On the Functional Calculus of Contractions with Nonvanishing Unitary Asymptotes

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1. Introduction

Let T be an absolutely continuous contraction on the (complex, separable) Hilbert space $\mathcal{H}$. The Sz.-Nagy–Foiaş functional calculus for T is a contractive, weak*-weak*-continuous algebra homomorphism from the Hardy space H^∞ into the set of all (bounded, linear) operators defined on $\mathcal{H}$:

$$\phi_T: H^\infty \to \mathcal{B}(\mathcal{H}), \qquad \phi_T: h \mapsto h(T)$$

(see [21]). We recall that the spaces H^∞ and $\mathcal{B}(\mathcal{H})$ can be identified as the duals of the factor space L^1/H_0^1 and the von Neumann–Schatten ideal $\mathcal{C}_1(\mathcal{H})$ of trace class operators, respectively (cf. [19]). (All function spaces are defined with respect to the normalized Lebesgue measure m on the unit circle ∂D, and H_0^1 means the set of functions with vanishing Fourier coefficients of nonpositive index in L^1.) The mapping ϕ_T induces a contractive, sesquilinear transformation $p_T: \mathcal{H} \times \mathcal{H} \to L^1/H_0^1$, where $p_T(x,y)$ is the unique element of L^1/H_0^1 possessing the property that $\langle h(T)x, y\rangle = \int_{\partial D} hf\,dm$ holds for every $h \in H^\infty$ and every integrable function f in the coset $p_T(x,y)$. The value $p_T(x,y)$ can be considered as the "local density function" of ϕ_T at x, y.

The powerful theorem asserting that p_T is surjective if ϕ_T is an isometry was proved simultaneously and independently by Bercovici [2] and Chevreau [8]. This theorem has many consequences. For example, it follows that any absolutely continuous contraction with an isometric functional calculus has a nontrivial invariant subspace, a theorem first proved by Brown, Chevreau, and Pearcy [5]. In [6], Brown and Chevreau have shown that the contractions with isometric functional calculus are even reflexive.

Let $T^{(a)} \in \mathcal{B}(\mathcal{H}^{(a)})$ denote the unitary asymptote of the contraction T (cf. [17]). $T^{(a)}$ is an absolutely continuous unitary operator, and $\mathcal{H}^{(a)} \neq \{0\}$ if and only if $\lim_{n\to\infty}\|T^n x\| \neq 0$ for some nonzero vector $x \in \mathcal{H}$. Let $\Gamma = \Gamma_T$ be a Borel set on the unit circle ∂D such that $\chi_\Gamma\,dm$ is a scalar spectral measure for $T^{(a)}$. (χ_Γ stands for the characteristic function of Γ.) It is not difficult to show (see Lemma 3) that if $m(\Gamma_T) = 1$ then ϕ_T is an isometry and so the Bercovici–Chevreau theorem applies. Thus $m(\Gamma_T) = 1$ implies that $\operatorname{ran} p_T = L^1/H_0^1 = \pi(L^1(\Gamma_T))$. (Here $\pi: L^1 \to L^1/H_0^1$; $\pi: f \mapsto [f]$ denotes the factor map-

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ping.) In [7], Chevreau, Exner, and Pearcy have shown that the inclusion $\operatorname{ran} p_T \supset \pi(L^1(\Gamma_T))$ is true in general; that is, for every function $f \in L^1(\Gamma)$ there exist vectors x and y in $\mathcal{H}$ such that $p_T(x,y) = [f]$. They gained the solution vectors x, y by an iterative process, as the limit of approximate solutions x_n, y_n: $p_T(x_n, y_n) \approx [f]$. (See [7, Prop. 2.3, Cor. 2.4, and Thm. 2.5].)

In the present paper we shall give a new proof for this theorem. Our proof is based on the functional representation of the canonical mapping X intertwining T and $T^{(a)}$ and provides the exact solutions x, y immediately. Discussing the consequences of the aforementioned relation, we find that the predual of the algebra generated by the unitary asymptote $T^{(a)}$ can be naturally embedded into the predual of the algebra generated by T. Furthermore, it turns out that the adjoint of this embedding coincides with the restriction of a canonical mapping between the commutants of T and $T^{(a)}$.

2. Preliminaries and the Main Results

1. In the sequel let $T \in \mathcal{B}(\mathcal{H})$ be an absolutely continuous contraction; this means that $\|T\| \le 1$ and the unitary part of T is absolutely continuous (see [21, §I.3]). Following [1] we introduce the functional calculus ϕ_T via the unitary dilation of T. Therefore, let us consider the minimal unitary dilation $U \in \mathcal{B}(\mathcal{K})$ of T. U is an absolutely continuous unitary operator on the space $\mathcal{K}$ containing $\mathcal{H}$, $\vee\{U^n\mathcal{H}: n \in Z\} = \mathcal{K}$, and $T^n = P_{\mathcal{H}} U^n | \mathcal{H}$ holds for every $n \in N$. (See [21, §I.4].) The mapping

$$\tilde{\phi}_T: L^\infty \to \mathcal{B}(\mathcal{H}), \qquad \tilde{\phi}_T: f \mapsto f(T) := P_{\mathcal{H}} f(U) | \mathcal{H}$$

is clearly contractive, linear and weak*-weak*-continuous. For any two vectors $x, y \in \mathcal{H}$, let $\omega_{x,y} \in \mathcal{B}(\mathcal{H})^*$ denote the functional defined by $\omega_{x,y}(A) = \langle Ax, y\rangle$ $(A \in \mathcal{B}(\mathcal{H}))$. Since the composite mapping $\omega_{x,y} \circ \tilde{\phi}_T$ is a weak*-continuous linear functional on L^∞ there exists a unique function in L^1, denoted by $(x \cdot y)_T$, such that

$$\langle f(T)x, y\rangle = (\omega_{x,y} \circ \tilde{\phi}_T)(f) = \int_{\partial D} f (x \cdot y)_T \, dm$$

holds for every function $f \in L^\infty$. It is obvious that

$$\tau_T: \mathcal{H} \times \mathcal{H} \to L^1, \qquad \tau_T(x,y) := (x \cdot y)_T$$

is a continuous, sesquilinear transformation; $\|\tau_T(x,y)\|_1 \le \|x\|\|y\|$. If T is a unitary operator with spectral measure E, then

$$(x \cdot y)_T dm = E_{x,y} := \langle E(\cdot)x, y\rangle.$$

The dilation property of U and the weak*-weak*-continuity of $\tilde{\phi}_T$ imply that the restriction

$$\phi_T := \tilde{\phi}_T | H^\infty : H^\infty \to \mathcal{B}(\mathcal{H})$$

is a weak*-weak*-continuous algebra homomorphism, called the Sz.-Nagy-Foiaş functional calculus for T.

DEFINITION 1. Let p_T be the composition of the mappings π and τ_T; that is,

$$p_T\colon \mathcal{H}\times\mathcal{H}\to L^1/H_0^1, \qquad p_T(x,y)=[(x\cdot y)_T].$$

It is clear that p_T is a continuous sesquilinear transformation, and $p_T(x,y)$ is the unique vector in L^1/H_0^1 that represents the functional $\omega_{x,y}\circ\phi_T$.

