Invariant Diagonal Operator Ranges

BEN MATHES

Introduction

It is a well-known fact that a subspace M of a Hilbert space $\mathcal{H}$ is invariant under a set $\mathcal{S}$ of operators on $\mathcal{H}$ if and only if $M^{\perp}$ is invariant under the set $\mathcal{S}^* \equiv \{T^*: T \in \mathcal{S}\}$. There is no similar statement for operator ranges. Indeed, if $\mathcal{A}$ is the algebra of operators that are lower triangular relative to an orthonormal basis $\mathcal{E} = \{e_0, e_1, e_2, \dots\}$ of $\mathcal{H}$, and if $\mathrm{Lat}_{1/2}\,\mathcal{A}$ denotes the lattice of operator ranges invariant under $\mathcal{A}$, then

$$\mathrm{Lat}_{1/2}\,\mathcal{A} = \mathrm{Lat}\,\mathcal{A} \equiv \{M: M \text{ a closed invariant subspace of } \mathcal{A}\}.$$

On the other hand, $\mathrm{Lat}_{1/2}\,\mathcal{A}^*$ properly contains $\mathrm{Lat}\,\mathcal{A}^*$ (for the proofs of these assertions, we refer the reader to [1], [2], and [9]). All of the invariant ranges of these algebras may be obtained as ranges of diagonal operators. The purpose of this paper is to replace $\mathcal{A}$ with small subalgebras, the commutants of certain strictly cyclic weighted shifts, and then characterize the ranges of diagonal operators invariant under these smaller algebras. In a paper to appear as a sequel to the one in hand, we will investigate the ranges of diagonal operators that are invariant under the adjoints of these smaller algebras. Our results suggest that the difference between the ranges of diagonal operators invariant under these smaller algebras and those invariant under their adjoints is the same difference seen when passing from $\mathcal{A}$ to $\mathcal{A}^*$.

Preliminaries

Assume for the moment that $\mathcal{A}$ is the commutant of the unilateral shift operator S, that is, the operator defined by $Se_i = e_{i+1}$ $(i = 0, 1, \dots)$. We assert that there are no nontrivial invariant ranges of diagonal operators under $\mathcal{A}$. By nontrivial, we mean ranges other than the obvious invariant closed subspaces of all lower triangular operators. This may be seen by assuming that $D = \mathrm{diag}(d_i)$ is invariant under $\mathcal{A}$, then proving that there exists $m \geq 0$ and $\epsilon > 0$ such that $d_i = 0$ for all $0 \leq i < m$, and $d_i \geq \epsilon$ whenever $i \geq m$ (there is no

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loss of generality in assuming that (d_i) is a positive sequence). The existence of m such that $d_i=0$ for all $0\le i<m$ and $d_i>0$ whenever $i\ge m$ follows immediately from the fact that S leaves the range of D invariant. The fact that the range of D is invariant under $\mathcal{A}$ gives us that the map $A\mapsto D_0^{-1}AD$ is a bounded homomorphism of $\mathcal{A}$ into $\mathcal{B}(\mathcal{H})$, where D_0 is the restriction of D to its cokernel and $\mathcal{B}(\mathcal{H})$ denotes the algebra of (bounded) operators on $\mathcal{H}$ (see [2]). If one writes down the matrix of $D_0^{-1}S^nD$, the fact that

$$\sup_{n\ge 1}\|D_0^{-1}S^nD\|<\infty$$

is easily seen to imply that

$$\sup_{\substack{n\ge 1\\ i\ge m}}\frac{d_i}{d_{i+n}}<\infty,$$

which gives us the existence of ϵ. This is the essence of the argument used in [10] to construct a range invariant under S but not under $\mathcal{A}$.

