Polynomial Proper Holomorphic Mappings between Balls, II

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Introduction

The purpose of this paper is to give a complete classification of all polynomial proper holomorphic maps between balls in complex vector spaces. The results here are extensions of the author's results in [D2; D4], and clarify the partial results in many other papers [CS2; Fa1; Fa2; Fo3; R3; W]. Furthermore, generalizations of these results apply in the rational case. The rational case is the natural one, because of the result of Forstneric [Fo2] that a sufficiently differentiable proper holomorphic map between balls is necessarily rational. The author has obtained partial results that he believes will lead to a complete classification of the rational proper maps. These will appear elsewhere. It is probably impossible to give such a classification of the proper maps that are not smooth at the boundary. See [Dor; H] for the construction of such proper maps.

The first result of the present paper gives a factorization of a proper polynomial map between balls. The author gave another version of this in [D2], but the present statement and proof are more transparent. The proof is the same without regard to the range or domain dimension, including the 1-dimensional case, or even the degree of the polynomial. The operations involved are either linear transformations, tensor products, or inverses of these. Analysis of the proof gives a complete description. The tensor product operation is as follows. Suppose that f is a holomorphic map from C^n to C^N, and that A is a linear subspace of C^N of dimension k. Let z denote the identity operation on C^n. We form a new holomorphic map

$$E_{(A,z)}f\colon C^n \to C^{N+k(n-1)}$$

as follows. Write $f = f_A \oplus f_{A^\perp}$ for the orthogonal direct sum decomposition of f determined by that on its range C^N, and define the tensor product operation by

$$E_{(A,z)}f = (f_A \otimes z) \oplus f_{A^\perp}.$$

See Section I for more details. Let B_n denote the unit ball in C^n. If we omit the dependence on A in the notation, our first result can be stated as follows.

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THEOREM. *Let $p: B_n \to B_N$ be a proper holomorphic polynomial map of degree m. Then p admits the finite composition product factorization*

$$p = (E^{-1})^m L E^m(1).$$

Here the operation E denotes the tensor product, E^{-1} denotes its inverse, and L is linear. The number 1 denotes the constant map to the point 1 on the unit circle. Each of the intermediate maps is a proper holomorphic map from B_n to some B_k. The proof of the theorem shows exactly how to choose the various subspaces. By keeping careful track of this, one obtains a complete classification of all proper polynomial maps between balls.

Different choices of the subspaces in the above factorization will in general lead to inequivalent maps. The author showed in [D1] that polynomial maps that preserve the origin are spherically equivalent (equivalent up to automorphisms of both the domain and range balls) only when they are unitarily equivalent. By combining the first result with this, and using some linear algebra, we determine on precisely how many parameters the family of proper polynomial maps of degree m from B_n to any larger dimensional unit ball depends. Let us say that an $m-1$ jet

$$s_{m-1}: C^n \to C^N$$

is allowable if there is a proper polynomial map

$$p: B_n \to B_N$$

whose $m-1$ jet is s_{m-1}. Assuming that N is large enough, the second main result tells us that there is an open subset in the space of jets of order $m-1$ consisting of allowable jets. The $m-1$ jet determines the map up to equivalence. From this we determine the number of parameters. See the statement of Theorem 18 for more detail.

THEOREM. *Let $p(z) = \sum_{|\alpha|=0}^{m} A_\alpha z^\alpha$ be a polynomial of degree m that is a proper holomorphic map from B_n to some B_N. Suppose also that the $m-1$ jet is generic* (*see Section IV*). (*This requires N to be sufficiently large.*) *Then the $m-1$ jet of p is otherwise arbitrary except that the numbers $\langle A_\alpha, A_\beta \rangle$ for $|\alpha|, |\beta| \leq m-1$ must satisfy the finite number of polynomial inequalities determined by* 18.3. *These determine p up to unitary equivalence. The set of allowable C^N-valued jets of order $m-1$ contains an open neighborhood in that space. The number of independent parameters is*

$$\frac{D(D+1)}{2} \quad \textit{where } D = \binom{n+m-1}{n}.$$

One very special case of this result is the following result from [D4], where the number is adjusted to account for the fact that we assume in this corollary that there is no constant term.

COROLLARY. *The set of spherical equivalence classes for proper holomorphic quadratic polynomials from B_n preserving the origin has dimension $d = n(n+1)/2$. These d numbers are the inner products of the column vectors of the derivative at the origin. (Thus n of them must be real.)*

The theorem is the analog of this corollary for polynomials of higher degree m, where one must consider the $m-1$ jet at the origin, and allows the possibility of a constant term.

