$ defines a tempered distribution on $\mathfrak{g}^*$,
(ii) ψ_O is nonzero, and
(iii) $\operatorname{Stab}_G(l)$ is nilpotent, for some (hence for every) $l \in O$.

Then the complement of U has Plancherel measure zero (cf. Lemme 5.2.2 of [4] and the remarks in 5.2.3 of [4]). The character formula for solvable groups together with the above are used to prove a Plancherel formula for general solvable type 1 groups (Proposition 5.2.4 and Théorème 5.2.7 of [4]). We summarize the main results for the completely solvable case.

THEOREM 1.1. *Let G be a connected, simply connected, and completely solvable Lie group. For each O in U, let $A_{\psi,O}$ be the positive semi-invariant operator of weight Δ for π_O associated with $|\psi|$* ($A_{\psi,O}$ is constructed for $\pi_O = \rho(l, \mathfrak{h})$ with $H \subset \ker(\Delta)$ in Lemme 3.2.2 of [4], and is determined up to unitary equivalence by O and $|\psi|$ (cf. also Theorem 5 of [3])). *Then there is a positive* $\operatorname{Ad}(G)$*-invariant function α on $\mathfrak{g}$, such that if $O \in U$ then the formula*

$$\operatorname{Tr}(A_{\psi,O}^{-1/2}\pi_O(\phi)A_{\psi,O}^{-1/2}) = \int_O (\alpha(\phi \circ \exp))^\wedge(l)|\psi(l)|^{-1}\, d\beta_O(l)$$

holds for every smooth function ϕ on G with compact support. For each such ϕ one has

$$\phi(e) = \int_{\mathfrak{g}^*/G} \operatorname{Tr}(A_{\psi,O}^{-1/2}\pi_O(\phi)A_{\psi,O}^{-1/2})\, dm_\psi(O).$$

The results stated in Proposition 2.3 and Theorem 2.4 of [2] provide detailed information about the structure of $\mathfrak{g}^*/G$, and our task here is to give a description of the measure m_ψ in terms of this data. We summarize the relevant aspects of these results as follows.

THEOREM 1.2. *Let G be a connected, simply connected, and completely solvable Lie group. There is a Zariski open subset Ω of $\mathfrak{g}^*$, a subset $J=\{j_1<j_2<\cdots<j_d\}$ of $\{1,2,\ldots,n\}$, a subset $\varphi=\{j_{k_1}<j_{k_2}<\cdots<j_{k_t}\}$ of J, for each j in φ a real-valued rational function q_j (nonsingular and nonvanishing on Ω), and n real analytic functions P_j $(1\le j\le n)$ in the real variables $w_1,w_2,\ldots,w_d,l_1,l_2,\ldots,l_n$, such that the following hold.*

(1) *For each $\epsilon\in\{-1,1\}^t$, the set $\Omega_\epsilon=\{l\in\Omega:\text{sign of } q_{j_{k_r}}(l)=\epsilon_r,\ 1\le r\le t\}$ is a nonempty open subset in $\mathfrak{g}^*$.*

(2) *Define $W\subset\mathbf{R}^d$ by $W=\prod R_k$, where $R_k=(0,\infty)$ if $j_k\in\varphi$ and $R_k=\mathbf{R}$ otherwise. Let $\epsilon\in\{-1,1\}^t$ and for $w\in W$ define $\epsilon w\in\mathbf{R}^d$ by $(\epsilon w)_j=\epsilon_r w_j$ if $j=j_{k_r}\in\varphi$ and $(\epsilon w)_j=w_j$ otherwise. Then for each l in Ω_ϵ, the mapping $w\to\sum_j P_j(\epsilon w,l)e_j$ is a diffeomorphism of W with the coadjoint orbit of l. The functions P_j also have the properties:*

- (i) *when l is fixed, each function $P_j(\cdot,l)$ depends only on the variables $w_1,w_2,\ldots,w_k$, where $j_k=\max\{j'\in J: j'\le j\}$;*
- (ii) *for each l and for each $j=j_k\in J$, $P_{j_k}(w,l)=w_k(w_1,w_2,\ldots,w_{k-1})$, and if $j_k\notin\varphi$ then $P_{j_k}\equiv w_k$; and*
- (iii) *for each j, and for each fixed w, $P_j(w,\cdot)$ is a G-invariant function on Ω.*

(3) *Define subspaces V_0 and V_φ of $\mathfrak{g}^*$ by $V_0=\operatorname{span}_{\mathbf{R}}\{e_i: i\notin J\}$ and $V_\varphi=\operatorname{span}_{\mathbf{R}}\{e_j: j\in\varphi\}$. Then the set $\Sigma=\{l\in(V_0+V_\varphi)\cap\Omega: |q_j(l)|=1,\ j\in\varphi\}$ is a cross-section for the coadjoint orbits in Ω. For each j in φ, the rational function q_j is of the form $q_j(l)=l_j+p_j(l_1,l_2,\ldots,l_{j-1})$, where p is a rational function, and q_j is G-semi-invariant with multiplier μ_j.*

(4) *For each $l\in\Omega$ let $\epsilon(l)\in\{-1,1\}^t$ such that $l\in\Omega_{\epsilon(l)}$. Then the mapping $P: W\times\Sigma\to\Omega$, defined by $P(w,l)=\sum_j P_j(\epsilon(l)w,l)e_j$, is a diffeomorphism.*

The above also follows from Proposition 2.7 and Theorem 2.8 of [1]. It follows from Theorem 2.9 of [1] that the cross-section Σ is a submanifold of $\mathfrak{g}^*$. If (but not only if) G is nilpotent, then $\varphi=\emptyset$, in which case it is understood that $\Omega_\epsilon=\Omega$. For nilpotent groups, a version of the above theorem was first given by Pukanszky in Proposition II.4.1 of [8].

