A Hypercontractive Estimate for the Heat Semigroup on $\mathbf{P}^2$

J. MICHAEL PEARSON

1. Introduction

Hypercontractive estimates for diffusion semigroups first appeared in the work of Nelson [7] in connection with certain problems in constructive quantum field theory. Perhaps the most important tool used to obtain this type of estimate is the logarithmic Sobolev inequality, introduced by Gross in [4]. Such inequalities can be viewed as limiting cases of Sobolev inequalities, and that is the approach we take in this work.

Denote by $\mathbf{P}^n$ n-dimensional real projective space, which we will view as the n-sphere S^n with antipodal points identified. We take $d\zeta$ to be normalized Lebesgue measure on either $\mathbf{P}^n$ or S^n. Functions on $\mathbf{P}^n$ will then be naturally identified with even functions on S^n. For such a function f we will write

$$\|f\|_{L^p(\mathbf{P}^n)} = \left(\int_{\mathbf{P}^n} |f|^p \, d\zeta\right)^{1/p} = \left(\int_{S^n} |f|^p \, d\zeta\right)^{1/p} = \|f\|_{L^p(S^n)}.$$

Using an appropriate logarithmic Sobolev inequality, Mueller and Weissler [6] obtained the following hypercontractive estimate for the heat semigroup on S^n: for $1 \le p \le q < \infty$ and $t \ge (1/2^n)\ln[(q-1)/(p-1)]$,

$$\|e^{t\Delta} f\|_{L^q(S^n)} \le \|f\|_{L^p(S^n)}. \tag{1.1}$$

Subsequently, Bakry and Emery [1] gave an alternative proof in terms of second-order estimates for the semigroup generator and Beckner [2] proved the stronger result that the Poisson semigroup is hypercontractive.

From the previous discussion, we see that the estimate (1.1) will also hold for functions on $\mathbf{P}^n$. A natural question to ask is whether or not any improvement on this estimate is possible for $\mathbf{P}^n$. The purpose of this note is to prove the following theorem which gives such an improvement on $\mathbf{P}^2$.

MAIN THEOREM (THEOREM 3.1). *Let $f \in L^p(\mathbf{P}^2)$. Then for $1 \le p \le q < \infty$ and $t \ge \frac{1}{8}\ln[(q-1)/(p-1)]$,*

$$\|e^{t\Delta} f\|_{L^q(\mathbf{P}^2)} \le \|f\|_{L^p(\mathbf{P}^2)}.$$

Received August 7, 1989. Revision received March 28, 1990.
Michigan Math. J. 37 (1990).

We will show in Section 3 that this result is equivalent to a logarithmic Sobolev inequality which will be derived in Section 2. The proof of the logarithmic inequality will depend on a sharp isoperimetric inequality for $\mathbf{P}^2$. While the isoperimetric inequality seems to have been recognized before (see, e.g., [9]), a proof does not seem to be readily accessible in the literature, and we have included one in Section 2.

I would like to acknowledge the advice and encouragement given by William Beckner during the course of this work. His friendship and help have been invaluable.

2. The Logarithmic Sobolev Inequality

The logarithmic Sobolev inequality corresponding to the hypercontractive estimate for the heat semigroup on S^n, given by Mueller and Weissler in [6], is

$$\int_{S^n} |f|^2 \ln|f|\, d\zeta - \|f\|^2_{L^2(S^n)} \ln\|f\|_{L^2(S^n)} \le \frac{1}{n}\int_{S^n} |\nabla f|^2\, d\zeta. \tag{2.1}$$

If we let $f=\sum_{k=0}^{\infty} Y_k$ denote the expansion of f in spherical harmonics on S^n and use the fact that for Y_k a spherical harmonic of degree k, $-\Delta Y_k = k(n+k-1)Y_k$, then we can rewrite (2.1) as

$$\int_{S^n} |f|^2 \ln|f|\, d\zeta - \|f\|^2_{L^2(S^n)} \ln\|f\|_{L^2(S^n)} \le \sum_{k=1}^{\infty} \frac{k(n+k-1)}{n}\int_{S^n} |Y_k|^2\, d\zeta. \tag{2.2}$$

In [2], Beckner derived the following inequality, which is stronger than (2.2):

$$\int_{S^n} |f|^2 \ln|f|\, d\zeta - \|f\|^2_{L^2(S^n)} \ln\|f\|_{L^2(S^n)} \le \sum_{k=1}^{\infty} k\int_{S^n} |Y_k|^2\, d\zeta. \tag{2.3}$$

Inequality (2.3) corresponds to the following hypercontractive estimate for the Poisson semigroup:

$$|r| \le \sqrt{\frac{p-1}{q-1}} \Rightarrow \|P_r f\|_{L^q(S^n)} \le \|f\|_{L^p(S^n)}.$$

In this section, we will show that the following logarithmic Sobolev inequality holds on $\mathbf{P}^2$.