The paper is organized so that those formal identities that apply in the rational case are proved in that generality. The ideas in the rational case are similar, but there one must consider Hermitian forms of mixed signature. The polynomial case is easier because of positive semidefiniteness.

I. Preliminaries

Recall that B_n denotes the unit ball in C^n. A holomorphic map $f: B_n \to B_N$ is called *proper* if $f^{-1}(K)$ is a compact subset of the domain ball whenever K is a compact subset of the range ball. Equivalently, f is proper if and only if, whenever a sequence tends to the boundary of the domain ball, the image sequence tends to the boundary of the range ball. If f extends continuously to the boundary, this condition is implied by the statement that $\|f(z)\| = 1$ whenever $\|z\| = 1$, together with the assumption that $f: B_n \to B_N$. It follows from the implicit function theorem that, for $N < n$, there can be no examples. In case $n = N \geq 2$, the proper maps of the unit ball are the automorphisms of the ball [A]. They are linear fractional transformations of the form

$$U \frac{a - L_a z}{1 - \langle z, a \rangle}.$$

Here U is unitary, $\langle \, , \rangle$ denotes the usual Hermitian inner product, a is a point in B_n, and L_a is the linear map defined by the formula

$$L_a z = \frac{\langle z, a \rangle a}{s+1} + sz.$$

In this formula, s is the positive number satisfying $|s|^2 = 1 - \|a\|^2$. In case $n = N = 1$, the proper maps are the finite Blaschke products

$$e^{i\theta} \prod \frac{a_j - z}{1 - \overline{a_j} z}.$$

Each a_j is a (not necessarily distinct) point in the disk and the product is finite. Thus in this case the only proper polynomial maps are monomials. It depends on n and N whether there are polynomial maps that are not equivalent to monomial maps. For example, for each n there is a proper polynomial map $p: B_n \to B_{4n-2}$ that is not equivalent to a monomial. We discuss this briefly in Section V. Here is the precise definition of (spherical) equivalence.

1. DEFINITION. Suppose $f, g: B_n \to B_N$ are proper holomorphic maps. They are called *spherically equivalent* if there are automorphisms ζ, ξ of the respective balls so that $f = \xi g \zeta$.

As noted above, the author [D1] has proved that proper polynomial maps that preserve the origin are spherically equivalent only when they are unitarily equivalent.

We review some additional considerations from linear algebra. Let $H(n, m)$ denote the vector space of homogeneous (holomorphic) complex-valued polynomials of degree m in n variables, and let $d(n, m)$ denote its dimension. Let $V(n, m)$ denote the vector space of (holomorphic) complex-valued polynomials of degree at most m in n variables, and let $D(n, m)$ denote its dimension. Thus $D(n, m) = \sum_0^m d(n, k)$. We also wish to consider polynomials that take values in C^N. Suppose $p(z) = \sum_{|\alpha|=0}^m A_\alpha z^\alpha = \sum_0^m \mathbf{p}_k$ is such a polynomial map. Its squared norm $\|p(z)\|^2$ is of course a real-valued polynomial in the variables z and $\bar{z}$ on C^n. For any real-valued polynomial function of degree at most m in each of these variables, say $\sum_0^m c_{\alpha\beta} z^\alpha \bar{z}^\beta$, we have the Hermitian form on $V(n, m)$ corresponding to the matrix $(c_{\alpha\beta})$. Thus to the polynomial p we assign the form Q_p that arises from its squared norm.

2. $$Q_p \approx \sum_0^m c_{\alpha\beta} z^\alpha \bar{z}^\beta = \sum_0^m \langle A_\alpha, A_\beta \rangle z^\alpha \bar{z}^\beta = \|p(z)\|^2.$$

Notice that if we replace p by Up, for U unitary, the form Q_p is unchanged. Conversely, if two such forms Q_p and Q_g are the same, then $p = Ug$ for some unitary U. This is because $\|p\| = \|g\|$ for vector-valued holomorphic maps if and only if there is a unitary map such that $p = Ug$ [D3]. From 2 it follows that the matrix of Q_p depends on D real numbers and $D(D-1)/2$ additional complex parameters. These parameters are the inner products of the coefficients of p, namely $\langle A_\alpha, A_\beta \rangle$.

Consider also a real-valued polynomial w on C^n that is homogeneous of degree m in both z and $\bar{z}$. As above one can think of w as a Hermitian form on $H(n, m)$. It is then trivial to verify that w is positive semidefinite if and only if $w(z, \bar{z})$ can be written as $\sum |f_k(z)|^2$ for holomorphic homogeneous polynomials f_k of degree m. Thus positive semidefiniteness for such a form is equivalent to the form coming from a norm squared of a holomorphic vector-valued polynomial. It is also easy to see that this condition is not implied simply by the nonnegativity of the values of $w(z, \bar{z})$.