It was proved in [2] that if ϕ_T is an isometry then p_T is a surjective, open mapping; more precisely: For every $L\in L^1/H_0^1$ and $\eta>0$ there exist vectors $x,y\in\mathcal{H}$ such that $[(x\cdot y)_T]=L$ and $\|x\|\|y\|\le(1+\eta)\|L\|$. The class of all absolutely continuous contractions having the latter property was denoted by $A_1(1)$.

2. A new semi-inner product can be introduced on $\mathcal{H}$ by

$$\langle x,y\rangle_T:=\lim_{n\to\infty}\langle T^n x, T^n y\rangle \quad (x,y\in\mathcal{H}).$$

Factorization and completion yield a Hilbert space $\mathcal{H}_+^{(a)}$, where T acts as an isometry $T_+^{(a)}$. Let $T^{(a)}\in\mathcal{B}(\mathcal{H}^{(a)}=\mathcal{H}_T^{(a)})$ be the minimal unitary extension of $T_+^{(a)}$; $T^{(a)}$ is called the *unitary asymptote* of T. (Using a generalized Banach limit, this operator can be introduced even in the case where T is a power bounded operator; see [17].)

The canonical embedding $X=X_T$ of $\mathcal{H}$ into $\mathcal{H}^{(a)}$ intertwines the contraction T with its unitary asymptote $T^{(a)}$: $XT=T^{(a)}X$, whence by continuity $Xh(T)=h(T^{(a)})X$ follows for every $h\in H^\infty$. This relation allows us to make a first attempt to relate the mappings p_T and $p_{T^{(a)}}$.

LEMMA 2. *For every vector $x\in\mathcal{H}$ and $v\in\mathcal{H}^{(a)}$ we have:*

$$[(x\cdot X^*v)_T]=[(Xx\cdot v)_{T^{(a)}}].$$

Proof. Indeed, let $h\in H^\infty$ be arbitrary. Then

$$\begin{aligned}[][(x\cdot X^*v)_T],h] &= \langle h(T)x, X^*v\rangle \\ &= \langle Xh(T)x, v\rangle = \langle h(T^{(a)})Xx, v\rangle = [[(Xx\cdot v)_{T^{(a)}}],h]. \qquad \Box\end{aligned}$$

Since $\bigvee\{T^{(a)-n}X\mathcal{H}: n\in N\}=\mathcal{H}^{(a)}$, by Lemma 2 and [2, Lemma 4.1] we conclude that $\{p_T(x,X^*v): x\in\mathcal{H}, v\in\mathcal{H}^{(a)}\}$ is a dense set in $\pi(L^1(\Gamma_T))$. Our aim is to show that this set actually coincides with $\pi(L^1(\Gamma_T))$.

The intertwining transformation X induces a homomorphism between the commutants of the operators T and $T^{(a)}$ in the following way. Let A be an arbitrary operator in the commutant $\{T\}':=\{C\in\mathcal{B}(\mathcal{H}): CT=TC\}$ of T. Then by [17, Thm. 2] there exists a unique operator $B\in\{T^{(a)}\}'$ such that $XA=BX$. It can be easily seen that the mapping

$$\gamma_T\colon \{T\}'\to\{T^{(a)}\}', \qquad \gamma_T\colon A\mapsto B$$

is an algebra homomorphism, and that $\gamma_T(h(T))=h(T^{(a)})$ holds for every $h\in H^\infty$.

For any Borel set $\alpha \subset \partial D$ and any function $f \in L^\infty$ we shall use the notation $\|f\|_{\alpha,\infty}$ for $\|\chi_\alpha f\|_\infty$. The following lemma provides us a lower estimate for the norm of the functions of T.

LEMMA 3. *The inequality $\|h\|_{\Gamma,\infty} \le \|h(T)\|$ is true for every function $h \in H^\infty$.*

Proof. Since γ_T is a homomorphism, it follows that the spectrum of $h(T^{(a)})$ is included in the spectrum of $h(T)$. Taking into account that $h(T^{(a)})$ is normal and $\|h(T^{(a)})\| = \|h\|_{\Gamma,\infty}$, we obtain the desired inequality. □

Therefore, ϕ_T is an isometry if $m(\Gamma_T) = 1$, and by [2] we obtain the following corollary, which was already proved in [7] (see Theorem 2.5 there).

COROLLARY 4. *If $m(\Gamma_T) = 1$ then $T \in A_1(1)$.*

There is a natural way to enlarge the operator classes $A_1(r)$ studied in [3].

DEFINITION 5. Let $r \ge 1$ be a real number and let α be a Borel set on the unit circle ∂D. The absolutely continuous contraction T is said to be of class $A_1(r,\alpha)$ if, given any function $f \in L^1(\alpha) := \chi_\alpha L^1$ and $\eta > 0$, there exist vectors $x, y \in \mathcal{H}$ such that

$$[(x \cdot y)_T] = [f] \quad \text{and} \quad \|x\|\|y\| \le (r+\eta)\|f\|_1.$$

It is obvious that $A_1(r) = A_1(r, \partial D)$. (Related operator classes were investigated in [7]; see Definition 3.2.)

The result of Chevreau, Exner, and Pearcy [7], cited in the Introduction, can now be stated as follows.

THEOREM 6. *Every absolutely continuous contraction T is of class $A_1(1,\Gamma_T)$.*

3. Proof in the Functional Model

The following lemma allows us to reduce the proof of Theorem 6 to the case where T is completely non-unitary.

LEMMA 7.

(a) *Given any $\epsilon > 0$ there exists an affinity $Q \in \mathcal{B}(\mathcal{H})$ such that QTQ^{-1} is a completely non-unitary contraction and $\|Q\|\|Q^{-1}\| < 1+\epsilon$.*

(b) *Suppose $S = QTQ^{-1}$. Then for all vectors $x, y \in \mathcal{H}$ we have $[(x\cdot y)_S] = [(Q^{-1}x \cdot Q^* y)_T]$.*

(c) *The unitary asymptotes of the operators $S = QTQ^{-1}$ and T are unitarily equivalent; hence $\Gamma_S = \Gamma_T$.*

Proof. (a) The contraction T splits into the orthogonal sum $T = T_1 \oplus T_2$ of a completely non-unitary contraction T_1 and an absolutely continuous unitary operator T_2. Furthermore, T_2 is unitarily equivalent to an orthogonal sum

$\oplus_k M_{\alpha_k}$, where $M_\alpha \in \mathcal{B}(L^2(\alpha))$ denotes the operator of multiplication by the identical function χ $(\chi(z)=z)$. For every k, let us choose an outer function $\Theta_k \in H^\infty$ such that $|\Theta_k|=[1+\epsilon/2]^{-1}\chi_{\alpha_k}+\chi_{\partial D\setminus\alpha_k}$ holds a.e. (cf. [12]). Then $\|\Theta(z)^{-1}\|<1+\epsilon$ $(z\in D)$ is true for the diagonal matrix $\Theta=\operatorname{diag}(\Theta_k)$. Hence by [21, Thm. IX.1.2] there exists an affinity $\tilde{Q}$ such that $\tilde{Q}T\tilde{Q}^{-1}=T_1\oplus S(\Theta)$ and $\|\tilde{Q}\|\|\tilde{Q}^{-1}\|<1+\epsilon$. Here $S(\Theta)$ denotes the model-operator corresponding to Θ, and so it is completely non-unitary. Let us consider the polar decomposition $\tilde{Q}=V|\tilde{Q}|$; it is immediate that the affinity $Q=|\tilde{Q}|$ possesses the required properties.

