Isomorphisms of Alg $\mathcal{L}_n$ and Alg $\mathcal{L}_\infty$

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Let $\mathcal{H}$ be a complex Hilbert space and let $\mathcal{L}_{2n}$ ($\mathcal{L}_{2n+1}$) be the subspace lattice of orthogonal projections generated by $\{[e_1], [e_3], \dots, [e_{2n-1}], [e_1, e_2, e_3], [e_3, e_4, e_5], \dots, [e_{2n-3}, e_{2n-2}, e_{2n-1}], [e_{2n-1}, e_{2n}]\}$ (respectively, $\{[e_1], [e_{2i+1}], [e_{2i-1}, e_{2i}, e_{2i+1}]: i = 1, 2, \dots, n\}$) with an orthonormal basis $\{e_1, e_2, \dots, e_{2n}\}$ ($\{e_1, e_2, \dots, e_{2n+1}\}$).

In this paper the following are proved:

(1) If Φ: Alg $\mathcal{L}_{2n} \to$ Alg $\mathcal{L}_{2n}$ is an isomorphism, then there exists an invertible operator T in Alg $\mathcal{L}_{2n}$ such that $\Phi(A) = TAT^{-1}$ for all A in Alg $\mathcal{L}_{2n}$.

(2) If Φ: Alg $\mathcal{L}_{2n+1} \to$ Alg $\mathcal{L}_{2n+1}$ is an isomorphism, then there exists an invertible operator S in Alg $\mathcal{L}_{2n+1}$ such that either $\Phi(A) = SAS^{-1}$ or $\Phi(A) = SUAUS^{-1}$, where U is a $(2n+1) \times (2n+1)$ matrix whose $(k, 2n-k+2)$-component is 1 for $k = 1, 2, \dots, 2n+1$ and all other entries are 0.

(3) A map Φ: Alg $\mathcal{L}_\infty \to$ Alg $\mathcal{L}_\infty$ is an isomorphism if and only if there exists an invertible operator (not necessarily bounded) T such that $\Phi(A) = TAT^{-1}$ for all A in Alg $\mathcal{L}_\infty$.

1. Introduction

The study of non-self-adjoint operator algebras on Hilbert space was begun in 1974 by Arveson [1]. Recently, such algebras have been found to be of use in physics, in electrical engineering, and in general systems theory. Of particular interest to mathematicians are reflexive algebras with commutative lattices of invariant subspaces. The algebras Alg $\mathcal{L}_n$ and Alg $\mathcal{L}_\infty$ are important classes of such algebras. These algebras possess many surprising properties related to isometries, isomorphisms, cohomology, and extreme points. In this paper, we shall investigate the isomorphisms of these algebras.

First, we introduce the terminologies used in this paper. Let $\mathcal{H}$ be a complex Hilbert space and let $\mathcal{A}$ be a subset of $\mathcal{B}(\mathcal{H})$, the class of all bounded operators acting on $\mathcal{H}$. If $\mathcal{A}$ is a vector space over $\mathbf{C}$ and if $\mathcal{A}$ is closed under the composition of maps, then $\mathcal{A}$ is called an algebra. $\mathcal{A}$ is called a self-

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adjoint algebra provided A^* is in $\mathcal{A}$ for every A in $\mathcal{A}$; otherwise, $\mathcal{A}$ is called a non-self-adjoint algebra. If $\mathcal{L}$ is a lattice of orthogonal projections acting on $\mathcal{H}$, then Alg $\mathcal{L}$ denotes the algebra of all bounded operators acting on $\mathcal{H}$ that leave invariant every orthogonal projection in $\mathcal{L}$. A subspace lattice $\mathcal{L}$ is a strongly closed lattice of orthogonal projections acting on $\mathcal{H}$, containing 0 and 1. Dually, if $\mathcal{A}$ is a subalgebra of $\mathcal{B}(\mathcal{H})$, then Lat $\mathcal{A}$ is the lattice of all orthogonal projections invariant for each operator in $\mathcal{A}$. An algebra $\mathcal{A}$ is reflexive if $\mathcal{A} = \text{Alg Lat } \mathcal{A}$ and a lattice $\mathcal{L}$ is reflexive if $\mathcal{L} = \text{Lat Alg } \mathcal{L}$. A lattice $\mathcal{L}$ is a commutative subspace lattice, or CSL, if each pair of projections in $\mathcal{L}$ commutes; Alg $\mathcal{L}$ is then called a CSL-algebra. If $x_1, x_2, \ldots, x_n$ are vectors in some Hilbert space, then $[x_1, x_2, \ldots, x_n]$ denotes the closed subspace generated by the vectors $x_1, x_2, \ldots, x_n$.