The procedure for constructing the objects in Theorem 1.2 resembles the proof of Proposition II.1.1 of [8]. It was shown in Lemma 1.1 of [2] that for each j in J there is a rational function $r_j:\Omega\to\mathfrak{g}$, with $r_j(l)\in\mathfrak{g}^l_{j-1}\sim\mathfrak{g}^l_j$ and

$$[r_j(l),l]=e_j \bmod(\mathfrak{g}_j^\perp)$$

for each l in Ω. For each $j_k\in J$, $k=1,2,\ldots,d$, set $g_k(t_k,l)=\exp(t_k r_{j_k}(l))$ and set $g(t,l)=g_1(t_1,l)g_2(t_2,l)\ldots g_d(t_d,l)$, for $t\in\mathbf{R}^d$. Denote by σ_l the diffeomorphism of $\mathbf{R}^d$ with Gl given by $\sigma_l(t)=g(t,l)l$. Write $\sigma_l(t)=\sum_j Q_j(t,l)e_j$

for $t \in \mathbf{R}^d$. The properties of the functions Q_j allow the successive definition of w_k and expression of t_k in terms of $w_1, w_2, \dots, w_k$, $k = 1, 2, \dots, d$. Thus in Proposition 2.3 of [2], for each l we obtained a function $\Phi(\cdot, l) : W \to \mathbf{R}^d$ such that $t_k = \Phi_k(w, l)$ is defined explicitly for each k, and the functions P_j are obtained explicitly as $P_j(\epsilon w, l) = Q_j(\Phi(w, l), l)$ for l in Ω_ϵ and $1 \le j \le n$. Thus for $l \in \Omega$ and $w \in W$, one has

$$P(w, l) = \sigma_l(\Phi(w, l)).$$

For each l in $\mathfrak{g}^*$ let $d\beta_l$ denote the canonical measure on Gl. For l belonging to the cross-section Σ, we shall compute $d\beta_l$ in terms of the coordinates $(w_1, w_2, \dots, w_d) \in W$.

Fix l in Σ. Then $\Phi(\cdot, l) : W \to \mathbf{R}^d$ is an analytic diffeomorphism. Let dw denote Lebesgue measure on W. Let $P_J(l)$ denote the Pfaffian of the matrix $[l([Z_{j_r}, Z_{j_k}])]_{1 \le r, k \le d}$; then $P_J(l) \ne 0$ for each $l \in \Omega$. For each s in G, define $\mu_J(s) = \prod_{j \in J} \mu_j(s)$.

LEMMA 1.3. *For each β_l-integrable function f on Gl,*

$$\int_{Gl} f \, d\beta_l = (2\pi)^d |P_J(l)|^{-1} \int_W f(P(w, l)) |\mu_J(g(\Phi(w, l), l))|^{-1} \, dw.$$

Proof. The proof is a simple application of Lemma 2.1.3 of [5], from which we have that

$$\int_{Gl} f \, d\beta_l = (2\pi)^d |P_j(l)|^{-1} \int_{\mathbf{R}^d} f(\sigma_l(t)) \prod_{k<r} |\mu_{j_k}(g_r(t_r, l))|^{-1} \, dt_1 dt_2 \dots dt_d$$

holds for any β_l-integrable function f on Gl. Hence for each such f, we have

$$\int_{Gl} f \, d\beta_l = (2\pi)^d |P_j(l)|^{-1} \int_W f(P(w, l)) \prod_{k<r} |\mu_{j_k}(g_r(\Phi_r(w, l), l))|^{-1} |J_\Phi(l)| \, dw,$$

where $J_\Phi(l)$ is the Jacobian matrix of $\Phi(\cdot, l)$. It remains for us to compute $|J_\Phi(l)|$.

For each $k = 1, 2, \dots, d$, $\Phi_k(w, l)$ depends only on the variables $w_1, w_2, \dots, w_k$. In fact, let $w = (w_1, w_2, \dots, w_d) \in W$ and let $t = \Phi(w, l)$. From the construction in Proposition 2.3 of [2], if $j_k \notin \varphi$ then

$$w_k = Q_{j_k}(t, l) = \mu_{j_k}(g_1(t_1, l) g_2(t_2, l) \dots g_{k-1}(t_{k-1}, l)) t_k + S(t_1, t_2, \dots, t_{k-1}, l)$$

and hence

$$t_k = \Phi_k(w, l) = \mu_{j_k}(g_1(\Phi_1(w, l), l) g_2(\Phi_2(w, l), l) \dots g_{k-1}(\Phi_{k-1}(w, l), l))^{-1} w_k$$

modulo $(w_1, w_2, \dots, w_{k-1})$. If $j = j_k = j_{k_r} \in \varphi$, then

$$\begin{aligned} w_k &= \epsilon_r(l) \mu_{j_k}(g_1(t_1, l) g_2(t_2, l) \dots g_k(t_k, l)) q_j(l)^{-1} \\ &= \mu_{j_k}(g_1(t_1, l) g_2(t_2, l) \dots g_k(t_k, l)) \\ &= \mu_{j_k}(g_1(t_1, l) g_2(t_2, l) \dots g_{k-1}(t_{k-1}, l)) \exp(t_k \epsilon_r(l)), \end{aligned}$$

where we have used the fact that $q_j(l)=\gamma_j(r_j(l))=\epsilon_r(l)$ since $l\in\Sigma_\epsilon$. Hence

$$\Phi_k(w,l)=\epsilon_r(l)\ln(w_k)$$

modulo $(w_1, w_2, \ldots, w_{k-1})$. It follows immediately that

$$|J_\Phi(l)|=\prod_{j_k\notin\varphi}|\mu_{j_k}(g_1(\Phi_1(w,l),l)g_2(\Phi_2(w,l),l)\ldots g_{k-1}(\Phi_{k-1}(w,l),l))|^{-1}$$
$$\times\prod_{j_k\in\varphi}|\mu_{j_k}(g_1(\Phi_1(w,l),l)g_2(\Phi_2(w,l),l)\ldots g_k(\Phi_k(w,l),l))|^{-1}.$$