THEOREM 2.1. *For* $f \in L^p(\mathbf{P}^2)$,

$$\int_{\mathbf{P}^2} f^2 \ln|f|\, d\zeta - \|f\|^2_{L^2(\mathbf{P}^2)} \ln\|f\|_{L^2(\mathbf{P}^2)} \le \frac{1}{4}\int_{\mathbf{P}^2} |\nabla f|^2\, d\zeta.$$

In Section 3, we will show that Theorem 2.1 is equivalent to the hypercontractive estimate for the heat semigroup on $\mathbf{P}^2$ which was stated in the introduction. Theorem 2.1 will be obtained as a limiting case of the Sobolev inequality for $\mathbf{P}^2$ given by Theorem 2.2.

THEOREM 2.2. *For* $p \geq 2$,

$$\|f\|_{L^p(\mathbf{P}^2)}^2 \leq \frac{p-2}{4} \int_{\mathbf{P}^2} |\nabla f|^2 \, d\zeta + \int_{\mathbf{P}^2} |f|^2 \, d\zeta.$$

In [3], Beckner obtained the following Sobolev inequality for functions on S^n:

$$\|f\|_{L^p(S^n)}^2 \leq \frac{p-2}{n} \int_{S^n} |\nabla f|^2 \, d\zeta + \int_{S^n} |f|^2 \, d\zeta, \tag{2.4}$$

where $p \in [2, 2n/(n-2)]$ if $n \geq 3$ and $p \geq 2$ if $n = 1, 2$. To prove Theorem 2.2, we will use inequality (2.4) and the following sharp isoperimetric inequality for $\mathbf{P}^2$, which we state in terms of symmetric sets on S^2.

THEOREM 2.3. *Let* $A \subset S^2$ *be any set which is antipodally symmetric. Then*

$$\sqrt{2} \cdot \sigma(\partial A^*) \leq \sigma(\partial A).$$

Here, A^* *denotes a spherical cap on* S^2 *with* $m(A^*) = m(A)$ *and* σ *denotes 1-dimensional Hausdorff measure.*

In order to prove Theorem 2.3, we will need the following lemmas.

LEMMA 2.4. *Suppose that* A, B *are spherical caps on* S^n *and that* $m(A) = 2m(B)$. *Then*

$$2\sigma(\partial B) \geq 2^{1/n} \sigma(\partial A).$$

Here, σ *denotes* $(n-1)$*-dimensional Hausdorff measure. This is the best possible inequality.*

LEMMA 2.5. *If* $A \subset S^2$ *is an open, connected, antipodally symmetric set such that* $(\bar{A})^c \neq \emptyset$, *then no point of* $(\bar{A})^c$ *lies in the same component of* $(\bar{A})^c$ *as its antipode.*

To obtain Theorem 2.3 from the lemmas, first note that for any $A \subset S^2$, $\partial A = \partial(A^c)$. Since $m(\bar{A}) > m(A)$ implies $\sigma(\partial A) = \infty$ and $\partial(\mathrm{Int}(\bar{A})) \subset \partial(A)$, we may as well assume that $m(A) = m(\bar{A})$ and that $\mathrm{Int}(A) = \mathrm{Int}(\bar{A}) = A$. Now by Lemma 2.5 either A or A^c, say A, naturally decomposes into two disjoint equimeasurable pieces A_+ and A_- with $A_+ = -A_-$. Furthermore, $\sigma(\partial A) = \sigma(\partial A_+) + \sigma(\partial A_-)$. The classical isoperimetric inequality for spherical geometry gives $\sigma(\partial A_+) \geq \sigma(\partial A_+^*)$. By Lemma 2.4,

$$2\sigma(\partial A_+^*) \geq \sqrt{2}\sigma(\partial A^*).$$

Hence,

$$\sigma(\partial A) = \sigma(\partial A_+) + \sigma(\partial A_-) \geq 2\sigma(\partial A_+^*) \geq \sqrt{2}\sigma(\partial A^*).$$