(b) Since for every function $h\in H^\infty$ we have

$$\int_{\partial D} h(x\cdot y)_S\,dm=\langle h(S)x,y\rangle=\langle Qh(T)Q^{-1}x,y\rangle=\langle h(T)Q^{-1}x,Q^*y\rangle$$
$$=\int_{\partial D} h(Q^{-1}x\cdot Q^*y)_T\,dm,$$

it follows that $[(x\cdot y)_S]=[(Q^{-1}x\cdot Q^*y)_T]$.

(c) Let $S^{(a)}\in\mathcal{B}(\mathcal{H}_S^{(a)})$ be the unitary asymptote of the power bounded operator $S=QTQ^{-1}$, and let $X'\in\mathcal{B}(\mathcal{H},\mathcal{H}_S^{(a)})$ be the natural embedding of $\mathcal{H}$ into $\mathcal{H}_S^{(a)}$. By [17, Thm. 2] there exists an operator $Q'\in\mathcal{B}(\mathcal{H}_S^{(a)},\mathcal{H}_T^{(a)})$, intertwining $S^{(a)}$ and $T^{(a)}$, such that $Q'X'=XQ$. It is clear that

$$\operatorname{ran} Q'\supset\operatorname{ran}(XQ)=\operatorname{ran} X.$$

On the other hand, by the intertwining relation $T^{(a)}Q'=Q'S^{(a)}$ we infer that $(T^{(a)})^{-n}Q'=Q'(S^{(a)})^{-n}$, and so

$$\operatorname{ran} Q'\supset\operatorname{ran}((T^{(a)})^{-n}Q')\supset\operatorname{ran}((T^{(a)})^{-n}X)$$

is true, for every positive integer n. Since $\bigvee_{n\in\mathbb{N}}\operatorname{ran}((T^{(a)})^{-n}X)=\mathcal{H}_T^{(a)}$, we have that Q' has dense range. Thus $T^{(a)}$ is unitarily equivalent to the restriction of $S^{(a)}$ to a reducing subspace (see [11, Lemma 4.1]).

By symmetry we obtain also that $S^{(a)}$ is unitarily equivalent to the restriction of $T^{(a)}$ to a reducing subspace. Hence $T^{(a)}$ and $S^{(a)}$ must be unitarily equivalent (see [13]). □

Therefore, in proving Theorem 6 it can be supposed that T is a completely non-unitary contraction, or (what is essentially the same) that $T=S(\Theta)$ is a model-operator (cf. [21, Chap. VI]). Hence let $\mathcal{E}$ and $\mathcal{E}_*$ be separable Hilbert spaces, and let us consider a contraction-valued analytic function $\Theta:\partial D\to\mathcal{B}(\mathcal{E},\mathcal{E}_*)$. Assume that Θ is purely contractive; this means that the values of the analytic extension of Θ to the unit disc are strict contractions. Let us form the spaces $\mathcal{K}=L^2(\mathcal{E}_*)\oplus(\Delta L^2(\mathcal{E}))^-$ and $\mathcal{K}_+=H^2(\mathcal{E}_*)\oplus(\Delta L^2(\mathcal{E}))^-$, where $\Delta(z)=(I-\Theta(z)^*\Theta(z))^{1/2}$ $(z\in\partial D)$, and let $U\in\mathcal{B}(\mathcal{K})$ denote the operator of multiplication by χ. The compression $T=P_{\mathcal{H}}U|\mathcal{H}$ of U to the subspace $\mathcal{H}=\mathcal{K}_+\ominus\{\Theta w\oplus\Delta w: w\in H^2(\mathcal{E})\}$ is a completely non-unitary contraction, called the model-operator associated with Θ and denoted by $T=S(\Theta)$. The operator U is the minimal unitary dilation of T.

As can be expected in a function space, a short computation shows that

$$(x\cdot y)_T(z)=\langle x(z),y(z)\rangle_{\mathcal{E}_*\oplus\mathcal{E}}\quad (z\in\partial D)$$

for all vectors $x,y\in\mathcal{K}$. Furthermore, there is a functional representation for the pair $(X,T^{(a)})$ also. Namely, let $\tilde{R}_*$ be the operator of multiplication by χ on the function space $\tilde{\mathfrak{R}}_*=(\Delta_*L^2(\mathcal{E}_*))^-$, where $\Delta_*(z)=(I-\Theta(z)\Theta(z)^*)^{1/2}$ $(z\in\partial D)$, and let $\tilde{X}\in\mathcal{B}(\mathcal{K},\tilde{\mathfrak{R}}_*)$ denote the restriction of the transformation given by the matrix $[-\Delta_*\ \ \Theta]$, with respect to the decomposition $H^2(\mathcal{E}_*)\oplus(\Delta L^2(\mathcal{E}))^-$, to the subspace $\mathcal{K}$. Then $\tilde{X}$ intertwines T with the unitary operator $\tilde{R}_*$: $\tilde{R}_*\tilde{X}=\tilde{X}T$, and there exists a unitary transformation $Z\in\mathcal{B}(\tilde{\mathfrak{R}}_*,\mathcal{K}^{(a)})$ such that $Z\tilde{R}_*=T^{(a)}Z$ and $X=Z\tilde{X}$. Thus the pairs $(X,T^{(a)})$ and $(\tilde{X},\tilde{R}_*)$ are equivalent (see [15] and [17]). As an immediate consequence we obtain

$$\Gamma_T=\{z\in\partial D:\Delta_*(z)\neq 0\}.$$

It will turn out that the cases $\|\Delta_*(z)\|=1$ and $\|\Delta_*(z)\|<1$ can be handled differently; hence we provide the following definition.

DEFINITION 8. For any model-operator $T=S(\Theta)$ let $\Gamma_0=\Gamma_{0,T}$ and $\Gamma_1=\Gamma_{1,T}$ denote the sets

$$\Gamma_0=\{z\in\partial D:0<\|\Delta_*(z)\|<1\}\quad\text{and}\quad\Gamma_1=\{z\in\partial D:\|\Delta_*(z)\|=1\}.$$

It is evident that $\Gamma_0\cup\Gamma_1=\Gamma$ and $\Gamma_0\cap\Gamma_1=\emptyset$.

We shall need yet a simple fact about defect operators. Let A be a contractive transformation from a Hilbert space $\mathcal{F}$ into a Hilbert space $\mathcal{G}$. Let us consider the defect operator $D_A=(I-A^*A)^{1/2}$ and the defect space $\mathfrak{D}_A=\mathcal{F}\ominus\ker D_A$, and let us introduce the space $\mathfrak{D}_{0,A}=\mathfrak{D}_A\ominus\ker(D_A-I)$ and the operator $D_{0,A}:=D_A|\mathfrak{D}_{0,A}\in\mathcal{B}(\mathfrak{D}_{0,A})$. Taking the polar decomposition of A, we have the following connection between the defect operators of A and its adjoint A^*.