By Proposition 1.3 of [2], if $j\notin\varphi$ then $\gamma_j(r_j(l))=0$, and so from the formula above we have

$$|J_\Phi(l)|=\prod_{k\geq r}|\mu_{j_k}(g_r(\Phi_r(w,l),l))|^{-1}.$$

Thus $|J_\Phi(l)|\prod_{k<r}|\mu_{j_k}(g_r(\Phi_r(w,l),l))|^{-1}=|\mu_J(g(\Phi(w,l),l))|^{-1}$. □

2. Computation of the Plancherel Measure

Fix a Euclidean measure dl on $\mathfrak{g}^*$ suitably normalized so that, for any continuous compactly supported function f on $\mathfrak{g}$,

$$f(0)=\int_{\mathfrak{g}^*}\hat{f}(l)\,dl,$$

where $\hat{f}(l)=\int_{\mathfrak{g}} f(X)e^{il(X)}\,dX$. Since Ω is Zariski open, the complement of Ω has measure zero, and by Theorem 1.2 we have the explicit diffeomorphism $P\colon W\times\Sigma\to\Omega$. The general idea is to compute $|\psi(l)|\,dl$ in terms of a product measure on $W\times\Sigma$, where $\psi(l)$ is a rational satisfying $\psi(sl)=\Delta(s)^{-1}\psi(l)$ for each s in G, and then to use Lemma 1.3 to read off the measure m_ψ as a measure on Σ. First we determine global coordinates for Σ.

If the subset φ of the index set J introduced in Theorem 1.2 is empty, then $\Sigma=V_0\cap\Omega$ and coordinates for Σ are obtained by identifying V_0 with $\mathbf{R}^{n-d}$. On the other hand, suppose that $\varphi\neq\emptyset$, and let t be the number of elements in φ. Recall that, by part (1) of Theorem 1.2, for each $\epsilon\in\{-1,1\}^t$ there is a nonempty open subset Ω_ϵ of Ω such that Ω is the disjoint union of the sets Ω_ϵ. Let $\epsilon\in\{-1,1\}^t$ and set $\Sigma_\epsilon=\Sigma\cap\Omega_\epsilon$. From parts (1) and (3) of Theorem 1.2 we have

$$\Sigma_\epsilon=\{l\in(V_0\oplus V_\varphi)\cap\Omega\colon \text{for each } j=j_{k_r}\in\varphi,\ l_j=\epsilon_r-p_j(l_1,l_2,\ldots,l_{j-1})\},$$

where, for each j in φ, p_j is a rational nonsingular function on Ω. From now on we assume that $\varphi\neq\emptyset$; the modifications for the case $\varphi=\emptyset$ amount to little more than dropping the subscript ϵ, and we leave them to the reader.

LEMMA 2.1. *Let $\epsilon\in\{-1,1\}^t$. Then there is a Zariski open subset Λ_ϵ of V_0 and a rational function $p_\epsilon\colon\Lambda_\epsilon\to V_\varphi$ such that* $\Sigma_\epsilon=\text{graph}(p_\epsilon)$.

Proof. Write $\{1,\dots,n\}\sim J=\{i_1<i_2<\cdots<i_{n-d}\}$ and for $\lambda\in V_0$ write $\lambda=\sum_\alpha \lambda_\alpha e_{i_\alpha}$. Define the rational function p_ϵ as follows. Let j be the smallest index in φ, and let i_α be the largest of those indices $i\notin J$ such that $i<j$. Then p_j is a function of the variables $\lambda_1,\lambda_2,\dots,\lambda_{i_\alpha}$. Set $p_{\epsilon,j}(\lambda)=\epsilon_1-p_j(\lambda_1,\lambda_2,\dots,\lambda_{i_\alpha})$. (If $j=1$, then $p_1=0$ and $p_{\epsilon,j}(\lambda)=\epsilon_1$.) Now let $j=j_k\in\varphi$ and assume that $p_{\epsilon,j'}(\lambda)$ is defined for all $j'\in\varphi$, $j'<j$. For each $i<j$, set $l_i(\lambda)=\lambda_{\alpha'}$ if $i=i_{\alpha'}\notin J$, $l_i(\lambda)=p_{\epsilon,i}(\lambda)$ if $i\in\varphi$, and $l_i(\lambda)=0$ otherwise. Set

$$p_{\epsilon,j}(\lambda)=\epsilon_r-p_j(l_1(\lambda),l_2(\lambda),\dots,l_{j-1}(\lambda))$$

and

$$p_\epsilon(\lambda)=\sum_{j\in\varphi}p_{\epsilon,j}(\lambda)e_j.$$

Set $\Lambda_\epsilon=\{\lambda\in V_0\colon p_\epsilon(\lambda)$ is defined at λ and $(\lambda,p_\epsilon(\lambda))\in\Omega\}$; Λ_ϵ is a Zariski open subset of V_0. It follows from the definition of Σ_ϵ above that $\Sigma_\epsilon=\operatorname{graph}(p_\epsilon)$. □

Thus projection of Σ_ϵ into V_0 parallel to V_1 defines a rational diffeomorphism π_ϵ of Σ_ϵ with Λ_ϵ. For $\lambda\in\Lambda_\epsilon$, let $O_{\lambda,\epsilon}$ denote the coadjoint orbit through $\pi_\epsilon^{-1}(\lambda)$, and let $\beta_{\lambda,\epsilon}$ denote the canonical measure on $O_{\lambda,\epsilon}$.