This is Theorem 2.3. □

Proof of Lemma 2.4. Denote $\zeta \in S^n$ by $\zeta = (\zeta' \sin\theta, \cos\theta)$, where $\zeta' \in S^{n-1}$ and θ is the polar angle, $0 \le \theta \le \pi$. Let $A = \{\zeta \in S^n \mid \theta \le \theta_0\}$ and $B = \{\zeta \in S^n \mid \theta \le \theta_1\}$ be two spherical caps on S^n with $m(A) = 2m(B)$. Then

$$\int_0^{\theta_0} (\sin\theta)^{n-1}\, d\theta = 2\int_0^{\theta_1} (\sin\theta)^{n-1}\, d\theta \quad \text{and} \quad \frac{\sigma(\partial B)}{\sigma(\partial A)} = \frac{(\sin\theta_1)^{n-1}}{(\sin\theta_0)^{n-1}}.$$

To prove the lemma it suffices to show that

$$2(\sin\theta_1)^{n-1} \ge 2^{1/n}(\sin\theta_0)^{n-1},$$

or equivalently,

$$2(\sin\theta_1)^n \ge (\sin\theta_0)^n.$$

Now,

$$\begin{aligned}(\sin\theta_0)^n - 2(\sin\theta_1)^n &= n\left\{\int_0^{\theta_0} (\sin\theta)^{n-1}\cos\theta\, d\theta - 2\int_0^{\theta_1} (\sin\theta)^{n-1}\cos\theta\, d\theta\right\} \\ &= n\left\{\int_{\theta_1}^{\theta_0} (\sin\theta)^{n-1}\cos\theta\, d\theta - \int_0^{\theta_1} (\sin\theta)^{n-1}\cos\theta\, d\theta\right\} \\ &\le n\cos\theta_1\left\{\int_{\theta_1}^{\theta_0} (\sin\theta)^{n-1}\, d\theta - \int_0^{\theta_1} (\sin\theta)^{n-1}\, d\theta\right\} = 0.\end{aligned}$$

Hence $2(\sin\theta_1)^n \ge (\sin\theta_0)^n$ as desired. That this estimate is the best possible is easily seen by taking limits as $m(A) \to 0$. □

Proof of Lemma 2.5. Take $A \subset S^2$ to be an open, antipodally symmetric, connected set such that $\bar{A} \ne S^2$. If $x \in \bar{A}^c$ then by making a rotation we may assume that x is the north pole $(0,0,1)$. We must also have that the south pole $(0,0,-1) \in \bar{A}^c$. We will show that $(0,0,1)$ is not in the same component of $\bar{A}^c$ as $(0,0,-1)$.

We use stereographic projection $P: S^2 \to R^2$ given by

$$P(x,y,z) = \left(\frac{x}{1+z}, \frac{y}{1+z}\right)$$

to recast the problem. Note that $P(0,0,1) = (0,0)$ and that P maps $(0,0,-1)$ to infinity. Since some neighborhood of the north pole $(0,0,1)$ on S^2 is contained in $\bar{A}^c$, some neighborhood of the origin $(0,0)$ in R^2 will be contained in $P(\bar{A})^c$. In fact, we must have that, for some $\epsilon > 0$, $P(A)$ is a connected open set contained in the annulus $\{p \in R^2 \mid \epsilon < |p| < 1/\epsilon\}$. Moreover, A antipodally symmetric implies that if $p \in P(A)$, then $-p/|p|^2 \in P(A)$. Because $P(A)$ is connected, there must be a point $p_0 \in P(A)$ with $|p_0| = 1 = |-p_0|$ and $-p_0 \in P(A)$. Without loss of generality, $p_0 = (-1,0)$ and $-p_0 = (1,0)$. Since $P(A)$ is connected there is a path $\beta: [0,1] \to P(A)$ with $\beta(0) = (-1,0)$ and $\beta(1) = (1,0)$. Let $\beta(t) = (r(t), \theta(t))$ be the representation of the path β in polar coordinates. Since $P(A) \subset \{p \in R^2 \mid \epsilon < |p| < 1/\epsilon\}$, $r(t) > \epsilon$ for $t \in [0,1]$ and $\theta: [0,1] \to R$ is well defined on $[0,1]$. We may take $\theta(0) = \pi$ and assume that θ is continuous. Since $\beta(1) = (1,0)$, we must have that $\theta(1) = \pi + (k + \frac{1}{2})2\pi$ for some $k \in \mathbf{Z}$. Because A is symmetric on S^2, the path $\tilde{\beta}$: $[0,1] \to R^2$ given by

$$\tilde{\beta}(t)=\left(\frac{1}{r(t)}, \theta(t)+\left(k+\frac{1}{2}\right)2\pi\right)$$

will be well defined with $\tilde{\beta}(t)\in P(A)$, $t\in[0,1]$. Note that $\beta(0)=(1,0)$ and $\beta(1)=(-1,0)$.