LEMMA 9. *For any contraction $A\in\mathcal{B}(\mathcal{F},\mathcal{G})$, the operators $D_{0,A}$ and D_{0,A^*} are unitarily equivalent: $D_{0,A}\cong D_{0,A^*}$.*

The following lemma is a common generalization of the lemmas in [16] and [22], and plays a key role in our proof of Theorem 6.

LEMMA 10. *Given any $0<\rho<1$, there exist functions $u\in H^2(\mathcal{E}_*)$ and $v\in(\Delta L^2(\mathcal{E}))^-$ such that:*

(1) $$\big|\|(u\oplus v)(z)\|_{\mathcal{E}_*\oplus\mathcal{E}}-\chi_\Gamma(z)\big|<\rho\quad\textit{for a.e. } z\in\partial D;$$

(2) $$\frac{\sqrt{2}}{2}-\rho<\|(-\Delta_*u+\Theta v)(z)\|_{\mathcal{E}_*}<1+\rho\quad\textit{for a.e. } z\in\Gamma_0;$$

and

(3) $$1-\rho<\|(-\Delta_*u+\Theta v)(z)\|_{\mathcal{E}_*}<1+\rho\quad\textit{for a.e. } z\in\Gamma_1.$$

Proof. Let us be given an arbitrary real number $0<\epsilon<1$ and a dense sequence $\{x_k\}_{k\in N}$ in the unit sphere of the Hilbert space $\mathcal{E}_*$.

Let us decompose the set Γ_0 into the union of the measurable sets $\alpha=\{z\in\Gamma_0:\|\Delta_*(z)\|>\sqrt{2}/2\}$ and $\beta=\Gamma_0\backslash\alpha$. For any $k\in N$, let $\alpha_{k,0}^{(0)}$ denote the set $\{z\in\alpha:\|\Delta_*(z)x_k\|>\sqrt{2}/2\}$, and let us consider the sequence $\{\alpha_{k,0}\}_{k\in N}$ of disjoint measurable sets defined by $\alpha_{1,0}=\alpha_{1,0}^{(0)}$ and $\alpha_{k,0}=\alpha_{k,0}^{(0)}\backslash[\bigcup_{j=1}^{k-1}\alpha_{j,0}]$ for $k\geq 2$. It is clear that $\bigcup_k\alpha_{k,0}=\alpha$.

In the same way, let us define the sequence $\{\alpha_{k,1}\}_{k\in N}$ as $\alpha_{1,1}=\alpha_{1,1}^{(0)}$ and $\alpha_{k,1}=\alpha_{k,1}^{(0)}\backslash[\bigcup_{j=1}^{k-1}\alpha_{j,1}]$ for $k\geq 2$, where $\alpha_{k,1}^{(0)}=\{z\in\Gamma_1:\|\Delta_*(z)x_k\|>1-\epsilon\}$. This sequence gives a measurable partition of the set Γ_1.

Finally, let us be given a sequence $\{\epsilon_k\}_{k\in N}$ of positive numbers such that $\sum_k\epsilon_k<\epsilon$.

For every $k\in N$ and $l\in\{0,1\}$ there exists a function $\hat{u}_{k,l}\in H^\infty$ such that

$$|\hat{u}_{k,l}|=\chi_{\alpha_{k,l}}+\epsilon_k\chi_{\partial D\backslash\alpha_{k,l}}\text{ a.e.};$$

let $u_{k,l}$ denote the vector-valued function $\hat{u}_{k,l}x_k\in H^2(\mathcal{E}_*)$. Since

$$\sum_{k=1}^{\infty}\sum_{l=0}^{1}\|u_{k,l}(z)\|=2\sum_{k=1}^{\infty}\epsilon_k<2\epsilon\quad\text{for a.e. } z\in\partial D\backslash(\alpha\cup\Gamma_1)$$

and

$$\sum_{k=1}^{\infty}\sum_{l=0}^{1}\|u_{k,l}(z)\|=1+2\sum_{k=1}^{\infty}\epsilon_k-\epsilon_j<1+2\epsilon\quad\text{for a.e. } z\in\alpha_{j,l}$$

$(j\in N,\ l\in\{0,1\})$, we infer that the series $\sum_{k=1}^{\infty}\sum_{l=0}^{1}u_{k,l}$ converges pointwise and in L^2-norm to a function $u\in H^2(\mathcal{E}_*)$ such that

$$\big|\|u(z)\|_{\mathcal{E}_*}-\chi_{\alpha\cup\Gamma_1}(z)\big|<2\epsilon\quad\text{for a.e. } z\in\partial D. \tag{4}$$

For any $z\in\partial D$ let us introduce the spaces $\mathcal{E}_0(z):=\mathfrak{D}_{0,\Theta(z)}$, $\mathcal{E}_{*,0}(z):=\mathfrak{D}_{0,\Theta(z)^*}$, and the operators $\Delta_0(z):=D_{0,\Theta(z)}\in\mathcal{B}(\mathcal{E}_0(z))$, $\Delta_{*,0}(z):=D_{0,\Theta(z)^*}\in\mathcal{B}(\mathcal{E}_{*,0}(z))$. $\mathcal{E}_0(\cdot)$, $\mathcal{E}_{*,0}(\cdot)$ are measurable subspace functions and $\Delta_0(\cdot)$, $\Delta_{*,0}(\cdot)$ are measurable operator functions (see [10]).

Let us suppose that $z\in\beta$. Then $0<\|\Delta_*(z)\|\leq\sqrt{2}/2$; thus by Lemma 9 it follows that $\|\Delta_0(z)\|=\|\Delta_{*,0}(z)\|=\|\Delta_*(z)\|$, whence we gain that $0<\|\Delta_0(z)\|\leq\sqrt{2}/2$. Therefore there exists a function $v\in L^2(\mathcal{E})$ such that $v(z)\in\mathcal{E}_0(z)$ for a.e. $z\in\beta$ and

$$\|v(z)\|_{\mathcal{E}}=\chi_\beta(z)\quad\text{for a.e. } z\in\partial D. \tag{5}$$

We conclude that $v\in(\Delta L^2(\mathcal{E}))^-$ and

$$\begin{aligned}\|\Theta(z)v(z)\|_{\mathcal{E}_*}&=(1-\|\Delta_0(z)v(z)\|_{\mathcal{E}}^2)^{1/2}\geq\left[1-\frac{1}{2}\right]^{1/2}\\&=\frac{\sqrt{2}}{2}\quad\text{for a.e. } z\in\beta.\end{aligned} \tag{6}$$

From (4) and (5) we infer that

$$\big|\|u(z)\oplus v(z)\|_{\mathcal{E}_*\oplus\mathcal{E}}-\chi_\Gamma(z)\big|<2\epsilon\quad\text{for a.e. } z\in\partial D. \tag{7}$$