2. Isomorphisms of Alg $\mathcal{L}_{2n}$, Alg $\mathcal{L}_{2n+1}$, and Alg $\mathcal{L}_\infty$

Let $\mathcal{L}_1$ and $\mathcal{L}_2$ be commutative subspace lattices. By an isomorphism Φ: Alg $\mathcal{L}_1 \to$ Alg $\mathcal{L}_2$ we mean a strictly algebraic isomorphism, that is, a bijective, linear, multiplicative map. An isomorphism Φ: Alg $\mathcal{L}_1 \to$ Alg $\mathcal{L}_2$ is said to be spatially implemented if there is a bounded invertible operator T such that $\Phi(A) = TAT^{-1}$ for all A in Alg $\mathcal{L}_1$. Let Ω_{2n} be the tridiagonal algebras discovered by Gilfeather and Larson [4]; that is, Ω_{2n} is an algebra consisting of bounded operators acting on $2n$-dimensional complex Hilbert space $\mathcal{H}$ of the form

$$\begin{bmatrix} * & * & & & & * \\ & * & & & & \\ & * & * & * & & \\ & & & * & & \\ & & & & \ddots & * \\ & & & & & * \end{bmatrix},$$

where all non-starred entries are zero, for some fixed basis $\{e_1, e_2, \ldots, e_{2n}\}$ of $\mathcal{H}$. Automorphisms of Ω_{2n} need not be spatially implemented [5]. Let $\mathcal{L}_{2n}$ (or, respectively, $\mathcal{L}_{n+1}$) be the subspace lattice of orthogonal projections generated by $\{[e_1], [e_3], \ldots, [e_{2n-1}], [e_1, e_2, e_3], [e_3, e_4, e_5], \ldots, [e_{2n-3}, e_{2n-2}, e_{2n-1}], [e_{2n-1}, e_{2n}]\}$ or $\{[e_1], [e_{2i+1}], [e_{2i-1}, e_{2i}, e_{2i+1}]: i = 1, 2, \ldots, n\}$, and let $\mathcal{B}_{2n}$ or $\mathcal{B}_{2n+1}$ be the algebra consisting of all bounded operators, acting on $2n$- or $(2n+1)$-dimensional complex Hilbert space $\mathcal{H}$, that are of the form

$$\begin{bmatrix} * & * & & & \\ & * & & & \\ & * & * & * & \\ & & & * & \\ & & & * & \\ & & & & \ddots & * \\ & & & & & * \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} * & * & & & \\ & * & & & \\ & * & * & * & \\ & & & * & \\ & & & * & \\ & & & & \ddots & \\ & & & & & * & * \end{bmatrix},$$

where all non-starred entries are zero and with an orthonormal basis $\{e_1, e_2, \ldots, e_{2n}\}$ or $\{e_1, e_2, \ldots, e_{2n+1}\}$, respectively.

Let $\mathcal{L}_\infty$ be the subspace lattice of orthogonal projections generated by $\{[e_{2i-1}], [e_{2i-1}, e_{2i}, e_{2i+1}]: i = 1, 2, \ldots\}$, and let $\mathcal{B}_\infty$ be the algebra consisting of all bounded operators acting on separable infinite-dimensional Hilbert space $\mathcal{H}$ of the form

$$\begin{bmatrix} * & * & & & \\ & * & & & \\ & * & * & * & \\ & & & * & \\ & & & * & \\ & & & & \ddots \end{bmatrix},$$

where all non-starred entries are 0 and with an orthonal basis $\{e_1, e_2, \ldots\}$.

LEMMA 2.1.