3.1. EXAMPLE. $(|z_1|^2 - |z^2|^2)^2$ is nonnegative as a function, but the corresponding form has a negative eigenvalue on $H(2, 2)$.

3.2. ELEMENTARY WARNING. The form corresponding to the real-valued polynomial $\sum_0^m c_{\alpha\beta} z^\alpha \bar{z}^\beta$ is positive definite if and only if all the principal minor determinants are positive; in the semidefinite case one has the analogous statement if one remembers to take *all* possible principal minor determinants. For example, with the 3×3 matrix

$$C=(c_{ij})=\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

it is not enough to note that $c_{11}=0$, $c_{11}c_{22}-|c_{12}|^2=0$, and $\det(C)=0$. The matrix has in fact a negative eigenvalue. One must consider all the principal minors, as $c_{22}c_{33}-|c_{23}|^2<0$.

As remarked above, the condition of positive semidefiniteness can be given as a finite number of inequalities on the complex numbers $c_{\alpha\beta}$. In the theorems on proper maps, we take as elementary the tedious process of deciding whether given Hermitian forms are in fact positive semidefinite.

Suppose that $p=\sum c_\alpha z^\alpha=\sum \mathbf{p}_k$ is a vector-valued polynomial. We often use orthogonality conditions on the coefficients. Note that the statement $\langle \mathbf{p}_k, \mathbf{p}_j\rangle=0$ means (as it must) that $\langle c_\alpha, c_\beta\rangle=0$ whenever $|\alpha|=k$ and $|\beta|=j$.

Finally, we need to describe the tensor product operation used in this paper. If g and h are two vector-valued maps with the same domain and with components g_j and h_k, then the tensor product $g\otimes h$ is the map whose components are all possible g_jh_k. Suppose that we write $C^N=A\oplus A^\perp$ as an orthogonal sum of linear subspaces. If f is a map that takes values in C^N then we can write $f=(f_A, f_B)=f_A\oplus f_{A^\perp}$ for the corresponding decomposition of f. Suppose now that g is a map with the same domain as f and with range yet another complex vector space. We define a new mapping $E_{(A,g)}f$ (or Ef, if g and A are understood) as follows.

4. DEFINITION. The tensor product of f by g on the subspace A is given by $E_{(A,g)}f=(f_A\otimes g)\oplus f_{A^\perp}$.

We have the following simple fact.

5. LEMMA. *If Ω is a domain in C^n, and if $f:\Omega\to B_N$ and $g:\Omega\to B_K$ are proper maps, then, for any choice of subspace A, $E_{(A,g)}f$ is a proper map from Ω to some B_j.*

Proof. Note that $\|E_{(A,g)}f\|^2=\|g\otimes f_A\|^2+\|f_{A^\perp}\|^2=\|g\|^2\|f_A\|^2+\|f_{A^\perp}\|^2$. If we choose a sequence of points in Ω tending to its boundary, then $\|g\|^2$ tends to 1 on this sequence, as g is a proper map to a ball. Thus $\|E_{(A,g)}f\|^2$ tends to $\|f_A\|^2+\|f_{A^\perp}\|^2=\|f\|^2$ which tends to 1 since f is proper. Thus $E_{(A,g)}f$ is also proper. □

II. Identities Satisfied by Rational Proper Maps between Balls

Suppose that $f=p/q$ is a rational proper holomorphic map from B_n to B_N. We can write $p=\sum p_\alpha z^\alpha$ and $q=\sum q_\alpha z^\alpha$, where each p_α is a point in C^N and each q_α is a complex number. We assume without loss of generality that

$q_0 = q(0) = 1$, and that p and q have no nontrivial common factors. Consider the following identity.

6. $$\|p(z)\|^2 = \|q(z)\|^2 \quad \text{whenever } \|z\|^2 = 1.$$

If we substitute $z = re^{i\theta}$ (this is multi-index notation, so that $r = (r_1, \ldots, r_n)$ and $\theta = (\theta_1, \ldots, \theta_n)$) into 6, we obtain

7. $$\sum (\langle p_\alpha, p_\beta \rangle - q_\alpha \bar{q}_\beta) r^{\alpha+\beta} e^{i\theta(\alpha-\beta)} = 0 \quad \text{whenever } \|r\|^2 = 1.$$

Our goal is to describe all choices of the vectors p_α and scalars q_α that are solutions to 7. The first simplication arises from observing that we may equate Fourier coefficients in 7. Write $\alpha - \beta = \gamma$. We obtain