Identify V_0 with $\mathbf{R}^{n-d}$ via the basis $\{e_i\colon i\notin J\}$, and let $d\lambda$ denote the Lebesgue measure on V_0. (If $V_0=(0)$ let $d\lambda$ be point mass measure.) Define $\Theta_\epsilon\colon W\times\Lambda_\epsilon\to\Omega_\epsilon$ by $\Theta_\epsilon(w,\lambda)=P(w,\pi_\epsilon^{-1}(\lambda))$. Then Θ is a diffeomorphism, and for each integrable function f on $\mathfrak{g}^*$ we have

$$\int_{\mathfrak{g}^*}f(l)\,dl=\int_\Omega f(l)\,dl=\sum_{\epsilon\in\{-1,1\}^t}\int_{\Omega_\epsilon}f(l)\,dl$$

and

$$\int_{\Omega_\epsilon}f(l)\,dl=\int_{W\times\Lambda_\epsilon}f(\Theta_\epsilon(w,\lambda))\,|J_{\Theta_\epsilon}(w,\lambda)|\,dw\,d\lambda,$$

where $J_{\Theta_\epsilon}(w,\lambda)$ is the Jacobian of Θ_ϵ. We need to compute $|J_{\Theta_\epsilon}(w,\lambda)|$; to do this, we shall need several preliminary results which will provide detailed information about the mapping Θ_ϵ.

For each l in Ω, set $\rho_0(Z,l)=Z$. In [2], integers i_k, $j(i_k)\in J$ are defined for each k $(1\le k\le d/2)$, and in Lemma 1.3 of [1], a function $\rho_k\colon\mathfrak{g}\times\Omega\to\mathfrak{g}$ is defined for each k $(1\le k\le d/2)$ recursively. In the completely solvable case the recursion formula for ρ_k is simply

$$\rho_k(Z,l)=\rho_{k-1}(Z,l)-c_k(Z,l)\,\rho_{k-1}(Z_{i_k},l)-d_k(Z,l)\,\rho_{k-1}(Z_{j(i_k)},l),$$

where $c_k(Z,l)$ and $d_k(Z,l)$ are given by

$$c_k(Z,l)=\frac{l([Z,\rho_{k-1}(Z_{j(i_k)},l)])}{l([\rho_{k-1}(Z_{i_k},l),\rho_{k-1}(Z_{j(i_k)},l)])},$$

$$d_k(Z,l)=\frac{l([Z,\rho_{k-1}(Z_{i_k},l)])}{l([\rho_{k-1}(Z_{j(i_k)},l),\rho_{k-1}(Z_{i_k},l)])}.$$

For each Z in $\mathfrak{g}$, $l \to \rho_k(Z, l)$ is a rational function. As is shown in [1], the functions ρ_k have the properties that, for each l in Ω, $l([\rho_k(Z,l), W]) = l([Z, \rho_k(W,l)])$ and $\rho_k(\rho_k(Z,l),l) = \rho_k(Z,l)$ for any Z and W in $\mathfrak{g}$. For each k, one has $i_k < j(i_k)$ and $J = \{i_1, i_2, \dots, i_{d/2}, j(i_1), j(i_2), \dots, j(i_{d/2})\}$. The connection between the functions $\{\rho_k : 1 \le k \le d/2\}$ of [1] and the functions $\{r_j : j \in J\}$ of Lemma 1.1 of [2] (cf. the remarks preceding Lemma 1.3 above) is that for each k $(1 \le k \le d/2)$ and each l in Ω,

$$r_{i_k}(l) = \frac{\rho_{k-1}(Z_{j(i_k)}, l)}{l([Z_{i_k}, \rho_{k-1}(Z_{j(i_k)}, l)])}$$

and

$$r_{j(i_k)}(l) = \frac{\rho_{k-1}(Z_{i_k}, l)}{l([Z_{j(i_k)}, \rho_{k-1}(Z_{i_k}, l)])}.$$

For each k $(1 \le k \le d/2)$ and for each i, j $(1 \le i, j \le n)$, set $p_{i,j,k}(l) = l([\rho_k(Z_i, l), Z_j])$ for $l \in \Omega$. We have the formula

$$p_{i,j,k}(l) = p_{i,j,k-1}(l) - c_k(Z_i, l)\, p_{i_k,j,k-1}(l) - d_k(Z_i, l)\, p_{j(i_k),j,k-1}(l).$$

LEMMA 2.2. *For every i, j such that $1 \le i < j \le n$, and for each k, the function $p_{i,j,k}(l)$ depends only on $\pi_i(l)$.*

Proof. We proceed by induction on k. Since $\rho_0 = \mathrm{id}$, the lemma holds for $k = 0$. Suppose that for every $1 \le i < j \le n$, the function $p_{i,j,k-1}(l)$ depends only on $\pi_i(l)$, and fix such indices i and j. We consider three cases.