On the other hand, for a.e. $z\in\alpha_{j,0}$ $(j\in N)$ we have

$$\begin{aligned}1+2\epsilon &> \|u(z)\| \ge \|-\Delta_*(z)u(z)+\Theta(z)v(z)\| = \|\Delta_*(z)u(z)\| \\ &\ge \|\Delta_*(z)u_{j,0}(z)\| - \sum_{k\neq j}\|\Delta_*(z)u_{k,0}(z)\| - \sum_k \|\Delta_*(z)u_{k,1}(z)\| \\ &> \frac{\sqrt{2}}{2} - 2\epsilon,\end{aligned} \tag{8}$$

whereas for a.e. $z\in\alpha_{j,1}$ $(j\in N)$ we have

$$\begin{aligned}1+2\epsilon &> \|-\Delta_*(z)u(z)+\Theta(z)v(z)\| \\ &\ge \|\Delta_*(z)u_{j,1}(z)\| - \sum_{k\neq j}\|\Delta_*(z)u_{k,1}(z)\| - \sum_k \|\Delta_*(z)u_{k,0}(z)\| \\ &> 1-\epsilon-2\epsilon = 1-3\epsilon.\end{aligned} \tag{9}$$

Finally, in view of (6), for a.e. $z\in\beta$:

$$\begin{aligned}1+2\epsilon &> \|u(z)\|+\|v(z)\| \ge \|-\Delta_*(z)u(z)+\Theta(z)v(z)\| \\ &\ge \|\Theta(z)v(z)\| - \|u(z)\| > \frac{\sqrt{2}}{2} - 2\epsilon.\end{aligned} \tag{10}$$

Now choosing $\epsilon=\rho/3$, it follows by the inequalities (7)–(10) that the functions u, v satisfy the conditions (1)–(3). □

We are ready to prove Theorem 6.

Proof of Theorem 6. As was pointed out at the beginning of this section (see Lemma 7), we can assume that T is a model-operator: $T=S(\Theta)$. Let $f\in L^1(\Gamma)$ and $0<\eta<1$ be given. It can be supposed that $f\neq 0$.

We are looking for vectors x and y in $\mathcal{H}$ such that

$$[(x\cdot y)_T]=[f] \tag{11}$$

and

$$\|x\|\|y\| \le (1+\eta)\|f\|_1. \tag{12}$$

(a) Vectors satisfying (11) and (12) can be especially easily found if Θ is an inner function (see [21, §V.2]). Hence let us assume first that $\Theta(z)$ is an isometry for a.e. $z\in\partial D$. Then $\Delta_*(z)$ is an orthogonal projection for a.e. $z\in\partial D$, and so $\Gamma=\Gamma_1$.

Let us consider a real number $0<\rho<1$. By Lemma 10 there exists a vector $x_0=u\oplus v\in H^2(\mathcal{E}_*)\oplus(\Delta L^2(\mathcal{E}))^-$ such that

$$\big|\|x_0(z)\|-\chi_\Gamma(z)\big|<\rho \quad \text{for a.e. } z\in\partial D, \tag{13}$$

and for the vector $h=-\Delta_* u+\Theta v$ we also have:

$$\big|\|h(z)\|-\chi_\Gamma(z)\big|<\rho \quad \text{for a.e. } z\in\partial D. \tag{14}$$

Let us choose a function $\theta\in H^\infty$ such that

$$|\theta(z)| = \chi_\Gamma(z)\|h(z)\|^{-1}+\chi_{\partial D\setminus\Gamma} \quad \text{for a.e. } z\in\partial D. \tag{15}$$

Let γ denote the set $\{z\in\Gamma:|f(z)|^{1/2}\geq\rho\}$ and let $\xi\in H^2$ and $\eta\in L^2(\Gamma)$ be functions satisfying the conditions

$$|\xi|=\chi_\gamma|f|^{1/2}+\rho\chi_{\partial D\setminus\gamma}\quad\text{and}\quad \xi\bar{\eta}=f. \tag{16}$$

Then the vector x is defined as the projection of $\xi\theta x_0\in\mathcal{K}_+$ onto the subspace $\mathcal{H}$:

$$x:=P_{\mathcal{H}}(\xi\theta x_0). \tag{17}$$

It can be easily verified that $\tilde{X}x=-\Delta_*(\xi\theta u)+\Theta(\xi\theta v)=\xi\theta h$.

Introducing the vector $y_0=\eta\theta h\in\tilde{\mathfrak{R}}_*$, we have that

$$(\tilde{X}x\cdot y_0)_{\tilde{R}_*}(z)=(\xi\theta h\cdot\eta\theta h)_{\tilde{R}_*}(z)=\xi(z)\overline{\eta(z)}=f(z)\quad\text{for a.e. } z\in\Gamma.$$

Therefore, defining $y\in\mathcal{H}$ by

$$y:=\tilde{X}^*y_0=P_{\mathcal{H}}(-\Delta_*y_0\oplus\Theta^*y_0), \tag{18}$$

on account of Lemma 2 we conclude that (11) is true.

On the other hand, by virtue of (13)–(18) it follows that

$$\begin{aligned}\|x\|^2&\leq\|\xi\theta x_0\|^2\leq(1+\rho)^2(1-\rho)^{-2}\int_{\partial D}|\xi(z)|^2\,dm(z)\\&\leq(1+\rho)^2(1-\rho)^{-2}(\|f\|_1+\rho^2)\end{aligned} \tag{19}$$

and

$$\|y\|^2\leq\|y_0\|^2=\int_\Gamma|\eta(z)|^2\,dm(z)\leq\|f\|_1. \tag{20}$$

But (19) and (20) immediately imply (12), provided that ρ is sufficiently small.

(b) Now, let $T=S(\Theta)$ be an arbitrary model-operator and let us consider a real number $0<\rho<\sqrt{2}/2$. By Lemma 10 there exists a vector $x_0=u\oplus v\in H^2(\mathcal{E}_*)\oplus(\Delta L^2(\mathcal{E}))^-$ satisfying condition (13) and such that

$$\frac{\sqrt{2}}{2}-\rho<\|h(z)\|<1+\rho\quad\text{for a.e. } z\in\Gamma \tag{14'}$$

is true for the vector $h=-\Delta_*u+\Theta v$. Defining θ, ξ, η, x, y by formulas (15)–(18), the relations (11) and (20) will hold; furthermore, we infer that

$$\|\tilde{X}x\|^2=\|\xi\theta h\|^2\leq\int_{\partial D}|\xi(z)|^2\,dm(z)\leq\|f\|_1+\rho^2. \tag{19*}$$

Given a positive integer n, let us introduce the vectors

$$x':=T^nx \tag{17'}$$

and

$$y':=\tilde{X}^*\tilde{R}_*^ny_0. \tag{18'}$$

In view of Lemma 2 we obtain

$$\begin{aligned}[(x'\cdot y')_T]&=[(T^nx\cdot\tilde{X}^*\tilde{R}_*^ny_0)_T]=[(\tilde{X}T^nx\cdot\tilde{R}_*^ny_0)_{\tilde{R}_*}]\\&=[(\tilde{R}_*^n\tilde{X}x\cdot\tilde{R}_*^ny_0)_{\tilde{R}_*}]=[(\tilde{X}x\cdot y_0)_{\tilde{R}_*}]\\&=[(x\cdot y)_T]=[f].\end{aligned} \tag{11'}$$

Taking into account that $\lim_{n\to\infty}\|T^n x\| = \|Xx\| = \|\tilde{X}x\|$, it follows from (19*) that

(19′) $$\|x'\|^2 \le \|f\|_1 + 2\rho^2$$

if n is large enough. Since the relation

(20′) $$\|y'\|^2 \le \|y_0\|^2 \le \|f\|_1$$

is obviously true, we conclude that

(12′) $$\|x'\|\|y'\| \le (1+\eta)\|f\|_1$$

holds if ρ is sufficiently small. □

4. The Preduals of the Algebras Generated by T and $T^{(a)}$

Let $T \in \mathcal{B}(\mathcal{H})$ be again an arbitrary absolutely continuous contraction.