8. $$\sum (\langle p_{\beta+\gamma}, p_\beta \rangle - q_{\beta+\gamma} \bar{q}_\beta) r^{\gamma+2\beta} = 0 \text{ on } \|r\|^2 = 1 \quad \text{for each multi-index } \gamma.$$

Since this holds for each γ, we can write this as

9. $$\sum (\langle p_{\beta+\gamma}, p_\beta \rangle - q_{\beta+\gamma} \bar{q}_\beta) r^{2\beta} = 0 \text{ on } \|r\|^2 = 1 \quad \text{for all } \gamma.$$

There is another version of 9 that arises if we write $p = \sum \mathbf{p}_k$ and $q = \sum \mathbf{q}_k$, where now the summands stand for homogeneous polynomials. In this case, we replace z by $ze^{i\theta}$ in 6 and equate Fourier coefficients again. The result is the set of identities 10, which hold on $\|z\| = 1$.

10.1. $$\sum (\langle \mathbf{p}_k, \mathbf{p}_{j+k} \rangle - \mathbf{q}_k \bar{\mathbf{q}}_{j+k}) = 0 \quad \text{for each } j > 0.$$

10.2. $$\sum (\|\mathbf{p}_k\|^2 - \|\mathbf{q}_k\|^2) = 0.$$

In the particular case that f is a polynomial, we have $q = \mathbf{q}_0 = 1$, and the identities 10 and 7 simplify. Assuming that p is of degree m, we have the following.

11.1. On the sphere,

$$\begin{aligned}
&\langle \mathbf{p}_0, \mathbf{p}_m \rangle = 0 \\
&\langle \mathbf{p}_0, \mathbf{p}_{m-1} \rangle + \langle \mathbf{p}_1, \mathbf{p}_m \rangle = 0 \\
&\langle \mathbf{p}_0, \mathbf{p}_{m-2} \rangle + \langle \mathbf{p}_1, \mathbf{p}_{m-1} \rangle + \langle \mathbf{p}_2, \mathbf{p}_m \rangle = 0 \\
&\vdots \\
&\sum \|\mathbf{p}_k\|^2 = 1.
\end{aligned}$$

These identities hold on the sphere, but by homogenizing them we obtain identities that hold everywhere. The resulting identities are

11.2. $$\begin{aligned}
&\langle \mathbf{p}_0, \mathbf{p}_m \rangle = 0 \\
&\langle \mathbf{p}_0, \mathbf{p}_{m-1} \rangle \|z\|^2 + \langle \mathbf{p}_1, \mathbf{p}_m \rangle = 0 \\
&\langle \mathbf{p}_0, \mathbf{p}_{m-2} \rangle \|z\|^4 + \langle \mathbf{p}_1, \mathbf{p}_{m-1} \rangle \|z\|^2 + \langle \mathbf{p}_2, \mathbf{p}_m \rangle = 0 \\
&\vdots
\end{aligned}$$

11.3. $$\sum \|\mathbf{p}_k\|^2 \|z\|^{2m-2k} = \|z\|^{2m}.$$

The formulas for polynomials corresponding to identity 7 are

11.4. $\sum\langle p_\alpha, p_\beta\rangle r^{\alpha+\beta}=0$ whenever $\|r\|^2=1$ and $\alpha\neq\beta$;

11.5. $\sum\|p_\alpha\|^2 r^{2\alpha}=1$ whenever $\|r\|^2=1$.

III. Proof of the Factorization Theorem

The idea of the factorization result is that every polynomial proper map can be made homogeneous by applying enough tensor products. The homogeneous case is then easy to understand. The following result of Rudin [R3] (see also [D2]) follows almost immediately from 11.3. We include our very simple proof because the result plays an important part in the classification problem.

12. THEOREM. *Suppose that $p: B_n \to B_N$ is a homogeneous proper polynomial mapping of degree m with linearly independent components. Then there is a unitary map on C^N such that $p=Uf$, where f is the monomial mapping $f(z)=(\ldots, c_\alpha z^\alpha, \ldots)$. Here $|c_\alpha|^2$ is the multinomial coefficient $\binom{m}{\alpha}$, and N is the dimension of the vector space $H(n,m)$ of homogeneous monomials of degree m in n variables.*

Proof. Note that the last identity in 11.1 follows immediately from the hypotheses, as p maps the sphere to the sphere. Then 11.3 holds, as it is just the homogenized version of 11.1. But, by the homogeneity, the left side is just $\|p(z)\|^2$. Expand the right side of 11.3 according to the multinomial theorem. The result is

$$\|p(z)\|^2=\|f(z)\|^2=\|(\ldots, c_\alpha z^\alpha, \ldots)\|^2.$$

Now we have two vector-valued holomorphic functions with the same number of components and whose norms are the same. Then (see [D3]) there is a unitary matrix of constants so that $p=Uf$. □