(1) Alg $\mathcal{L}_{2n} = \mathcal{B}_{2n}$, Alg $\mathcal{L}_{2n+1} = \mathcal{B}_{2n+1}$, *and* Alg $\mathcal{L}_\infty = \mathcal{B}_\infty$.
(2) Lat $\mathcal{B}_{2n} = \mathcal{L}_{2n}$, Lat $\mathcal{B}_{2n+1} = \mathcal{L}_{2n+1}$, *and* Lat $\mathcal{B}_\infty = \mathcal{L}_\infty$.

Let i and j be positive integers. Then E_{ij} is the matrix whose (i, j)-component is 1 and all other components are 0.

THEOREM 2.2. *Let* Φ: Alg $\mathcal{L}_{2n} \to$ Alg $\mathcal{L}_{2n}$ *be an isomorphism such that* $\Phi(E_{ii}) = E_{ii}$ *for all* $i = 1, 2, \ldots, 2n$. *Then there exist nonzero complex numbers* α_{ij} *such that* $\Phi(E_{ij}) = \alpha_{ij} E_{ij}$ *for all* E_{ij} *in* Alg $\mathcal{L}_{2n}$.

Proof. Let $\Phi(E_{ii}) = E_{ii}$ for all i. Then $\Phi(E_{ij}) = \Phi(E_{ii} E_{ij} E_{jj}) = E_{ii} \Phi(E_{ij}) E_{jj}$. If $\Phi(E_{ij}) = \sum_{k,m} \alpha_{km} E_{km}$, then $\Phi(E_{ij}) = \alpha_{ij} E_{ij}$ for some nonzero complex number α_{ij}. □

By an argument similar to that of Theorem 2.2, we can obtain the following theorem.

THEOREM 2.3. *Let* Φ: Alg $\mathcal{L}_{2n+1} \to$ Alg $\mathcal{L}_{2n+1}$ (*resp.*, Alg $\mathcal{L}_\infty \to$ Alg $\mathcal{L}_\infty$) *be an isomorphism such that* $\Phi(E_{ii}) = E_{ii}$ *for all* $i = 1, 2, \ldots, n+1$ $(i = 1, 2, \ldots)$. *Then there exist nonzero complex numbers* α_{ij} *such that* $\Phi(E_{ij}) = \alpha_{ij} E_{ij}$ *for all* E_{ij} *in* Alg $\mathcal{L}_{2n+1}$ (Alg $\mathcal{L}_\infty$).

THEOREM 2.4. *Let* Φ: Alg $\mathcal{L}_{2n} \to$ Alg $\mathcal{L}_{2n}$ *be an isomorphism such that* $\Phi(E_{ii}) = E_{ii}$ *for all* $i = 1, 2, \ldots, 2n$, *and let* $\Phi(E_{ij}) = \alpha_{ij} E_{ij}$, $\alpha_{ij} \neq 0$, *for all* E_{ij} *in* Alg $\mathcal{L}_{2n}$. *Then* $\Phi(A) = TAT^{-1}$ *for all* A *in* Alg $\mathcal{L}_{2n}$, *where* T *is a* $2n \times 2n$ *diagonal operator whose*

(1) $(1,1)$-*component is* 1,
(2) $(2,2)$-*component is* α_{12}^{-1},
(3) $(2i-1, 2i-1)$-*component is* $(\prod_{k=1}^{i-1} \alpha_{2k-1,2k})^{-1} \prod_{k=1}^{i-1} \alpha_{2k+1,2k}$, *and*

(4) $(2i, 2i)$*-component is* $(\prod_{k=1}^{i} \alpha_{2k-1,2k})^{-1} \prod_{k=1}^{i-1} \alpha_{2k+1,2k}$ *for all* $i = 1, 2, \ldots, 2n$.

Proof. Let $A = (a_{ij})$ be in Alg $\mathcal{L}_{2n}$. Then $\Phi(A) = (\alpha_{ij} a_{ij})$ and $\alpha_{ii} = 1$ for all $i = 1, 2, \ldots, 2n$. Consider $\Phi(A)T$ and TA for the above T. Then for all i and j $(i, j = 1, 2, \ldots, 2n)$, the (i, j)-component of $\Phi(A)T$ and TA are the same. Hence $\Phi(A)T = TA$. □

By an argument similar to that of Theorem 2.4, we can derive the following theorem.