If we look at the commutant of weighted shift operators, then the story is completely different. We will see that some of these algebras have an abundance of invariant ranges that are ranges of diagonal operators. An operator Q is called a *unilateral weighted shift* with weight sequence (w_i) (relative to the basis $\mathcal{E}$) provided $Qe_i=w_ie_{i+1}$ $(i=0,1,\dots)$. We will always assume that the weights are all nonzero. A unilateral weighted shift operator is said to be *strictly cyclic* if there exists $x\in\mathcal{H}$ such that $\{Bx: B\in\mathcal{A}\}=\mathcal{H}$, where $\mathcal{A}$ is the strongly closed algebra generated by Q (note that $\mathcal{A}$ is also the commutant of Q). A survey of weighted shift operators is presented in [11] where a proof of the following fact may be found: If Q is a unilateral weighted shift with a monotonically decreasing weight sequence of positive numbers (w_i), then Q is strictly cyclic if and only if

$$(*)\qquad \sup_{k\ge 1}\sum_{i=0}^{k}\left[\frac{\beta_k}{\beta_i\beta_{k-i}}\right]^2<\infty,$$

where $\beta_0=1$ and $\beta_k=w_{k-1}w_{k-2}\cdots w_0$ $(k\ge 1)$. If (w_i) is an arbitrary sequence of positive (nonzero) numbers, then (*) is a sufficient condition that Q be strictly cyclic. In [3] an example of a strictly cyclic weighted shift that does not satisfy (*) is given, so (*) is not a necessary condition for strict cyclicity in general.

If Q is a strictly cyclic unilateral weighted shift, then Q induces a bounded bilinear form $\varphi_Q\colon \mathcal{H}\times\mathcal{H}\to\mathcal{H}$ defined by

$$\varphi_Q(x,y)\equiv\sum_{k=0}^{\infty}\left\{\sum_{i=0}^{k}\langle x,e_i\rangle\langle y,e_{k-i}\rangle\frac{\beta_k}{\beta_i\beta_{k-i}}\right\}e_k.$$

The following lemma is taken from [7] and sheds some more light on the condition (*).

LEMMA 1 [7]. *For all $0\le i\le k$, let λ_{ki} be complex numbers such that*

$$\varphi(x,y)\equiv\sum_{k=0}^{\infty}\left\{\sum_{i=0}^{k}\langle x,e_i\rangle\langle y,e_{k-i}\rangle\lambda_{ki}\right\}e_k$$

defines a bounded bilinear form $\varphi: \mathcal{H} \times \mathcal{H} \to \mathcal{H}$. *If* $\mathcal{H} \otimes_2 \mathcal{H}$ *is the Hilbert space tensor product of* $\mathcal{H}$ *with itself, then there exists* $\Omega \in \mathcal{B}(\mathcal{H} \otimes_2 \mathcal{H}, \mathcal{H})$ *such that* $\varphi(x,y) = \Omega(x \otimes y)$ *for all* $x, y \in \mathcal{H}$ *if and only if*

$$\sup_{k \geq 1} \sum_{i=0}^{k} |\lambda_{ki}|^2 < \infty.$$

A bounded bilinear form $\varphi: \mathcal{H} \times \mathcal{H} \to \mathcal{H}$ for which there exists $\Omega \in \mathcal{B}(\mathcal{H} \otimes_2 \mathcal{H}, \mathcal{H})$ such that $\varphi(x,y) = \Omega(x \otimes y)$ for all $x, y \in \mathcal{H}$ is called a *weak Hilbert-Schmidt map* (see [5]). Thus a paraphrase of the principal result in [3] is that there exists a strictly cyclic weighted shift Q such that φ_Q is not a weak Hilbert-Schmidt map.

We will now explain the reason for fussing whether φ_Q is or is not a weak Hilbert-Schmidt map. If $\mathcal{A}$ is any norm closed subalgebra of $\mathcal{B}(\mathcal{H})$, then there is a sublattice of $\mathrm{Lat}_{1/2}\,\mathcal{A}$, which we denote $\mathrm{Lat}_{\mathrm{cb}}\,\mathcal{A}$, that consists of those invariant operator ranges that induce completely bounded homomorphisms (see [8]). It is an open question whether or not one has $\mathrm{Lat}_{1/2}\,\mathcal{A} = \mathrm{Lat}_{\mathrm{cb}}\,\mathcal{A}$. If φ_Q is a weak Hilbert-Schmidt map and $\mathcal{A}$ is the commutant of Q, then there is an explicit description of $\mathrm{Lat}_{\mathrm{cb}}\,\mathcal{A}$ in [7]. Namely, one can make the identification