We now show how to reduce the classification of proper polynomial maps to the homogeneous case. Suppose now that p is a proper polynomial map of degree m from B_n to some B_N. According to 11.1, we have that $\mathbf{p}_0$ is orthogonal to $\mathbf{p}_m$. If $\mathbf{p}_0$ is zero, go to $\mathbf{p}_1$. If $\mathbf{p}_0$ is not zero, let A be the subspace generated by $\mathbf{p}_0$. Write $\mathbf{p}_m=\sum P_\alpha z^\alpha$ where $|\alpha|=m$. By choosing specific points on the sphere, we see that each P_α is orthogonal to A. We write $C^N=A\oplus A^\perp$, and consider the proper map $E_{(z,A)}p$. According to Lemma 5, it is a proper map $p^\#$ from B_n to some B_{N_1}. Furthermore $p^\#$ satisfies $p^\#(0)=\mathbf{p}_0^\#=0$, and is still of degree m. We now see from 11.1 that $\langle \mathbf{p}_1^\#, \mathbf{p}_m^\#\rangle=0$. Again we can write $C^{N_1}=A^\#\oplus A^{\#\perp}$, for the subspace $A^\#$ spanned by the vectors in $\mathbf{p}_1^\#$. We now form $E_{(z,A^\#)}p^\#=p^{\#\#}$. The map $p^{\#\#}$ is now a proper map from B_n to B_{N_2} that satisfies $\mathbf{p}_0^{\#\#}=\mathbf{p}_1^{\#\#}=0$. Proceeding by induction, we obtain a finite number of intermediate maps p, $p^\#$, $p^{\#\#}$, etc. and subspaces A, $A^\#$, $A^{\#\#}$, etc., so that

$p^{\#k} = E_{(z, A^{\#(k-1)})} p^{\#(k-1)}$. The resulting map $p^{\#m}$ is homogeneous of degree m. After a linear transformation L that perhaps decreases dimensions, we obtain a map $Lp^{\#m}$ that satisfies the hypotheses of Theorem 12. According to that result, we can write, after replacing L by UL if necessary, $Lp^{\#m} = f$, where f is the map in that theorem. Next, we observe that this map f can be built up from the identity by iterating the tensor product operation, where the tensor product is taken on the full range. In fact we can write $f = \otimes^m(1) = E^m(1)$, where 1 denotes the constant map from B_n to the point 1 in C.

We have proved the following theorem.

13. THEOREM. *Let p be a proper holomorphic map from B_n to B_N of degree m. Then p admits a finite composition product factorization*

13.1. $$p = (E^{-1})^m L E^m(1).$$

Here 1 is the constant mapping to the point 1 in C, and L is linear. Every intermediate composition is a proper map to the unit ball in some C^k. Each application of E is the full tensor product (the corresponding subspace A is the whole space), with $Ef = z \otimes f$. Each application of E^{-1} is defined by $E^{-1}((z \otimes g) \oplus h) = g \oplus h$. The map p is completely determined by the linear map L and the choices of subspaces on which the operation E^{-1} is applied.

It is perhaps useful to remark that one can choose a 0-dimensional subspace when applying E^{-1}. Thus $E^{-1}((z \otimes 0) \oplus h) = h$. This tactic enables one to use the same notation for all cases. The homogeneous polynomial maps of Theorem 12, for example, do not require the applications of E^{-1} used in Theorem 13, but still admit the factorization 13.1 if one uses this approach.

14. EXAMPLE (see [D4]). We describe all quadratic polynomials that map 0 to 0 and are proper maps between balls. Suppose that $f(z) = A(z, z) + Bz$. Start with the constant map 1. Apply the tensor product to get the identity map $z \to z$. We must have $B^*B \leq I$, where B^* is the usual adjoint of B. Apply a linear map to get $\sqrt{(I - B^*B)}z \oplus Bz$. It doesn't matter which square root we take, as the resulting maps will be unitarily equivalent. Apply the tensor product on the first slot to get $(z \otimes \sqrt{(I - B^*B)}z) \oplus Bz$. This is the map f.