(i) $i < i_k$. Then $\rho_k(Z_i, l) = \rho_{k-1}(Z_i, l)$, and hence $p_{i,j,k}(l) = p_{i,j,k-1}(l)$.

(ii) $i_k < i < j(i_k)$. Then $d_k(Z_i, l) = 0$, hence

$$\rho_k(Z_i, l) = \rho_{k-1}(Z_i, l) - c_k(Z_i, l)\, \rho_{k-1}(Z_{i_k}, l).$$

By induction, each of the functions $l \to c_k(Z_i, l)$, $p_{i_k,j,k-1}(l)$ and $p_{i,j,k-1}(l)$ depends only on $\pi_i(l)$, hence $p_{i,j,k}(l)$ does also.

(iii) $i > j(i_k)$. Here we apply the induction hypothesis to each of the functions $l \to c_k(Z_i, l)$, $l \to d_k(Z_i, l)$, $p_{i,j,k-1}(l)$, $p_{i_k,j,k}(l)$, and $p_{j(i_k),j,k}(l)$, and thus $p_{i,j,k}(l)$ depends only on $\pi_i(l)$. □

LEMMA 2.3. *Let k be a positive integer, $1 \le k \le d/2$, and let i and j be integers belonging to the set*

$$\{1, 2, \dots, n\} \sim \{i_1, i_2, \dots, i_k, j(i_1), j(i_2), \dots, j(i_k)\}, \quad i \ne j.$$

Then there is a rational functional $q(l)$ depending only on $\pi_{j-1}(l)$ such that $p_{i,j,k}(l) = -\gamma_j(\rho_k(Z_i, l)) l_j + q(l)$. Moreover, the function $l \to \gamma_j(\rho_k(Z_i, l))$ depends only on $\pi_{j-1}(l)$.

Proof. We proceed by induction on k, the lemma being clear for the case $k = 0$. Suppose that $k > 0$ and that, for any integers i and j belonging to

$$\{1, 2, \dots, n\} \sim \{i_1, i_2, \dots, i_{k-1}, j(i_1), j(i_2), \dots, j(i_{k-1})\}, \quad i \ne j,$$

we have $p_{i,j,k}(l) = -\gamma_j(\rho_k(Z_i, l)) l_j + q(l)$, with $q(l)$ and $\gamma_j(\rho_k(Z_i, l))$ as in the hypothesis. By Lemma 2.2, we may assume that $i>j$. As in the previous lemma, we consider three cases.

(i) $j<i_k$. Then $\rho_k(Z_j,l)=\rho_{k-1}(Z_j,l)$, hence $p_{i,j,k}(l)=l([\rho_k(Z_i,l),Z_j])= l([Z_i,\rho_k(Z_j,l)])=l([Z_i,\rho_{k-1}(Z_j,l)])=l([\rho_{k-1}(Z_i,l),Z_j])=p_{i,j,k-1}(l)$. The lemma follows from the induction hypothesis.

(ii) $i_k<j<j(i_k)$. Here we have $p_{i_k,j,k-1}(l)=0$. By Lemma 2.2, $l\to d_k(Z_i,l)$ depends only on $\pi_{i_k}(l)$, hence only on $\pi_{j-1}(l)$. If $i<j(i_k)$, then $d_k(Z_i,l)=0$, and hence $p_{i,j,k}(l)=p_{i,j,k-1}(l)$. Suppose that $i>j(i_k)$. Then applying the induction hypothesis we have rational functions q_0 and q_1 depending only on $\pi_{j-1}(l)$ such that

$$\begin{aligned}p_{i,j,k-1}(l)&=-\gamma_j(\rho_{k-1}(Z_i,l))l_j+q_0(l),p_{j(i_k),j,k-1}(l)\\&=-\gamma_j(\rho_{k-1}(Z_{j(i_k)},l))l_j+q_1(l).\end{aligned}$$

Now

$$\begin{aligned}p_{i,j,k}(l)&=p_{i,j,k-1}(l)-d_k(Z_i,l)p_{j(i_k),j,k-1}(l)\\&=-(\gamma_j(\rho_{k-1}(Z_i,l))-d_k(Z_i,l)\gamma_j(\rho_{k-1}(Z_{j(i_k)},l)))l_j\\&\quad+q_0(l)-d_k(Z_i,l)q_1(l)\\&=-\gamma_j(\rho_k(Z_i,l))l_j+q_0(l)-d_k(Z_i,l)q_1(l).\end{aligned}$$

By induction the functions $\gamma_j(\rho_{k-1}(Z_i,l))$ and $\gamma_j(\rho_{k-1}(Z_{j(i_k)},l))$ depend only on $\pi_{j-1}(l)$, hence $\gamma_j(\rho_k(Z_i,l))$ depends only on $\pi_{j-1}(l)$. We define $q(l)$ by $q(l)=q_0(l)-d_k(Z_i,l)q_1(l)$.