We now define a path $\alpha:[0,1)\cong S^1\to P(A)$ by:

$$\alpha(t)=\begin{cases}\beta(2t) & 0\le t\le \frac{1}{2},\\ \tilde{\beta}(2t-1) & \frac{1}{2}\le t<1.\end{cases}$$

Then α is a closed path with winding number about the origin $(2k+1)$, and so separates the origin from any point exterior to the annulus $\{p\in R^2 \mid \epsilon< |p|<1/\epsilon\}$.

In particular, this implies that the set $A\subset S^2$ separates the north pole $(0,0,1)$ from the south pole $(0,0,-1)$, and Lemma 2.5 follows. □

To obtain Theorem 2.2, we will use Theorem 2.3 and a variation of an argument of Pólya and Szegö [11] used to obtain estimates for capacity.

Proof of Theorem 2.2. Suppose that $f\ge 0$ is an even smooth function on S^2 and that $m\{\zeta\in S^2 \mid \nabla f(\zeta)=0\}=0$. Let f^* be the symmetric decreasing rearrangement of f. Then

$$m\{\zeta\in S^2\mid f(\zeta)>\lambda\}=m\{\zeta\in S^2\mid f^*(\zeta)>\lambda\}.$$

Differentiating in λ, we obtain

$$\int_{\{f=\lambda\}}\frac{1}{|\nabla f|}\,d\sigma=\int_{\{f^*=\lambda\}}\frac{1}{|\nabla f^*|}\,d\sigma=\frac{1}{|\nabla f^*(\theta_\lambda)|}\cdot\sigma(\{f^*=\lambda\}),$$

where $\theta_\lambda=\max\{\theta\mid f^*(\theta)\ge\lambda\}$. Using Jensen's inequality, we find

$$\frac{\int_{\{f=\lambda\}}|\nabla f|^2\,d\sigma/|\nabla f|}{\int_{\{f=\lambda\}}d\sigma/|\nabla f|}\ge\left(\frac{\int_{\{f=\lambda\}}|\nabla f|\,d\sigma/|\nabla f|}{\int_{\{f=\lambda\}}d\sigma/|\nabla f|}\right)^2=\left(\frac{\sigma\{f=\lambda\}}{\int_{\{f=\lambda\}}d\sigma/|\nabla f|}\right)^2.$$

Combining this with the previous result yields

$$\int_{\{f=\lambda\}}|\nabla f|^2\frac{d\sigma}{|\nabla f|}\ge\frac{\sigma(\{f^*=\lambda\})}{|\nabla f^*(\theta_\lambda)|}\left[|\nabla f^*(\theta_\lambda)|\frac{\sigma(\{f=\lambda\})}{\sigma(\{f^*=\lambda\})}\right]^2.$$

The isoperimetric inequality Theorem 2.3 implies that

$$\frac{\sigma(\{f=\lambda\})}{\sigma(\{f^*=\lambda\})}\ge\sqrt{2}$$

since $\{f=\lambda\}\supseteq\partial\{f>\lambda\}$, $\{f^*=\lambda\}=\partial\{f^*>\lambda\}$, and $\{f>\lambda\}$ is an open, antipodally symmetric set with $m\{f>\lambda\}=m\{f^*>\lambda\}$.

We now have

$$\int_{\{f=\lambda\}}|\nabla f|^2\frac{d\sigma}{|\nabla f|}\ge 2\int_{\{f^*=\lambda\}}|\nabla f^*|^2\frac{d\sigma}{|\nabla f^*|},$$

which we can write as

$$\frac{-d}{d\lambda}\int_{\{f>\lambda\}}|\nabla f|^2\,d\zeta \geq 2\left(\frac{-d}{d\lambda}\int_{\{f^*>\lambda\}}|\nabla f^*|^2\,d\zeta\right).$$

Integrating from $a=\min\{f(\zeta)\mid\zeta\in S^2\}$ to $b=\max\{f(\zeta)\mid\zeta\in S^2\}$ in λ, we have

$$\int_{S^2}|\nabla f|^2\,d\zeta \geq 2\int_{S^2}|\nabla f^*|^2\,d\zeta.$$

Combining this with inequality (2.4) yields Theorem 2.2. □

We note that the only dependence on dimension in any of the preceding arguments occurs in Lemma 2.5. For $n\geq 3$, Lemma 2.5 does not hold. In fact, S^3 can be written as the union of two tori which are antipodally symmetric and share a common boundary.