Alternatively, start with the map $z \otimes z$, and write the second z as

$$\sqrt{(I - B^*B)}z \oplus Bz.$$

We then have

$$\begin{aligned} 1 &\text{ goes to } z \otimes 1 = z \\ &\text{ goes to } z \otimes z \\ &\text{ goes to } z \otimes (\sqrt{(I - B^*B)}z \oplus Bz) = (z \otimes \sqrt{(I - B^*B)}z) \oplus z \otimes Bz \\ &\text{ goes to } (z \otimes \sqrt{(I - B^*B)}z) \oplus Bz = A(z, z) + Bz. \end{aligned}$$

This shows that the general quadratic without constant term can be written as $E^{-1}LEE(1)$. It is easy to see that two maps of the form $A(z, z) + Bz$ and

$A^{\#}(z,z)+B^{\#}z$ are equivalent if and only if $B^{\#}$ and B are unitarily equivalent. This shows that the number of independent parameters for maps of the form $A(z,z)+Bz$ equals $n(n+1)/2$, as this is the number of independent inner products of the column vectors of B, and these are the only unitary invariants.

IV. Allowable Jets and Number of Parameters

In this section we determine exactly how many parameters it takes to describe all proper polynomial maps of degree m from B_n to B_N. According to Example 14, there are $n(n+1)/2$ equivalence classes of quadratic polynomial maps from the n-ball that preserve the origin. The parameters are inner products of vectors in C^N.

Suppose first that $p: C^n \to C^N$ is a polynomial map of degree m. Then the number of parameters on which p depends is

15.1. $$ND(n,m)=N(1+d(n,1)+\cdots+d(n,m)).$$

Here $d(n,k)$ is the dimension of the vector space of homogeneous polynomials of degree k in v variables, and $D(n,m)$ is the dimension of the vector space $V(n,m)$ of complex-valued polynomials of degree at most m in n variables. We prefer to think of the coefficients as elements in C^N, so that we can say that the coefficients of a polynomial map of degree m are $D(n,m)$ vectors in C^N. Suppose $p(z)=\sum_{|\alpha|=0}^{m} A_\alpha z^\alpha$ is such a polynomial map. Recall that we associate to p the Hermitian form Q_p defined by

2. $$Q_p \approx \sum_0^m \langle A_\alpha, A_\beta\rangle z^\alpha \bar{\mathbf{z}}^\beta = \|p(z)\|^2.$$

The matrix of Q_p depends on D real numbers and $D(D-1)/2$ additional complex parameters. These parameters are the inner products of the coefficients of p, namely $\langle A_\alpha, A_\beta\rangle$.

Now suppose also that $p=\sum A_\alpha z^\alpha = \sum \mathbf{p}_k$ is a proper map between balls. Recall the identities

11.4. $$\sum \langle A_\alpha, A_\beta\rangle r^{\alpha+\beta}=0 \quad \text{whenever } \|r\|^2=1 \text{ and } \alpha\neq\beta;$$

11.5. $$\sum \|A_\alpha\|^2 r^{2\alpha}=1 \quad \text{whenever } \|r\|^2=1;$$

11.2. $$\text{for each } j>0, \quad \sum_{k=0}^{m-j} \langle \mathbf{p}_k, \mathbf{p}_{j+k}\rangle \|z\|^{2m-2k}=0 \text{ on } C^n;$$

11.3. $$\text{for } j=0, \quad \sum_0^m \|p_k\|^2 \|z\|^{2m-2k}=\|z\|^{2m} \text{ on } C^n.$$

The identities 11.2 place conditions on the inner products of the vectors in each homogeneous part. One way to get around this difficulty is to assume that these ranges are orthogonal, so that these conditions are automatically satisfied. A weaker assumption is the following.

16. DEFINITION. A vector-valued polynomial $\sum A_\alpha z^\alpha$ is called *generic* if the collection $\{A_\alpha\}$ is linearly independent.

17. DEFINITION. A vector-valued $m-1$ jet $s: C^n \to C^N$ is called *allowable* if there is a proper holomorphic map $p: B_n \to B_N$ whose $m-1$ jet is s.

18. THEOREM. *Suppose that $p: B_n \to B_N$ is a proper polynomial map of degree at most m. Suppose also that p is generic. Write $p = \mathbf{p}_0 + \cdots + \mathbf{p}_m$ for its expansion in terms of homogeneous polynomials. The following hold:*

18.1. $\mathbf{p}_0$ *is arbitrary except for* $\|p_0\|^2 \leq 1$.