$$\mathcal{A} = \{M_y : y \in \mathcal{H}\},$$

where M_y is the operator defined by

$$M_y(x) = \varphi_Q(x,y).$$

This association of $\mathcal{A}$ with φ_Q is denoted in [7] by $\mathcal{A} \sim (\mathcal{H}, \varphi_Q)$. An element $\mathrm{ran}(T)$ of $\mathrm{Lat}_{1/2}\,\mathcal{A}$ then gives rise to a bounded bilinear map φ_T defined by

$$\varphi_T(x,y) \equiv T_0^{-1} \varphi_Q(Tx, y),$$

where T_0 is the restriction of T to its cokernel. The result in [7] states that $\mathrm{ran}(T)$ is in $\mathrm{Lat}_{\mathrm{cb}}\,\mathcal{A}$ if and only if φ_T is a weak Hilbert-Schmidt map. This gives a concrete test for determining whether the range of a diagonal operator is in $\mathrm{Lat}_{\mathrm{cb}}\,\mathcal{A}$.

LEMMA 2. *Assume* $D = \mathrm{diag}(d_i)$, $\mathrm{ran}(D) \neq \mathcal{H}$, $\mathcal{A}$ *is the commutant of a strictly cyclic unilateral weighted shift* Q, *and* φ_Q *is a weak Hilbert-Schmidt map. Then* $\mathrm{ran}(D) \in \mathrm{Lat}_{\mathrm{cb}}\,\mathcal{A}$ *if and only if*

(1) *there exists* $m \geq 1$ *such that* $d_i = 0$ *for all* $0 \leq i < m$ *but* $d_i \neq 0$ *for* $i \geq m$, *and*

(2) $\sup_{k \geq 1} \sum_{i=0}^{k} |\lambda_{ki}|^2 < \infty$, *where*

$$\lambda_{ki} = \begin{cases} (d_i/d_k)(\beta_k/\beta_i\beta_{k-i}) & \text{if } k \geq i \geq m, \\ 0 & \text{otherwise.} \end{cases}$$

Proof. Assume that $\mathrm{ran}(D) \in \mathrm{Lat}_{\mathrm{cb}}\,\mathcal{A}$. That there exists $m \geq 0$ such that $d_i = 0$ for all $0 \leq i < m$ but $d_i \neq 0$ for $i \geq m$ is an immediate consequence of the fact that Q leaves the range of D invariant. That $m \geq 1$ follows since $\mathcal{A}$ has no dense invariant linear manifolds (see [6]). Thus we have established (1).

To establish (2), it suffices to prove that

$$\varphi_D(x,y) \equiv \sum_{k=0}^{\infty} \left\{ \sum_{i=0}^{k} \langle x, e_i \rangle \langle y, e_{k-i} \rangle \lambda_{ki} \right\} e_k$$

defines a bilinear form that is in fact a weak Hilbert–Schmidt map, by Lemma 1. To this end, write $\mathfrak{A} = \{M_y : y \in \mathcal{H}\}$ and let $\Phi_D : \mathfrak{A} \to \mathcal{B}(\mathcal{H})$ be the homomorphism defined by $\Phi_D(A) = D_0^{-1} AD$. Then it follows that

$$\Phi_D(M_y)(x) = \sum_{k=0}^{\infty} \left\{ \sum_{i=0}^{k} \langle x, e_i \rangle \langle y, e_{k-i} \rangle \lambda_{ki} \right\} e_k,$$

so φ_D is a bounded bilinear form. By hypothesis, Φ_D is a completely bounded map, which implies that φ_D is a weak Hilbert–Schmidt map by Theorem 3 of [7].

Conversely, if (1) and (2) hold, then we may define a weak Hilbert–Schmidt map by the formula

$$\varphi_D(x,y) \equiv \sum_{k=0}^{\infty} \left\{ \sum_{i=0}^{k} \langle x, e_i \rangle \langle y, e_{k-i} \rangle \lambda_{ki} \right\} e_k,$$

which evidently satisfies $M_y D(x) = D(\varphi_D(x,y))$ for all $x, y \in \mathcal{H}$. It follows, again from Theorem 3 of [7], that $\operatorname{ran}(D) \in \operatorname{Lat}_{\mathrm{cb}} \mathfrak{A}$. □

Implicit in the above proof is a condition that the range of a diagonal operator be invariant under the commutant of a strictly cyclic weighted shift, which we now record.