We are now in a position to derive the logarithmic Sobolev inequality of Theorem 2.1. We first rewrite the Sobolev inequality for $\mathbf{P}^2$ given in Theorem 2.2 as

$$\frac{\|f\|^2_{L^p(\mathbf{P}^2)}-\|f\|^2_{L^2(\mathbf{P}^2)}}{p-2} \leq \frac{1}{4}\int_{\mathbf{P}^2}|\nabla f|^2\,d\zeta.$$

Taking a limit as $p\downarrow 2$ yields

$$\frac{d}{dp}(\|f\|^2_{L^p(\mathbf{P}^2)})\big|_{p=2} \leq \frac{1}{4}\int_{\mathbf{P}^2}|\nabla f|^2\,d\zeta. \tag{2.5}$$

Since

$$\|f\|^2_{L^p(\mathbf{P}^2)}=\left(\int_{\mathbf{P}^2}|f|^p\,d\zeta\right)^{2/p}$$
$$=\exp\left\{\frac{2}{p}\ln\left[\int_{\mathbf{P}^n}\exp(p\ln|f|)\,d\zeta\right]\right\},$$

we can use logarithmic differentiation to see that (2.5) is equivalent to

$$\int_{\mathbf{P}^2}|f|^2\ln|f|\,d\zeta-\|f\|^2_{L^2(\mathbf{P}^2)}\ln\|f\|_{L^2(\mathbf{P}^2)} \leq \frac{1}{4}\int_{\mathbf{P}^2}|\nabla f|^2\,d\zeta. \tag{2.5'}$$

This is precisely the inequality of Theorem 2.1, as desired. □

3. The Hypercontractive Estimate

The logarithmic Sobolev inequality (Theorem 2.1) is equivalent to the following hypercontractive estimate for the heat semigroup on $\mathbf{P}^2$.

THEOREM 3.1. *Let* $f\in L^p(\mathbf{P}^2)$, $p\geq 1$. Then for $t\geq\frac{1}{8}\ln[(q-1)/(p-1)]$, *we have*

$$\|e^{t\Delta}f\|_{L^q(\mathbf{P}^2)} \leq \|f\|_{L^p(\mathbf{P}^2)}.$$

The relationship between these two types of estimates was first recognized by Gross [4]. As indicated in the first two sections, this has previously been used to obtain hypercontractive estimates for other semigroups. The proof of the equivalence of the two types of estimates for the heat semigroup on $\mathbf{P}^2$ is essentially the same as that given by Mueller and Weissler [6] for the heat semigroup on S^n, and the interested reader is directed there for details.

4. Additional Remarks

A. The question of sharpness for the hypercontractive estimate given by Theorem 3.1 remains open, though the author believes that it is the best possible. The following discussion provides some evidence for this point of view. There is also the question of analogous results for dimensions other than 2.

The natural generalization of the isoperimetric inequality (Theorem 2.3) for $\mathbf{P}^n$ would yield the following Sobolev inequality for $\mathbf{P}^n$:

$$\|f\|^2_{L^p(\mathbf{P}^n)} \le \frac{p-2}{n\cdot 2^{2/n}}\int_{\mathbf{P}^n}|\nabla f|^2\,d\zeta + \int_{\mathbf{P}^n}|f|^2\,d\zeta, \tag{4.1}$$

where $p\in[2, 2n/(n-2)]$ if $n\ge 3$ and $p\ge 2$ if $n=1,2$. From this, the method used to prove Theorem 3.1 could easily be applied to obtain the following hypercontractive estimate for the heat semigroup on $\mathbf{P}^n$: for $1\le p\le q<\infty$ and $t\ge[n\cdot 2^{(n+2)/n}]^{-1}\ln[(q-1)/(p-1)]$,

$$\|e^{t\Delta}f\|_{L^q(\mathbf{P}^n)} \le \|f\|_{L^p(S^n)}. \tag{4.2}$$

For $n=1$, this estimate can be realized directly by a change of variables in the logarithmic inequality (2.1). Note that the obvious diffeomorphism from S^1 to $\mathbf{P}^1$ does not preserve the Laplace operator. For $n\ge 3$, however, additional information is needed: consideration of fractional integration on $\mathbf{P}^n$ indicates that (4.1) is not sharp for higher dimensions [10]. In fact, the method used by Beckner [2] to obtain inequality (2.3) can be applied to show that