(iii) $j>j(i_k)$. In this case we have that $d_k(Z_i,l)$ and $c_k(Z_i,l)$ depend only on $\pi_{j-1}(l)$ (by Lemma 2.2). Applying the induction hypothesis to each of the functions $p_{i,j,k-1}(l)$, $p_{i_k,j,k-1}(l)$, and $p_{j(i_k),j,k-1}(l)$, we have rational functions q_0, q_1, and q_2 depending only on $\pi_{j-1}(l)$ such that

$$\begin{aligned}p_{i,j,k}(l)&=p_{i,j,k-1}(l)-c_k(Z_i,l)p_{i_k,j,k-1}(l)-d_k(Z_i,l)p_{j(i_k),j,k-1}(l)\\&=-(\gamma_j(\rho_{k-1}(Z_i,l))-c_k(Z_i,l)\gamma_j(\rho_{k-1}(Z_{i_k},l))\\&\qquad-d_k(Z_i,l)\gamma_j(\rho_{k-1}(Z_{j(i_k)},l)))l_j\\&\quad+q_0(l)-c_k(Z_i,l)q_1(l)-d_k(Z_i,l)q_2(l)\\&=-\gamma_j(\rho_k(Z_i,l))l_j+q_0(l)-c_k(Z_i,l)q_1(l)-d_k(Z_i,l)q_2(l).\end{aligned}$$

By induction we see that the functions $\gamma_j(\rho_{k-1}(Z_i,l))$, $\gamma_j(\rho_{k-1}(Z_{i_k},l))$, and $\gamma_j(\rho_{k-1}(Z_{j(i_k)},l))$ depend only on $\pi_{j-1}(l)$, hence $\gamma_j(\rho_k(Z_i,l))$ depends only on $\pi_{j-1}(l)$. The function $q(l)$ is then defined by

$$q(l)=q_0(l)-c_k(Z_i,l)q_1(l)-d_k(Z_i,l)q_2(l). \qquad \square$$

COROLLARY 2.4. *Let $j_k\in J$ and let i be an integer such that $1\le i\le j_k$ and $i\notin J$. Then $\mu_i(g_k(\Phi_k(w,l),l))=1$ holds for every w in W and l in Ω.*

Proof. Since $i<j_k$, $l([r_{j_k}(l),Z_i])=0$ holds for each l in Ω. Now by definition of $r_{j_k}(l)$, there is j' in J such that for each l, $r_{j_k}(l)$ is a multiple of $\rho_{k-1}(Z_{j'},l)$ and hence $p_{j',i,k-1}=0$ on Ω. Since Ω is dense and open, Lemma 2.3 implies that $\gamma_i(\rho_{k-1}(Z_{j'},l))=0$. Thus $\gamma_i(r_{j_k}(l))=0$ and $\mu_i(g_k(\Phi_k(w,l),l))= \exp(\Phi_k(w,l)\gamma_i(r_{j_k}(l)))=1$. $\square$

COROLLARY 2.5. *Let $i \notin J$, $1 \le i \le n$. Then $P_i(w, l)$ is of the form*

$$P_i(w,l) = \mu_i(g_1(\Phi_1(w,l),l)g_2(\Phi_2(w,l),l)\dots g_k(\Phi_k(w,l),l))l_i + R_i(w_1, w_2, \dots, w_k, l_1, l_2, \dots, l_{i-1}),$$

where k is such that $j_k < i < j_{k+1}$.

Proof. Here we use the following principle: If $l \to X(l)$ is a rational function from Ω into $\mathfrak{g}_i$ such that $l \to l(X(l))$ depends only on $\pi_i(l)$, then $l \to X(l)$ depends only on $\pi_i(l)$. It follows from this and from Lemma 2.3 that, for each j in J and for any i $(1 \le i \le n)$, there is a function Y from Ω into $\mathfrak{g}_{i-1}$ such that for each l in Ω,

$$[r_j(l), Z_i] = -\gamma_i(r_j(l))Z_i + Y(l_1, l_2, \dots, l_{i-1}).$$

Hence by Lemma 2.2 and the above, for each n there is a function Y_n from Ω into $\mathfrak{g}_{i-1}$ such that for each l in Ω,

$$\mathrm{ad}^n r_j(l)(Z_i) = (-\gamma_i(r_j(l)))^n Z_i + Y_n(l_1, l_2, \dots, l_{i-1}).$$

It follows that for each i $(1 \le i \le n)$, and for each $t \in \mathbf{R}^d$, $Q_i(t,l) = \sigma_l(t)(Z_i)$ depends only on $\pi_i(l)$; hence, for each r $(1 \le r \le k)$, $\Phi_r(w,l)$ depends on the variables $w_1, w_2, \dots, w_r$ and $l_1, l_2, \dots, l_{j_r}$ only. Hence there is a function $y = y(w_1, w_2, \dots, w_k, l_1, l_2, \dots, l_{i-1})$ into $\mathfrak{g}_{i-1}$ such that

$$\begin{aligned}&\mathrm{Ad}(g_1(\Phi_1(w,l),l)g_2(\Phi_2(w,l),l)\dots g_k(\Phi_k(w,l),l))^{-1}Z_i \\ &\quad = \mu_i(g_1(\Phi_1(w,l),l)g_2(\Phi_2(w,l),l)\dots g_k(\Phi_k(w,l),l))Z_i \\ &\qquad + y(w_1, w_2, \dots, w_k, l_1, l_2, \dots, l_{i-1}).\end{aligned}$$

From this we see that

$$\begin{aligned}P_i(w,l) &= \sigma_l(\Phi(w,l))(Z_i) \\ &= l(\mathrm{Ad}(g_1(\Phi_1(w,l),l)g_2(\Phi_2(w,l),l)\dots g_k(\Phi_k(w,l),l))^{-1}Z_i)\end{aligned}$$

has the desired form. □

We can now compute $|J_{\Theta_\epsilon}(w,\lambda)|$. For the remainder of the paper we simplify notation as follows. When $\epsilon \in \{-1,1\}^t$ is fixed, and for each w in W and λ in Λ_ϵ, set $\Phi_\epsilon(w,\lambda) = \Phi(w, \pi_\epsilon^{-1}(\lambda))$ and $g_\epsilon(w,\lambda) = g(\Phi_\epsilon(w,\lambda), \pi_\epsilon^{-1}(\lambda))$.