First of all, let us recall that for any vectors $x, y \in \mathcal{H}$ the expression $x \otimes y$ denotes the operator of rank 1 defined by $(x\otimes y)(h) := \langle h, y\rangle x$ $(h \in \mathcal{H})$. It is obvious that the sesquilinear transformation

$$\omega: \mathcal{H} \times \mathcal{H} \to \mathcal{C}_1(\mathcal{H}), \qquad \omega(x,y) = x \otimes y$$

is continuous; indeed, $\|\omega\| := \sup\{\|\omega(x,y)\|_1 = \|x\|\|y\| : \|x\| \le 1, \|y\| \le 1\} = 1$. This mapping is universal; in fact, the pair $(\omega, \mathcal{C}_1(\mathcal{H}))$ can be considered as the projective tensor product of the Hilbert space $\mathcal{H}$ by itself (see [18, p. 93]).

LEMMA 11. *Given any continuous, sesquilinear mapping ρ from the Cartesian product $\mathcal{H} \times \mathcal{H}$ into a Banach space $\mathcal{G}$, there exists a unique transformation $R \in \mathcal{B}(\mathcal{C}_1(\mathcal{H}), \mathcal{G})$ such that $\rho = R \circ \omega$. Furthermore, $\|R\| \le \|\rho\|$.*

Proof. For the sake of completeness we sketch the proof. Let us define the linear transformation R_0 on the set of finite rank operators $\mathcal{F}(\mathcal{H})$ by $R_0[\sum_{i=1}^n x_i \otimes y_i] := \sum_{i=1}^n \rho(x_i, y_i)$. To show that R_0 is well defined let us assume that $\sum_{i=1}^n x_i \otimes y_i = 0$, and let us consider an arbitrary linear functional $\Lambda \in \mathcal{G}^*$. The composition $\Lambda \circ \rho$ is a continuous, sesquilinear form, so it can be represented by an operator $A \in \mathcal{B}(\mathcal{H})$; that is, $(\Lambda\circ\rho)(x,y) = \langle Ax, y\rangle$ for every $x, y \in \mathcal{H}$. Then

$$\Lambda\left[\sum_{i=1}^n \rho(x_i, y_i)\right] = \sum_{i=1}^n \Lambda(\rho(x_i, y_i)) = \sum_{i=1}^n \langle Ax_i, y_i\rangle = \operatorname{tr}\left[A \sum_{i=1}^n x_i \otimes y_i\right] = 0,$$

and since $\Lambda \in \mathcal{G}^*$ was arbitrary, it follows that $\sum_{i=1}^n \rho(x_i, y_i) = 0$. Therefore, R_0 is well defined on $\mathcal{F}(\mathcal{H})$.

Given any orthonormal sequences $\{e_i\}_{i=1}^n$, $\{f_i\}_{i=1}^n$ of vectors and positive numbers $\{s_i\}_{i=1}^n$, for the operator $C = \sum_{i=1}^n s_i e_i \otimes f_i$ we have that

$$\|R_0(C)\| = \left\|\sum_{i=1}^n s_i \rho(e_i, f_i)\right\| \le \|\rho\| \sum_{i=1}^n s_i = \|\rho\|\|C\|_1.$$

Thus R_0 is bounded: $\|R_0\| \leq \|\rho\|$, and so its continuous extension R to $\mathcal{C}_1(\mathcal{H})$ will have the required properties. □

On the basis of the previous lemma the homomorphism ϕ_T can be recovered from the sesquilinear mapping p_T as follows.

PROPOSITION 12. *There exists a unique transformation*

$$P_T \in \mathcal{B}(\mathcal{C}_1(\mathcal{H}), L^1/H_0^1)$$

such that $P_T \circ \omega = p_T$. *Furthermore,* $\|P_T\| \leq \|p_T\| \leq 1$, *and the adjoint of* P_T *coincides with the functional calculus* ϕ_T: $P_T^* = \phi_T$.

Proof. In view of Lemma 11 we have only to identify the adjoint of P_T. But given any vectors $x, y \in \mathcal{H}$ and function $h \in H^\infty$ we infer that

$$[P_T(x \otimes y), h] = [p_T(x,y), h] = [[(x \cdot y)_T], h]$$

$$= \int_{\partial D} h(x \cdot y)_T \, dm = \langle h(T)x, y \rangle.$$

At the same time,

$$[P_T(x \otimes y), h] = [x \otimes y, P_T^* h] = \operatorname{tr}((P_T^* h)(x \otimes y)) = \langle (P_T^* h)x, y \rangle;$$

therefore $\phi_T h = h(T) = P_T^* h$, which was to be proved. □

The linear span of ran p_T is dense in L^1/H_0^1 if and only if the linear transformation P_T has dense range, and by Proposition 12 this occurs precisely when $\ker \phi_T = \{0\}$. The latter condition means that the contraction is not of class C_0 (cf. [21, §III.4]).

Let us denote by $\mathcal{A}_T$ the closure of ran ϕ_T in the weak*-topology of $\mathcal{B}(\mathcal{H})$. Then the preannihilator

$${}^\perp\mathcal{A}_T := \{C \in \mathcal{C}_1(\mathcal{H}) : \operatorname{tr}(CA) = 0 \text{ for every } A \in \mathcal{A}_T\}$$

of $\mathcal{A}_T$ coincides with the kernel of the transformation P_T: ${}^\perp\mathcal{A}_T = \ker P_T$. Let us form the factor space $Q_T = \mathcal{C}_1(\mathcal{H})/{}^\perp\mathcal{A}_T$, and let $\pi_T: \mathcal{C}_1(\mathcal{H}) \to Q_T$ be the factor mapping: $\pi_T(C) = [C] := C + {}^\perp\mathcal{A}_T$ ($C \in \mathcal{C}_1(\mathcal{H})$). There exists a unique transformation $\varphi_T \in \mathcal{B}(Q_T, L^1/H_0^1)$ such that $\varphi_T \circ \pi_T = P_T$. It is clear that $\|\varphi_T\| \leq 1$ and $\ker \varphi_T = \{0\}$. Taking into account that the dual space of Q_T can be identified with $\mathcal{A}_T$ (see e.g. [9, §III.10]), we gain that the adjoint of φ_T is ϕ_T:

$$\varphi_T^* = \phi_T, \tag{21}$$

when ϕ_T is considered as a mapping into $\mathcal{A}_T$.