18.2. *For each k with* $0 \leq k \leq m-1$, *we have the statement that* $\mathbf{p}_k$ *is arbitrary except that the form* W_k *defined on* $H(n, m)$ *by* 18.3 *must be positive semidefinite.*

18.3. $W_k \approx \|z\|^{2m} - \|z\|^{2m}\|\mathbf{p}_0\|^2 - \|z\|^{2m-2}\|\mathbf{p}_1\|^2 - \cdots - \|z\|^{2m-2k}\|\mathbf{p}_k\|^2.$

18.4. *The highest-order part* $\mathbf{p}_m$ *is determined* (*up to a unitary map*) *by equations* 11.2 *and* 11.3. *Hence,*

$$\begin{aligned}
&\langle \mathbf{p}_0, \mathbf{p}_m\rangle = 0,\\
&\langle \mathbf{p}_1, \mathbf{p}_m\rangle = -\langle \mathbf{p}_0, \mathbf{p}_{m-1}\rangle\|z\|^2,\\
&\langle \mathbf{p}_2, \mathbf{p}_m\rangle = -\langle \mathbf{p}_0, \mathbf{p}_{m-2}\rangle\|z\|^4 - \langle \mathbf{p}_1, \mathbf{p}_{m-1}\rangle\|z\|^2,\\
&\quad\vdots\\
&\|\mathbf{p}_m\|^2 = \|z\|^{2m} - \sum_0^{m-1}\|\mathbf{p}_k\|^2\|z\|^{2m-2k}.
\end{aligned}$$

18.5. *A generic* $m-1$ *jet* $s = \sum A_\alpha z^\alpha$ *is allowable if the numbers* $\langle A_\alpha, A_\beta\rangle$ *satisfy the finite number of polynomial inequalities determined by* 18.2. *If N is sufficiently large, there is an open subset in the space of* C^N*-valued jets of order* $m-1$ *consisting of allowable jets.*

18.6. *Therefore the number of independent parameters is*

$$\frac{D(D+1)}{2} \quad \textit{where } D = \binom{n+m-1}{n} = D(n, m-1).$$

Of these numbers, D are (*nonnegative*) *real and the other* $D(D-1)/2$ *are complex.*

Proof. Statement 18.1 is obvious as $\mathbf{p}_0 = p(0)$ lies in B_N. To prove 18.2, we assume inductively that we have found $\mathbf{p}_0, \mathbf{p}_1, \ldots, \mathbf{p}_{k-1}$, and that $k < m$. According to identity 11.4, $\sum\|z\|^{2m-2k}\|\mathbf{p}_k\|^2 = \|z\|^{2m}$ holds for all z in C^n. Thus, by subtracting the first k terms from both sides of 11.2, we see that 18.3 is expressed as a sum of absolute values squared of holomorphic functions. According to the remarks in Section I, this holds if and only if the corresponding form is positive and semidefinite. The term $\|\mathbf{p}_m\|^2$ is determined by the last equality in 18.4. As long as $k < m$, the other equations determine the projections of $\mathbf{p}_m$ onto the subspaces spanned by the coefficients of $\{\mathbf{p}_0, \mathbf{p}_1, \ldots, \mathbf{p}_{m-1}\}$. Since the jet is generic, we have no compatibility conditions on

$\mathbf{p}_k$. We now determine $\mathbf{p}_m$ up to a unitary. All the inner products $\langle \mathbf{p}_k, \mathbf{p}_m \rangle$ (including $k=m$) are determined by the equations 11.3 and 11.4. These inner products actually determine $\langle A_\alpha, A_\beta \rangle$ for $|\alpha|=k$ and $|\beta|=m$, by equating coefficients of the real analytic function $\|p\|^2$. Thus, to define the map p, we must give the Hermitian form W_k for $k = 0, \ldots, m-1$. When $k=0$, this amounts to selecting $p(0)$. For other k, we select a Hermitian form on $H(n, k)$. According to 11.2, we need only concern ourselves with forms defined by $\|\mathbf{p}_k\|^2$.

To analyze this more fully, notice that the homogeneous polynomial map of degree m given by Theorem 12 has corresponding Hermitian form $\|\mathbf{p}_m\|^2 = \|z\|^{2m}$. The matrix corresponding to this form is diagonal and strictly positive definite, with eigenvalues given by the squares $\binom{m}{\alpha}^2$ of the multinomial coefficients. Thus, the minimum eigenvalue is 1. This shows that we can achieve all possible $\mathbf{p}_0, \ldots, \mathbf{p}_{m-1}$ allowed by 18.3, in particular, if all these forms are close to 0 and if the jet is generic. Since the genericity condition is also open, there is an allowable open subset (whose closure contains the origin) in the space of $m-1$ jets in n variables. Thus the parameters can be chosen to be precisely

$$\{\langle A_\alpha, A_\beta \rangle : |\alpha|, |\beta| \leq m-1\}.$$

These determine $\mathbf{p}_m$ up to a unitary, and hence p up to unitary equivalence. Thus the number of parameters is given exactly by 18.6. The last statement follows because those inner products that are squared norms must be positive. □