LEMMA 2.6. *Let $\epsilon \in \{-1,1\}^t$. Then $|J_{\Theta_\epsilon}(w,\lambda)| = \prod_{j \notin J} |\mu_j(g_\epsilon(w,\lambda))|$.*

Proof. Fix j, $1 \le j \le n$. We write $\{1, \dots, n\} \sim J = \{i_1 < i_2 < \dots < i_{n-d}\}$ and use the following notation. Set $w^{j-1} = (w_1, w_2, \dots, w_{k-1})$ where k is such that $j_{k-1} < j \le j_k$, and set $\lambda^{j-1} = (\lambda_1, \lambda_2, \dots, \lambda_{k-1})$ where k is such that $i_{k-1} < j \le i_k$. Write $\Theta_\epsilon(w,\lambda) = \sum_j \Theta_\epsilon(w,\lambda)_j e_j$. To see that

$$|J_{\Theta_\epsilon}(w,\lambda)| = \prod_{j \notin J} |\mu_j(g_\epsilon(w,\lambda))|,$$

one need only look at each $\Theta_\epsilon(w,\lambda)_j$. Suppose first that $j \in J$, $j = j_k$. If $j \notin \varphi$, then $\Theta_\epsilon(w,\lambda)_j = w_k$, while if $j \in \varphi$ and $j = j_{k_r}$ then

$$\Theta_\epsilon(w,\lambda)_j = \epsilon_r w_{k_r} \bmod (w^{j-1}, \lambda^{j-1}).$$

Now suppose that $j \notin J$. By Corollary 2.4, we have

$$\mu_j(g(w,l)) = \mu_j(g_1(w,l)\, g_2(w,l) \dots g_k(w,l)),$$

where $k = \max\{k' : j_{k'} < j\}$. Hence by Corollary 2.5,

$$\Theta_\epsilon(w,\lambda)_j = \mu_j(g(w,l))\lambda_j \bmod (w^{j-1}, \lambda^{j-1}). \qquad \square$$

Let ψ be any rational function on $\mathfrak{g}^*$ as in Theorem 1.1, so that

$$\psi(sl) = \Delta(s)^{-1}\psi(l)$$

for each s in G and each l in $\mathfrak{g}^*$.

PROPOSITION 2.7. *Let $\epsilon \in \{-1,1\}^t$. Then for each continuous function f on Ω_ϵ having compact support, we have*

$$\int_{\Omega_\epsilon} f(l)|\psi(l)|\,dl = \int_{\Lambda_\epsilon}\left(\int_{O_{\lambda,\epsilon}} f\,d\beta_{\lambda,\epsilon}\right)|\psi(\pi_\epsilon^{-1}(\lambda))|^{-1}|P_J(\pi_\epsilon^{-1}(\lambda))|(2\pi)^{-d}\,d\lambda.$$

Proof. By Lemma 2.6 we have

$$\int_{\Omega_\epsilon} f(l)\,dl = \int_{W\times\Lambda_\epsilon} f(\Theta_\epsilon(w,\lambda)) \prod_{j\notin J}|\mu_j(g_\epsilon(w,\lambda))|\,dw\,d\lambda$$

for each $\epsilon \in \{-1,1\}^t$. Now

$$\Theta_\epsilon(w,\lambda) = P(w,\pi_\epsilon^{-1}(\lambda)) = \sigma_{\pi_\epsilon^{-1}(\lambda)}(\Phi_\epsilon(w,\lambda)) = g_\epsilon(w,\lambda)\pi_\epsilon^{-1}(\lambda)$$

holds for each $w \in W$, $\lambda \in \Lambda_\epsilon$. Thus for each $\epsilon \in \{-1,1\}^t$,

$$\Delta(g_\epsilon(w,\lambda))\,\psi(\Theta_\epsilon(w,\lambda)) = \psi(\pi_\epsilon^{-1}(\lambda)).$$

Hence by Lemma 2.6 and Lemma 1.3 (and using the fact that $\Delta(g_\epsilon(w,\lambda)) = \prod_{j\notin J}|\mu_j(g_\epsilon(w,\lambda))|\,|\mu_J(g_\epsilon(w,\lambda))|$), we have

$$\int_{\Omega_\epsilon} f(l)|\psi(l)|\,dl$$

$$= \int_{W\times\Lambda_\epsilon} f(\Theta_\epsilon(w,\lambda))|\psi(\Theta_\epsilon(w,\lambda))| \prod_{j\notin J}|\mu_j(g_\epsilon(w,\lambda))|\,dw\,d\lambda$$

$$= \int_{\Lambda_\epsilon}\int_W (2\pi)^d|P_J(\pi_\epsilon^{-1}(\lambda))|^{-1} f(\Theta_\epsilon(w,\lambda))|\mu_J(g_\epsilon(w,\lambda))|^{-1}$$
$$\times\Delta(g_\epsilon(w,\lambda))|\psi(\Theta_\epsilon(w,\lambda))|\,|P_J(\pi_\epsilon^{-1}(\lambda))|(2\pi)^{-d}\,dw\,d\lambda$$

$$= \int_{\Lambda_\epsilon}\left((2\pi)^d|P_J(\pi_\epsilon^{-1}(\lambda))|^{-1}\int_W f(\Theta_\epsilon(w,\lambda))|\mu_J(g_\epsilon(w,\lambda))|^{-1}\,dw\right)$$
$$\times|\psi(\pi_\epsilon^{-1}(\lambda))|\,|P_J(\pi_\epsilon^{-1}(\lambda))|(2\pi)^{-d}\,d\lambda =$$