The following theorem illuminates the connection between the algebras $\mathcal{A}_T$ and $\mathcal{A}_{T^{(a)}}$ and their preduals Q_T and $Q_{T^{(a)}}$.

THEOREM 13.

(a) *There exists a unique transformation* $F_T \in \mathcal{B}(Q_{T^{(a)}}, Q_T)$ *such that* $\varphi_T \circ F_T = \varphi_{T^{(a)}}$, *$F_T$ is one-to-one, and*

(22) $$F_T[Xx\otimes v]=[x\otimes X^*v]$$

holds for every $x\in\mathcal{H}$ *and* $v\in\mathcal{H}^{(a)}$.

(b) *The adjoint of* F_T *coincides with the restriction of* γ_T, *introduced in Section 2, to the algebra* $\mathcal{A}_T$:

(23) $$F_T^*=\gamma_T\,|\,\mathcal{A}_T.$$

Therefore $F_T^*=\gamma_T\,|\,\mathcal{A}_T\colon\mathcal{A}_T\to\mathcal{A}_{T^{(a)}}$ *is a norm- and weak*-weak*-continuous algebra homomorphism.*

Proof. (a) By Lemma 11 there exists a uniquely determined transformation $P'_{T^{(a)}}\in\mathcal{B}(\mathcal{C}_1(\mathcal{H}^{(a)}),L^1)$ such that $\tau_{T^{(a)}}=P'_{T^{(a)}}\circ\omega_a$, where $\omega_a\colon\mathcal{H}^{(a)}\times\mathcal{H}^{(a)}\to\mathcal{C}_1(\mathcal{H}^{(a)})$ is defined by $\omega_a(u,v)=u\otimes v$ $(u,v\in\mathcal{H}^{(a)})$. It is evident that

$$\operatorname{ran}P'_{T^{(a)}}=L^1(\Gamma);$$

hence, taking into account that $\pi\circ P'_{T^{(a)}}=P_{T^{(a)}}=\varphi_{T^{(a)}}\circ\pi_{T^{(a)}}$, we obtain

(24) $$\operatorname{ran}\varphi_{T^{(a)}}=\pi(L^1(\Gamma)).$$

On the other hand, $\operatorname{ran}\varphi_T=\operatorname{ran}P_T\supset\operatorname{ran}p_T$ and by Theorem 6 the range of p_T contains $\pi(L^1(\Gamma))$. Thus

(25) $$\operatorname{ran}\varphi_T\supset\pi(L^1(\Gamma)).$$

Since φ_T and $\varphi_{T^{(a)}}$ are injective transformations, it follows from (24) and (25) that there exists a unique one-to-one linear mapping $F_T\colon Q_{T^{(a)}}\to Q_T$ satisfying the condition $\varphi_T\circ F_T=\varphi_{T^{(a)}}$. It can be easily verified that F_T is a closed transformation, hence F_T must be continuous.

Equation (22) is an easy consequence of Lemma 2.

(b) Given any vectors $x\in\mathcal{H}$, $v\in\mathcal{H}^{(a)}$ and any function $h\in H^\infty$, by (22) we have

$$\begin{aligned}[F_T[Xx\otimes v],h(T)]&=[[x\otimes X^*v],h(T)]=\operatorname{tr}(h(T)(x\otimes X^*v))\\&=\langle h(T)x,X^*v\rangle=\langle Xh(T)x,v\rangle=\langle h(T^{(a)})Xx,v\rangle\\&=\operatorname{tr}(h(T^{(a)})(Xx\otimes v))=[[Xx\otimes v],h(T^{(a)})].\end{aligned}$$

Since the vectors of type $[Xx\otimes v]$ form a total set in $Q_{T^{(a)}}$, it follows that $F_T^*h(T)=h(T^{(a)})$.

Let us now consider an arbitrary operator A in $\mathcal{A}_T$. There exists a net $\{h_\nu\}$ of H^∞ functions such that $h_\nu(T)\to A$ in the weak*-topology. Then the operators $h_\nu(T^{(a)})=F_T^*h_\nu(T)$ converge to the operator $F_T^*(A)$ in the weak*-topology also. We infer that the nets $\{Xh_\nu(T)\}$ and $\{h_\nu(T^{(a)})X\}$ of linear transformations converge to the mappings XA and $F_T^*(A)X$, respectively, in the weak operator topology. Since $Xh_\nu(T)=h_\nu(T^{(a)})X$ holds for every ν, it follows that $XA=F_T^*(A)X$. The operator $F_T^*(A)$ clearly commutes with $T^{(a)}$; hence $F_T^*(A)=\gamma_T(A)$. □

Since the mapping F_T is one-to-one, its adjoint has dense range: $(\operatorname{ran}F_T^*)^-=\mathcal{A}_{T^{(a)}}$. On the other hand, by the previous theorem we obtain a sufficient condition for the case when F_T has dense range. Let us recall that the contraction T is said to be of class $C_{1\cdot}$ if $\lim_{n\to\infty}\|T^n x\| \neq 0$ for every nonzero vector $x \in \mathcal{H}$.

COROLLARY 14. *If T is a $C_{1\cdot}$-contraction, then* $(\operatorname{ran} F_T)^- = Q_T$.

Proof. If $T \in C_{1\cdot}$, then γ_T is obviously one-to-one. Hence, by (23) we obtain that $\ker F_T^* = \{0\}$, which implies that $(\operatorname{ran} F_T)^- = Q_T$. □

Finally, we characterize the case when F_T is surjective.

PROPOSITION 15. *The following three conditions are equivalent:*

(a) $\operatorname{ran} F_T = Q_T$;
(b) $\operatorname{ran} \varphi_T = \pi(L^1(\Gamma_T))$;
(c) $\gamma_T \mid \mathcal{A}_T : \mathcal{A}_T \to \mathcal{A}_{T^{(a)}}$ *is an isomorphism.*

Moreover, if $m(\Gamma_T) < 1$ *then conditions* (a)–(c) *are all equivalent to:*

(d) T is similar to $T^{(a)}$.

Proof. Conditions (a) and (b) are equivalent because $\ker \varphi_T = \ker \varphi_{T^{(a)}} = \ker F_T = \{0\}$ and $\operatorname{ran} \varphi_{T^{(a)}} = \pi(L^1(\Gamma)) \subset \operatorname{ran} \varphi_T$. The equivalence of (a) and (c) follows from (23) using a theorem of Banach (see e.g. [9, Thm. VI.1.10]). Furthermore, it can be easily seen that (d) always implies (c).