19. REMARK. The conditions in Theorem 18.5 depend only on inner products, so are unitarily invariant. Using the fact that polynomial maps preserving the origin are spherically equivalent only when they are unitarily equivalent, we obtain spherical equivalence with this assumption. Without this assumption, one must be careful. The map $f: B_n \to B_{n+1}$ defined by $f(z) = (\cos\theta, \sin(\theta) z)$ is spherically equivalent to the map $f(z) = (0, z)$ by composition with an automorphism that sends $(\cos(\theta), 0)$ to $(0, 0)$. On the other hand, it is not unitarily equivalent. Thus, counting parameters under unitary equivalence gives the same value as counting them under spherical equivalence only when we consider polynomials preserving the origin. Otherwise the counts can differ. If we assume that $f(0) = 0$, we must subtract from the number $D(D+1)/2$ in Theorem 16 the number of inner products with the constant term. The result is $(D-1)(D-2)/2$.

V. Examples

Let us first give some examples of the factorization result. Consider the mapping p defined by ELE. For simplicity we assume $n=2$. We start with $(z, w) \to (z, zw, w^2)$, where the subspace A is the span of $(0, 1)$. Now apply a (specific) unitary linear map L to obtain $((z-zw)/\sqrt{2}, (z+zw)/\sqrt{2}, w^2)$. Now apply E on the subspace generated by $(1, 0, 0)$ to get the map ELE given by

20. $$p(z,w)=\left(\frac{z^2-z^2w}{\sqrt{2}}, \frac{zw-zw^2}{\sqrt{2}}, \frac{z+zw}{\sqrt{2}}, w^2\right).$$

As the map p includes six distinct monomials, but maps into B_4, it is not unitarily equivalent to any monomial map into B_4. By the result in [D1], it is not spherically equivalent to any monomial map either. Note that the proof of Theorem 13 gives a different factorization, namely $(E^{-1})^3LE^3$; as the last subspace is trivial, one could write $(E^{-1})^2LE^3$. Of course the proof of that result must work for all polynomials and hence does not always yield the simplest factorization. Generalizations of this example hold for any n.

Because the ideas are independent of the domain dimension n, it is worthwhile to look at Theorem 18 in case $n=1$. The polynomial map of degree m defined by $f(z)=\sum A_j z^j$ will be proper if the vectors A_j are orthogonal and also satisfy $\sum \|A_j\|^2=1$. If we wish to make f map into a specific ball of dimension less than $m+1$, equations 18.4 require that some of the A_j be zero. This phenomenon is completely general. Given m and n, we can choose N large enough so that we can get a maximal-dimensional family of dimension $D(D+1)/2$. If we wish to lower N, we must begin specifying some of the parameters.

Recall that we considered generic jets. The next example shows why we must do this.

21. EXAMPLE. Put $n=1$ for simplicity, and put $s(z)=(\epsilon+\delta z, \omega z, 0)$, where the parameters are small and (say) real. Suppose we wish to find a proper quadratic polynomial mapping whose 1-jet is s. In the notation of Theorem 18,

$$\mathbf{p}_0=(\epsilon,0,0) \quad \text{and} \quad \mathbf{p}_1=(\delta,\omega,0).$$

Letting $\mathbf{p}_2=(a,b,c)$, we see immediately from $\langle \mathbf{p}_0, \mathbf{p}_2\rangle=0$ that $a=0$. The second equation that $\langle \mathbf{p}_0, \mathbf{p}_1\rangle |z|^2+\langle \mathbf{p}_1, \mathbf{p}_2\rangle=0$ tells us that $\epsilon\delta+\omega\bar{b}=0$. The third equation tells us that $\epsilon^2+\delta^2+\omega^2+|b|^2+|c|^2=1$. If ω vanishes then the second equation cannot be satisfied unless one of ϵ or δ vanishes. Thus not every 1-jet is allowable. The condition that $\omega\neq 0$ is precisely the generic condition used in Theorem 18. The precise condition here is as follows:

$$\begin{aligned}
&\{\epsilon,\delta,\omega\}: \epsilon^2+\delta^2+\omega^2\leq 1, \quad \omega\neq 0, \qquad \text{or}\\
&\{\epsilon,0,0\}: \epsilon^2\leq 1, \qquad \text{or}\\
&\{0,\delta,0\}: \delta^2\leq 1.
\end{aligned}$$

Of course, the set of allowable 1-jets contains an open set.

22. EXAMPLE. For a general n, consider the quadratic polynomial maps $\mathbf{p}_0+\mathbf{p}_1+\mathbf{p}_2$. The independent parameters are all the inner products of the vectors in the constant and linear terms. There are $N=1+n+[n(n+1)]/2$ of these. This means that the most general quadratic proper map from B_n fits into B_N. To fit it into a lower-dimensional ball, one must specify some of these parameters.

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