THEOREM 2.5. *Let* Φ: Alg $\mathcal{L}_{2n+1} \to$ Alg $\mathcal{L}_{2n+1}$ *(resp.,* Alg $\mathcal{L}_\infty \to$ Alg $\mathcal{L}_\infty$*) be an isomorphism such that* $\Phi(E_{ii}) = E_{ii}$ *for all* $i = 1, 2, \ldots, 2n+1$ $(i = 1, 2, \ldots)$, *and let* $\Phi(E_{ij}) = \alpha_{ij} E_{ij}$, $\alpha_{ij} \neq 0$, *for all* E_{ij} *in* Alg $\mathcal{L}_{2n+1}$ (Alg $\mathcal{L}_\infty$). *Then there exists a diagonal operator* T *such that* $\Phi(A) = TAT^{-1}$ *for all* A *in* Alg $\mathcal{L}_{2n+1}$ (Alg $\mathcal{L}_\infty$).

LEMMA 2.6 [5]. *Let* $\mathcal{L}_1$ *and* $\mathcal{L}_2$ *be commutative subspace lattices on Hilbert spaces* $\mathcal{H}_1$ *and* $\mathcal{H}_2$, *respectively, and suppose that* Φ: Alg $\mathcal{L}_1 \to$ Alg $\mathcal{L}_2$ *is an algebraic isomorphism. Let* $\mathcal{M}$ *be a maximal abelian self-adjoint subalgebra (masa) contained in* Alg $\mathcal{L}_1$. *Then there exists a bounded invertible operator* Y: $\mathcal{H}_1 \to \mathcal{H}_2$ *and an automorphism* ρ: Alg $\mathcal{L}_1 \to$ Alg $\mathcal{L}_2$ *such that*

(i) $\rho(M) = M$ *for all* M *in* $\mathcal{M}$ *and*
(ii) $\Phi(A) = Y\rho(A)Y^{-1}$ *for all* A *in* Alg $\mathcal{L}_1$.

THEOREM 2.7. *Let* Φ: Alg $\mathcal{L}_{2n} \to$ Alg $\mathcal{L}_{2n}$ *be an isomorphism. Then there exists an invertible operator* T *such that* $\Phi(A) = TAT^{-1}$ *for all* A *in* Alg $\mathcal{L}_{2n}$.

Proof. Since (Alg $\mathcal{L}_{2n}$)$\cap$(Alg $\mathcal{L}_{2n}$)* is a masa of Alg $\mathcal{L}_{2n}$ and since E_{ii} is in (Alg $\mathcal{L}_{2n}$)$\cap$(Alg $\mathcal{L}_{2n}$)* for all $i = 1, 2, \ldots, 2n$, by Lemma 2.6 there exist an invertible operator Y in $\mathcal{B}(\mathcal{H})$ and an isomorphism ρ: Alg $\mathcal{L}_{2n} \to$ Alg $\mathcal{L}_{2n}$ such that $\rho(E_{ii}) = E_{ii}$ and $\Phi(A) = Y\rho(A)Y^{-1}$ for all $i = 1, 2, \ldots, 2n$. By Theorem 2.4, $\rho(A) = SAS^{-1}$ for some invertible operator S. Hence

$$\Phi(A) = Y\rho(A)Y^{-1} = YSAS^{-1}Y^{-1}.$$

Let $T = YS$. Then $\Phi(A) = TAT^{-1}$ for all A in Alg $\mathcal{L}_{2n}$. □

With the same proof as Theorem 2.7, we have the following theorem.

THEOREM 2.8. *Let* Φ: Alg $\mathcal{L}_{2n+1} \to$ Alg $\mathcal{L}_{2n+1}$ *(resp.,* Alg $\mathcal{L}_\infty \to$ Alg $\mathcal{L}_\infty$*) be an isomorphism. Then there exists an invertible operator* T *from* $\mathcal{H}$ *onto* $\mathcal{H}$ *such that* $\Phi(A) = TAT^{-1}$ *for all* A *in* Alg $\mathcal{L}_{2n+1}$ (Alg $\mathcal{L}_\infty$).