Let us assume now that $m(\Gamma_T) < 1$ and (c) is true. Since the Lebesgue measure m is not absolutely continuous with respect to the spectral measure of the unitary asymptote $T^{(a)}$, it follows that $T^{(a)}$ is a reductive operator, and so $(T^{(a)})^* \in \mathcal{A}_{T^{(a)}}$. (See [23] and [9, §IX.9].) Exploiting the assumption that the mapping $\gamma_T \mid \mathcal{A}_T : \mathcal{A}_T \to \mathcal{A}_{T^{(a)}}$ is an algebra isomorphism, we have that T is invertible and that $\|T^n\| = \|\gamma_T^{-1}((T^{(a)})^n)\| \leq \|F_T^{-1}\| \|(T^{(a)})^n\| = \|F_T^{-1}\|$ for every integer n. Therefore, both T and its inverse T^{-1} are power bounded operators, and hence a theorem of Sz.-Nagy implies that T is similar to a unitary operator (cf. [20]). Then, by [17, Thm. 2], T must be similar to its unitary asymptote $T^{(a)}$. □

Let us consider now an operator $S \in \mathcal{B}(\mathcal{H})$ which is similar to the absolutely continuous contraction T; that is, S is of the form $S = QTQ^{-1}$, where $Q \in \mathcal{B}(\mathcal{H})$ is an affinity. The Sz.-Nagy–Foiaş functional calculus can be trivially extended for S by $h(S) := Qh(T)Q^{-1}$ $(h \in H^\infty)$. The mapping $\phi_S : H^\infty \to \mathcal{B}(\mathcal{H})$, $\phi_S : h \mapsto h(S)$ is evidently a bounded, weak*-weak*-continuous algebra homomorphism. In view of Lemma 7(b) and (c), it is easy to verify that the results of this section are true also for the operator S.

5. Concluding Remarks

The technique used in our proof for Theorem 6 can be also applied to prove the following slightly stronger version of Theorem 6. This corollary played an important role and was proved by an iterative process in [7] (see Corollary 2.4 and Theorem 2.5 there).

COROLLARY 16. *Given any positive number η, positive integer k, and functions $f_1, \ldots, f_k \in L^1(\Gamma)$, there exist vectors $x, y_1, \ldots, y_k \in \mathcal{H}$ such that*

$$[(x \cdot y_j)_T] = [f_j] \quad \text{for } 1 \le j \le k, \tag{26}$$

$$\|y_j\| \le \|f_j\|_1^{1/2} \quad \text{for } 1 \le j \le k, \tag{27}$$

and

$$\|x\| \le (1+\eta)\left[\sum_{j=1}^{k} \|f_j\|_1\right]^{1/2} \tag{28}$$

Proof. Let us introduce the function $f = \sum_{j=1}^{k} |f_j| \in L^1(\Gamma)$. Let us define $\xi \in H^2$ by (16) and, for any $1 \le j \le k$, let $\eta_j \in L^2(\Gamma)$ denote a function such that $\xi\bar{\eta}_j = f_j$. Then, by the proof of Theorem 6, it can be seen that some vectors $x, y_1, \ldots, y_k$ defined through the formulas (17), (17′), (18), and (18′) will satisfy conditions (26)–(28). □

The following proposition shows that the sesquilinear mapping p_T is not necessarily surjective if $m(\Gamma_T) < 1$.

PROPOSITION 17. *Suppose that γ is a Borel set on the unit circle ∂D, $m(\gamma) > 0$, and $\gamma^- \ne \partial D$. Then there exists an absolutely continuous contraction $T \in \mathcal{B}(\mathcal{H})$ such that $T \in A_1(1, \gamma)$ but $[1] \notin \operatorname{ran} p_T$.*

Proof. By [14] there exists a contraction $T \in \mathcal{B}(\mathcal{H})$ of class C_{10} ($:= C_{1.} \cap C_{.0}$) such that $\Gamma_T = \gamma$ and $\sigma(T) \subset \gamma^-$. In view of Theorem 6 we have that $T \in A_1(1, \gamma)$.

On the other hand, since $\sigma(T) \subset \gamma^- \ne \partial D$ we infer that T is invertible and that its inverse can be approximated by the polynomials of T in norm. This means that the restriction of T to any invariant subspace is invertible, and hence $x \in \bigvee_{n \ge 1} T^n x$ holds for every vector $x \in \mathcal{H}$. But it is easy to verify that the latter property is equivalent to the condition $[1] \notin \operatorname{ran} p_T$. □

In contrast to the previous operator, a non-invertible C_{10}-contraction $T' \in \mathcal{B}(\mathcal{H}')$ can be found as well with $\Gamma_{T'} = \gamma$. Let $f \in \mathcal{H}'$ be a nonzero vector such that

$$T'' = \begin{bmatrix} T' & f \\ 0 & 0 \end{bmatrix} \in \mathcal{B}(\mathcal{H}' \oplus C)$$

is also a C_{10}-contraction (cf. [4]). It is clear that $\Gamma_{T''} = \Gamma_{T'} = \gamma$, but now $[1] \in \operatorname{ran} p_{T''}$.

Considering the unitary *-asymptote $T_*^{(a)} := ((T^*)^{(a)})^* \in \mathcal{B}(\mathcal{H}_*^{(a)})$ of the contraction T, we can obtain the duals of our previous results. Let $Y = Y_T := (X_{T^*})^* \in \mathcal{B}(\mathcal{H}_*^{(a)}, \mathcal{H})$ be the canonical intertwining mapping $TY = YT_*^{(a)}$, and let

$$\gamma_{*,T} : \{T\}' \to \{T_*^{(a)}\}', \qquad \gamma_{*,T} : A \mapsto C$$

be the algebra homomorphism, where C is the unique operator in $\{T_*^{(a)}\}'$ satisfying the condition $AY = YC$. Let $\Gamma_* = \Gamma_{*,T}$ denote the Borel set on the unit circle ∂D such that $\chi_{\Gamma_*} dm$ is a scalar spectral measure for $T_*^{(a)}$, and let $\Lambda = \Lambda_T$ denote the set $\Gamma_T \cup \Gamma_{*,T}$. Using these objects in place of X, γ_T, and Γ_T, we have the following statements.

LEMMA 2*. *For all vectors $u \in \mathcal{H}_*^{(a)}$ and $y \in \mathcal{H}$, it is true that*

$$[(Yu \cdot y)_T] = [(u \cdot Y^* y)_{T_*^{(a)}}].$$

LEMMA 3′. *For every function $h \in H^\infty$, the inequality $\|h\|_{\Lambda,\infty} \le \|h(T)\|$ is valid.*

THEOREM 6*. *Every absolutely continuous contraction T is of class $A_1(1, \Gamma_{*,T})$.*

These results raise the following question.

QUESTION 18. Is it true that $T \in A_1(1, \Lambda_T)$ for every absolutely continuous contraction T?

Using the technique of [1], it can be proved that "near factorizations" exist for the elements of $L^1(\Lambda_T)$ as follows.

PROPOSITION 19. *Given $\xi_1, \dots, \xi_p \in \mathcal{H}$, $\epsilon > 0$, and $f \in L^1(\Lambda_T)$, there exist vectors $x, y \in \mathcal{H}$ such that $\langle x, \xi_j \rangle = \langle y, \xi_j \rangle = 0$ for $j = 1, \dots, p$, $\|x\|\|y\| \le \|f\|_1$, and $\|f - (x \cdot y)_T\|_1 < \epsilon$.*

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