$$= \int_{\Lambda_\epsilon} \left((2\pi)^d |P_J(\pi_\epsilon^{-1}(\lambda))|^{-1} \int_W f(P(w, \pi_\epsilon^{-1}(\lambda))) |\mu_J(g_\epsilon(w, \lambda))|^{-1} dw \right) \times |\psi(\pi_\epsilon^{-1}(\lambda))| |P_J(\pi_\epsilon^{-1}(\lambda))| (2\pi)^{-d} d\lambda$$

$$= \int_{\Lambda_\epsilon} \left(\int_{O_{\lambda,\epsilon}} f \, d\beta_{\lambda,\epsilon} \right) |\psi(\pi_\epsilon^{-1}(\lambda))| |P_J(\pi_\epsilon^{-1}(\lambda))| (2\pi)^{-d} d\lambda. \qquad \square$$

COROLLARY 2.8. *For any integrable function F on $\mathfrak{g}^*/G$, the formula*

$$\int_{\mathfrak{g}^*/G} F(O) \, dm_\psi(O) = \sum_\epsilon \int_{\Lambda_\epsilon} F(O_{\lambda,\epsilon}) |\psi(\pi_\epsilon^{-1}(\lambda))| |P_J(\pi_\epsilon^{-1}(\lambda))| (2\pi)^{-d} d\lambda$$

holds.

Proof. The measure m_ψ is uniquely determined by the formula

$$\int_{\mathfrak{g}^*} f(l) |\psi(l)| \, dl = \int_{\mathfrak{g}^*/G} \left(\int_O f(l) \, d\beta_O(l) \right) dm_\psi(O),$$

and, by Proposition 2.7,

$$\int_{\mathfrak{g}^*} f(l) |\psi(l)| \, dl = \sum_\epsilon \int_{\Omega_\epsilon} f(l) |\psi(l)| \, dl = \sum_\epsilon \int_{\Lambda_\epsilon} \left(\int_{O_{\lambda,\epsilon}} f \, d\beta_{\lambda,\epsilon} \right) |\psi(\pi_\epsilon^{-1}(\lambda))|^{-1} |P_J(\pi_\epsilon^{-1}(\lambda))| (2\pi)^{-d} d\lambda. \qquad \square$$

Set $[\rho_{\lambda,\epsilon}] = K^{-1}(O_{\lambda,\epsilon})$ for $\epsilon \in \{-1, 1\}^t$ and $\lambda \in \Lambda_\epsilon$. For each nonzero rational function ψ on $\mathfrak{g}^*$ satisfying $\psi(sl) = \Delta(s)^{-1}\psi(l)$ for s in G and l in $\mathfrak{g}^*$, let $A_{\psi,\lambda,\epsilon}$ denote the semi-invariant operator of weight Δ for the irreducible representation $\rho_{\lambda,\epsilon}$ corresponding to the restriction of $|\psi|$ to $O_{\lambda,\epsilon}$. From Corollary 2.8 and Theorem 1.1 the following is immediate.

THEOREM 2.9. *Let G be a connected, simply connected, and completely solvable Lie group. Let $\{Z_1, Z_2, \ldots, Z_n\}$ be a Jordan–Holder basis for the Lie algebra $\mathfrak{g}$ of G. Fix a Euclidean measure on $\mathfrak{g}$, and let the Haar measure da and the modular function Δ be as above. Then each of the following holds.*

(1) *There is a finite collection of pairwise disjoint open subsets U_ϵ of $G^\wedge$ and there is a subspace V_0 of $\mathfrak{g}^*$ such that for each ϵ, U_ϵ is parameterized by a Zariski open subset Λ_ϵ of V_0, $\bigcup U_\epsilon$ is dense in $G^\wedge$, and the complement of $\bigcup U_\epsilon$ has Plancherel measure zero.*

(2) *Let ψ be any nonzero rational function on $\mathfrak{g}^*$ satisfying*

$$\psi(sl) = \Delta(s)^{-1}\psi(l)$$

for all s in G and l in $\mathfrak{g}^*$. For each ϵ, there is a rational function $r_{\psi,\epsilon}$ on V_0 such that for any smooth compactly supported function ϕ on G, the function

$$\lambda \to \mathrm{Tr}(A_{\psi,\lambda,\epsilon}^{-1/2}\rho_{\lambda,\epsilon}(\phi)A_{\psi,\lambda,\epsilon}^{-1/2})|r_{\psi,\epsilon}(\lambda)|$$

on Λ_ϵ *is Lebesgue integrable. For any such* ϕ *one has*

$$\phi(e) = \sum_\epsilon \int_{\Lambda_\epsilon} \mathrm{Tr}(A_{\psi,\lambda,\epsilon}^{-1/2}\rho_{\lambda,\epsilon}(\phi)A_{\psi,\lambda,\epsilon}^{-1/2})|r_{\psi,\epsilon}(\lambda)|\,d\lambda.$$

The function $r_{\psi,\epsilon}$ is given by $r_{\psi,\epsilon}(\lambda) = \psi(\pi_\epsilon^{-1}(\lambda))P_J(\pi_\epsilon^{-1}(\lambda))(2\pi)^{-d}$, where d is the maximal dimension of orbits in $\mathfrak{g}^*/G$, $J = \{j_1 < j_2 < \cdots < j_d\}$ is the subset of $\{1, 2, \ldots, n\}$ given in Theorem 1.2, and P_J is the Pfaffian of the skew-symmetric matrix $[l([Z_{j_r}, Z_{j_k}])]_{1 \le r, k \le d}$.

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