LEMMA 3. *Assume* $D = \operatorname{diag}(d_i)$, $\operatorname{ran}(D) \neq \mathcal{H}$, *and* $\mathfrak{A}$ *is the commutant of a strictly cyclic unilateral weighted shift* Q. *Then* $\operatorname{ran}(D) \in \operatorname{Lat}_{1/2} \mathfrak{A}$ *if and only if*

(1) *there exists* $m \geq 1$ *such that* $d_i = 0$ *for all* $0 \leq i < m$ *but* $d_i \neq 0$ *for* $i \geq m$, *and*

(2) *for all* $x, y \in \mathcal{H}$,

$$\sum_{k=0}^{\infty} \left| \sum_{i=0}^{k} \langle x, e_i \rangle \langle y, e_{k-i} \rangle \lambda_{ki} \right|^2 < \infty,$$

where

$$\lambda_{ki} = \begin{cases} (d_i/d_k)(\beta_k/\beta_i\beta_{k-i}) & \text{if } k \geq i \geq m, \\ 0 & \text{otherwise.} \end{cases}$$

Main Results

If Q is a unilateral weighted shift with weight sequence (w_i), then Q has the same range as the diagonal operator

$$\begin{bmatrix} 0 & 0 & 0 & \cdots \\ 0 & w_0 & 0 & \cdots \\ 0 & 0 & w_1 & \\ \vdots & \vdots & & \ddots \end{bmatrix}.$$

We will see that a necessary condition for the range of a diagonal operator $D=\operatorname{diag}(0, d_1, d_2, \dots)$ ($d_i \neq 0$ for $i \geq 1$) to be invariant under the commutant of a strictly cyclic weighted shift Q is that $\operatorname{ran}(Q) \subset \operatorname{ran}(D)$. If Q is strongly strictly cyclic—that is, if for all $i \geq 0$ the unilateral weighted shift with weight sequence $(w_i, w_{i+1}, w_{i+2}, \dots)$ is strictly cyclic—and if the sequence (w_i) is monotonically decreasing, then we will see that $\operatorname{ran}(Q) \subset \operatorname{ran}(D)$ if and only if the range of D is invariant under $\mathfrak{A}$. The following theorem completely categorizes the ranges of diagonal operators invariant under the commutant of such weighted shifts.

THEOREM 1. *Assume Q is a strictly cyclic weighted shift with positive weight sequence (w_i), and let $\mathfrak{A}$ be the commutant of Q. Let $D=\operatorname{diag}(d_i)$ be a diagonal operator such that $d_i = 0$ for all $0 \leq i < m$ but $d_i > 0$ for $i \geq m \geq 1$. Each statement below implies all the statements that follow it:*

(1) $\operatorname{ran}(D) \in \operatorname{Lat}_{\mathrm{cb}} \mathfrak{A}$;

(2) *for all $x, y \in \mathcal{H}$,*

$$\sum_{k=0}^{\infty} \left| \sum_{i=0}^{k} \langle x, e_i \rangle \langle y, e_{k-i} \rangle \lambda_{ki} \right|^2 < \infty,$$

where

$$\lambda_{ki} = \begin{cases} (d_i/d_k)(\beta_k/\beta_i \beta_{k-i}) & \text{if } k \geq i \geq m, \\ 0 & \text{otherwise;} \end{cases}$$

(3)
$$\sup_{\substack{i \geq m \\ j \geq 0}} \frac{d_i}{d_{i+j}} \frac{\beta_{i+j}}{\beta_i \beta_j} < \infty;$$

(4) $\operatorname{ran}(Q^m) \subset \operatorname{ran}(D)$ *and the range of D is invariant under Q.*

Furthermore, if Q is strongly strictly cyclic with a monotonically decreasing weight sequence, then the previous are equivalent.