$$\|f\|^2_{L^p(\mathbf{P}^n)} \le \frac{p-2}{2}\cdot\frac{p}{p+n}\int_{\mathbf{P}^n}|\nabla f|^2\,d\zeta + \int_{\mathbf{P}^n}|f|^2\,d\zeta, \tag{4.3}$$

where $n\ge 3$ and $2\le p\le 2n/(n-2)$. For p close to 2, this is stronger than (4.1). (Note that if we take $n=1$ or 2 in (4.3), the resulting inequality is weaker than (4.1) for all $p>2$.) Since these results are inherited from the sphere S^n, one expects that additional improvement is possible.

However, using the procedure described in Sections 2 and 3, (4.3) does yield the following hypercontractive estimate for the heat semigroup on $\mathbf{P}^n$: for $1\le p\le q<\infty$ and $t\ge[2(n+2)]^{-1}\ln[(q-1)/(p-1)]$,

$$\|e^{t\Delta}f\|_{L^q(\mathbf{P}^n)} \le \|f\|_{L^p(\mathbf{P}^n)}. \tag{4.4}$$

We note that, for $n=2$, this agrees with Theorem 3.1.

B. Another interesting limiting case of Sobolev inequalities on the sphere is the sharp form of the Moser–Trudinger inequality due to Onofri [8]:

(4.5) $$\int_{S^2} e^f \, d\zeta \leq \exp\left\{\int_{S^2} |f|^2 \, d\zeta + \frac{1}{4} \int_{S^2} |\nabla f|^2 \, d\zeta\right\}.$$

Using the technique of the proof of Theorem 2.2, we can immediately obtain the following version of inequality (4.5) for $\mathbf{P}^2$:

(4.6) $$\int_{\mathbf{P}^2} e^f \, d\zeta \leq \exp\left\{\int_{\mathbf{P}^2} |f|^2 \, d\zeta + \frac{1}{8} \int_{\mathbf{P}^2} |\nabla f|^2 \, d\zeta\right\}.$$

Inequality (4.6) first appeared in [9], where it is shown to be sharp.

References

1. D. Bakry and M. Emery, *Hypercontractivité de semi-groupes de diffusion,* C. R. Acad. Sci. Paris Sér. I Math. 299 (1984), 775–778.
2. W. Beckner, *Sobolev inequalities, the Poisson semigroup and analysis on the sphere S^n*, preprint.
3. ———, *Sharp Sobolev inequalities on the sphere and the Moser–Trudinger inequality,* preprint.
4. L. Gross, *Logarithmic Sobolev inequalities,* Amer. J. Math. 97 (1975), 1061–1083.
5. J. Moser, *On a nonlinear problem in differential geometry,* Dynamical Systems (Proc. Sympos., Univ. Bahia, Salvador, 1971), pp. 273–280, Academic Press, New York, 1973.
6. C. E. Mueller and F. B. Weissler, *Hypercontractivity for the heat semigroup for ultraspherical polynomials and on the n-sphere,* J. Funct. Anal. 48 (1982), 252–283.
7. E. Nelson, *The free Markoff field,* J. Funct. Anal. 12 (1973), 211–227.
8. E. Onofri, *On the positivity of the effective action in a theory of random surfaces,* Comm. Math. Phys. 86 (1982), 321–326.
9. B. Osgood, R. Phillips, and P. Sarnak, *Extremals of determinants of Laplacians,* J. Funct. Anal. 80 (1988), 148–211.
10. J. M. Pearson, *Sobolev inequalities for ultraspherical polynomials,* Ph.D. dissertation, Univ. of Texas - Austin, 1989.
11. G. Pólya and G. Szegö, *Isoperimetric inequalities in mathematical physics,* Princeton Univ. Press, Princeton, N.J., 1951.
12. F. B. Weissler, *Logarithmic Sobolev inequalities for the heat-diffusion semigroup,* Trans. Amer. Math. Soc. 237 (1978), 255–269.
13. ———, *Logarithmic Sobolev inequalities and hypercontractive estimates on the circle,* J. Funct. Anal. 37 (1980), 218–234.

Department of Mathematics
Florida International University
Miami, FL 33199