Proof. That (1)$\Rightarrow$(2) follows immediately from Lemma 3. To see that (2)$\Rightarrow$(3), note that the map

$$(x, y) \mapsto \sum_{k=0}^{\infty} \left\{ \sum_{i=0}^{k} \langle x, e_i \rangle \langle y, e_{k-i} \rangle \lambda_{ki} \right\} e_k$$

is bounded bilinear and thus there exists a constant γ such that

$$\frac{d_r}{d_s} \frac{\beta_s}{\beta_r \beta_{s-r}} = \left\| \sum_{k=0}^{\infty} \left\{ \sum_{i=0}^{k} \langle e_r, e_i \rangle \langle e_{s-r}, e_{k-i} \rangle \lambda_{ki} \right\} e_k \right\| < \gamma$$

for all $s \geq r \geq m$. To see that (3)$\Rightarrow$(4), let $j=1$ in (3) and choose γ such that

$$\frac{d_i}{d_{i+1}} \frac{\beta_{i+1}}{\beta_i \beta_1} < \gamma$$

for all $i \geq m$. It follows that

$$\frac{d_i w_i}{d_{i+1}} < w_0 \gamma$$

for all $i \geq m$, so $D_0^{-1}QD$ is bounded and the range of D is invariant under Q. If one lets $i=m$ in (3) and chooses γ such that

$$\frac{d_m}{d_{m+j}}\frac{\beta_{m+j}}{\beta_m\beta_j}<\gamma$$

for all $j \geq 0$, then by rearranging terms we see that

$$\sup_{j\geq 0}\frac{w_{m+j-1}\cdots w_j}{d_{m+j}}<\infty,$$

that is, that $D_0^{-1}Q^m$ is bounded. It follows that $\operatorname{ran}(Q^m)\subset\operatorname{ran}(D)$. Finally, assume that Q is strongly strictly cyclic with a monotonically decreasing weight sequence (w_i), and with no loss of generality assume that $w_0=1$. It follows that φ_Q is a weak Hilbert-Schmidt map (by Lemma 1), and by Lemma 2 it suffices to prove that (4) implies

$$\sup_{k\geq 1}\sum_{i=0}^{k}|\lambda_{ki}|^2<\infty.$$

Since Q is strongly strictly cyclic, the weighted shift with weight sequence $(w_m, w_{m+1}, w_{m+2}, \dots)$ is strictly cyclic. Let $v_i=w_{m+i}$, $\alpha_0=1$, and let $\alpha_k= v_0\cdots v_{k-1}$ for $k\geq 1$. It follows that

$$\sup_{k\geq 1}\sum_{i=0}^{k}\left[\frac{\alpha_k}{\alpha_i\alpha_{k-i}}\right]^2<\infty,$$

which implies that

$$\sup_{k\geq 2m}\sum_{i=m}^{k-m}\left[\frac{1}{w_{k-1}\cdots w_{k-m}}\frac{\beta_k}{\beta_i\beta_{k-i}}\right]^2<\infty.$$

To see the last implication note that

$$\sum_{i=m}^{k-m}\left[\frac{1}{w_{k-1}\cdots w_{k-m}}\frac{\beta_k}{\beta_i\beta_{k-i}}\right]^2=\frac{1}{w_{m-1}\cdots w_0}\sum_{i=0}^{k-2m}\left[\frac{\alpha_{k-2m}}{\alpha_i\alpha_{k-2m-i}}\right]^2.$$

If $k\geq 2m$, then

$$\begin{aligned}\sum_{i=0}^{k}|\lambda_{ki}|^2&=\sum_{i=m}^{k}\left[\frac{d_i}{d_k}\frac{\beta_k}{\beta_i\beta_{k-i}}\right]^2\\&=1+\sum_{i=1}^{m-1}\left[\frac{d_{k-i}}{d_k}\frac{w_{k-1}\cdots w_{k-i}}{w_0\cdots w_{i-1}}\right]^2+\sum_{i=m}^{k-m}\left[\frac{d_i}{d_k}\frac{\beta_k}{\beta_i\beta_{k-i}}\right]^2.\end{aligned}$$

If $1\leq i\leq m-1$, then Q^i leaves the range of D invariant, which implies that

$$\sup_{k\geq 2m}\frac{d_{k-i}}{d_k}\frac{w_{k-1}\cdots w_{k-i}}{w_0\cdots w_{i-1}}<\infty,$$

and hence

$$\sup_{k\geq 2m}\sum_{i=1}^{m-1}\left[\frac{d_{k-i}}{d_k}\frac{w_{k-1}\cdots w_{k-i}}{w_0\cdots w_{i-1}}\right]^2<\infty.$$

Since $\operatorname{ran}(Q^m)\subset\operatorname{ran}(D)$, there exists a constant γ such that

$$\frac{1}{d_k} < \gamma \frac{1}{w_{k-1} \cdots w_{k-m}}$$

for all $k \geq 2m$. It follows that

$$\sup_{k \geq 2m} \sum_{i=m}^{k-m} \left[\frac{d_i}{d_k} \frac{\beta_k}{\beta_i \beta_{k-i}} \right]^2 \leq (\|D\| \gamma)^2 \sup_{k \geq 2m} \sum_{i=m}^{k-m} \left[\frac{1}{w_{k-1} \cdots w_{k-m}} \frac{\beta_k}{\beta_i \beta_{k-1}} \right]^2 < \infty.$$

This completes the proof of the theorem. □

COROLLARY 1. *If $\mathfrak{A}$ is the commutant of a strongly strictly cyclic weighted shift Q with a monotonically decreasing weight sequence, and if* $D = \operatorname{diag}(0, d_1, d_2, \ldots)$ *with $d_i \neq 0$ for $i \geq 1$, then the range of D is invariant under $\mathfrak{A}$ if and only if* $\operatorname{ran}(Q) \subset \operatorname{ran}(D)$.

Proof. This is an immediate consequence of Theorem 1 and the fact that $\operatorname{ran}(Q) \subset \operatorname{ran}(D)$ implies the range of D is invariant under Q. □

COROLLARY 2. *If $\mathfrak{A}$ and Q are defined as in Corollary 1 and D is a diagonal operator, then* $\operatorname{ran}(D) \in \operatorname{Lat}_{1/2} \mathfrak{A}$ *implies* $\operatorname{ran}(D) \in \operatorname{Lat}_{\mathrm{cb}} \mathfrak{A}$.

An Example and Questions

In the search for an element of $\operatorname{Lat}_{1/2} \mathfrak{A}$ that is not in $\operatorname{Lat}_{\mathrm{cb}} \mathfrak{A}$, Corollary 2 tells us where not to look. On the other hand, the proof of Theorem 1 suggests that we might have some luck investigating the ranges of diagonal operators invariant under the commutant of a strictly cyclic weighted shift Q that is not strongly strictly cyclic. An example is given in [4] of a sequence $(w_0, w_1, \ldots)$ such that the associated weighted shift is strictly cyclic, but the weighted shift with weight sequence $(w_1, w_2, \ldots)$ is not strictly cyclic. With this sequence, we now construct an example of a diagonal operator that satisfies condition (4) but not condition (1) of Theorem 1.

EXAMPLE 1. Let $(w_0, w_1, \ldots)$ be a monotonically decreasing null sequence of positive numbers such that the weighted shift Q is strictly cyclic but the weighted shift with weight sequence $(w_1, w_2, \ldots)$ is not strictly cyclic (see [4]). We will construct a diagonal operator $D = \operatorname{diag}(0, d_1, d_2, \ldots)$, with $d_i \neq 0$ for $i \geq 1$, such that $\operatorname{ran}(Q) \subset \operatorname{ran}(D)$ but $\operatorname{ran}(D) \notin \operatorname{Lat}_{\mathrm{cb}} \mathfrak{A}$, where $\mathfrak{A}$ is the commutant of Q.

The fact that Q is strictly cyclic but the weighted shift with weight sequence $(w_1, w_2, \ldots)$ is not strictly cyclic implies that

$$\sup_{k \geq 1} \sum_{i=0}^{k} \left[\frac{\beta_k}{\beta_i \beta_{k-i}} \right]^2 < \infty$$

but

$$\sup_{k \geq 2} \sum_{i=1}^{k-1} \left[\frac{1}{w_{k-1}} \frac{\beta_k}{\beta_i \beta_{k-i}} \right]^2 = \infty.$$

Choose k_1 such that

$$2<\sum_{i=1}^{k_1-1}\left[\frac{1}{w_{k_1-1}}\frac{\beta_{k_1}}{\beta_i\beta_{k_1-i}}\right]^2.$$

Since (w_i) is a monotonically decreasing null sequence,

$$\lim_{n\to\infty}\frac{1}{w_{n-1}}\frac{\beta_n}{\beta_{k_1}\beta_{n-k_1}}=0.$$

Thus we may choose $k_2>k_1$ such that

$$\left[\frac{1}{w_{k_2-1}}\frac{\beta_{k_2}}{\beta_{k_1}\beta_{k_2-k_1}}\right]^2<1$$

and

$$2^2+1<\sum_{i=1}^{k_2-1}\left[\frac{1}{w_{k_2-1}}\frac{\beta_{k_2}}{\beta_i\beta_{k_2-i}}\right]^2.$$

It follows that

$$\sum_{\substack{i=1\\ i\neq k_1}}^{k_2-1}\left[\frac{1}{w_{k_2-1}}\frac{\beta_{k_2}}{\beta_i\beta_{k_2-i}}\right]^2\geq\sum_{i=1}^{k_2-1}\left[\frac{1}{w_{k_2-1}}\frac{\beta_{k_2}}{\beta_i\beta_{k_2-i}}\right]^2-1>2^2.$$

Having chosen k_{m-1} such that

$$2^{m-1}<\sum_{\substack{i=1\\ i\neq k_j\ \forall j<m-1}}^{k_{m-1}-1}\left[\frac{1}{w_{k_{m-1}-1}}\frac{\beta_{k_{m-1}}}{\beta_i\beta_{k_{m-1}-i}}\right]^2,$$

choose k_m such that

$$\left[\frac{1}{w_{k_m-1}}\frac{\beta_{k_m}}{\beta_{k_j}\beta_{k_m-k_j}}\right]^2<\frac{1}{m-1}\quad(j=1,\ldots,m-1)$$

and

$$2^m+1<\sum_{i=1}^{k_m-1}\left[\frac{1}{w_{k_m-1}}\frac{\beta_{k_m}}{\beta_i\beta_{k_m-i}}\right]^2.$$

Now let $D=\operatorname{diag}(0,d_1,d_2,\ldots)$, with $d_i=w_{k_j-1}$ if $i=k_j$ and $d_i=1$ otherwise. It is obvious that $\operatorname{ran}(Q)\subset\operatorname{ran}(D)$, and

$$\begin{aligned}\sum_{i=1}^{k_j}\left[\frac{d_i}{d_{k_j}}\frac{\beta_{k_j}}{\beta_i\beta_{k_j-1}}\right]^2&>\sum_{\substack{i=1\\ i\neq k_t}}^{k_j}\left[\frac{d_i}{d_{k_j}}\frac{\beta_{k_j}}{\beta_i\beta_{k_j-i}}\right]^2\\&=\sum_{\substack{i=1\\ i\neq k_t}}^{k_j-1}\left[\frac{1}{w_{k_j-1}}\frac{\beta_{k_j}}{\beta_i\beta_{k_j-i}}\right]^2>2^j,\end{aligned}$$

so $\operatorname{ran}(D)\notin\operatorname{Lat}_{cb}\mathcal{A}$ by Lemma 2. □

QUESTION 1. Is the range of the diagonal operator constructed in Example 1 invariant under $\mathcal{A}$?

An affirmative answer to Question 1 clearly gives an example of an invariant operator range of $\mathcal{A}$ that does not induce a completely bounded homomorphism.

QUESTION 2. Let Q be an arbitrary unilateral weighted shift (with a non-zero weight sequence), $\mathfrak{A}$ the commutant of Q, and $D=\operatorname{diag}(d_i)$ a diagonal operator such that $d_i=0$ for all $0\leq i<m$ but $d_i>0$ for $i\geq m\geq 1$. What (if any) part of Theorem 1 generalizes to this setting?

Our comments concerning the unilateral shift S at the beginning of the preliminaries say that D is invariant under the commutant of S if and only if $\operatorname{ran}(S^m)\subset\operatorname{ran}(D)$. This suggests that there might be a general connection between the invariance of $\operatorname{ran}(D)$ under the commutant of Q and the range inclusion $\operatorname{ran}(Q^m)\subset\operatorname{ran}(D)$.

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