THEOREM 2.9. *Let* Φ: Alg $\mathcal{L}_{2n} \to$ Alg $\mathcal{L}_{2n}$ *be an isomorphism. Then there exists an invertible operator* T *in* Alg $\mathcal{L}_{2n}$, *all of whose diagonal components are nonzero, such that* $\Phi(A) = TAT^{-1}$ *for all* A *in* Alg $\mathcal{L}_{2n}$.

Proof. Let Φ: Alg $\mathcal{L}_{2n} \to$ Alg $\mathcal{L}_{2n}$ be an isomorphism. By Theorem 2.7, there exists an invertible operator T such that $\Phi(A) = TAT^{-1}$ for all A in Alg $\mathcal{L}_{2n}$. Let $A = (a_{ij})$ and $\Phi(A) = (b_{ij})$ be in Alg $\mathcal{L}_{2n}$, and let $T = (t_{ij})$. Then

$$(*) \qquad \Phi(A)T = TA.$$

From equation (*) we have the following.

(2-1) $t_{2i,2j-1} = 0$ for all i and $j = 1, 2, \ldots, n$.

(2-2) If $t_{2i,2j} \neq 0$, then

- (1) $a_{2j,2j} = b_{2i,2i}$ for all i and $j = 1, 2, \ldots, n$,
- (2) $t_{2i,m} = 0$ for all m such that $m \neq 2j$, and
- (3) $t_{2k,2j} = 0$ for all k such that $k \neq i$.

(2-3) If $t_{2i-1,2j-1} \neq 0$, then

- (1) $a_{2j-1,2j-1} = b_{2i-1,2i-1}$ for all i and $j = 1, 2, \ldots, n$,
- (2) $t_{m,2j-1} = 0$ for all m such that $m \neq 2i-1$, and
- (3) $t_{2i-1,2k-1} = 0$ for all k such that $k \neq j$.

We will show that

(2-4) if $t_{11} \neq 0$, then T is in Alg $\mathcal{L}_{2n}$.

It is easy to check that if $t_{11} \neq 0$ then $t_{kk} \neq 0$ for all $k = 1, 2, \ldots, 2n$. Let $t_{ii} \neq 0$ for all $i = 1, 2, \ldots, 2n$. Then $t_{1,2j} = 0$ for $j = 2, 3, \ldots, n$, $t_{2k-1,2n} = 0$ for $k = 2, 3, \ldots, n-1$, and $t_{2k-1,2i} = 0$ for $k \neq i$ and $k \neq i+1$ ($k = 2, 3, \ldots, n$; $i = 1, 2, \ldots, n-1$). Thus T belongs to Alg $\mathcal{L}_{2n}$.

Finally, we show that $t_{11} \neq 0$. It is easily verified that

(2-5) (1) $t_{2i-1,1}$ and $t_{2i-2,2}$ cannot both be nonzero, and

(2) $t_{2i-1,1}$ and $t_{2i,2}$ cannot both be nonzero.

Now suppose that $t_{11} = 0$. Then $t_{2i-1,1} \neq 0$ for some i ($i = 2, 3, \ldots, n$). Suppose that $t_{2i-2,2} = 0$ and $t_{2i,2} = 0$. Comparing the $(2i-1, 2)$-component of $\Phi(A)T$ with that of TA, we have $t_{2i-1,2}(a_{11} - a_{22}) = t_{2i-1,1}(a_{12})$ which is a contradiction. Thus either $t_{2i-2,2} \neq 0$ or $t_{2i,2} \neq 0$. But this contradicts (2-5), and therefore $t_{11} \neq 0$. □

THEOREM 2.10. *Let* Φ: Alg $\mathcal{L}_{2n+1} \to$ Alg $\mathcal{L}_{2n+1}$ *be an isomorphism. Then there exists an invertible operator* S *in* Alg $\mathcal{L}_{2n+1}$ *whose diagonal components are all nonzero and such that either*

$$\Phi(A) = SAS^{-1} \quad \text{or} \quad \Phi(A) = SUAUS^{-1},$$

where U *is a* $(2n+1) \times (2n+1)$ *matrix whose* $(k, 2n-k+2)$*-component is* 1 *for* $k = 1, 2, \ldots, 2n+1$, *and all other entries are* 0.

Proof. Let Φ: Alg $\mathcal{L}_{2n+1} \to$ Alg $\mathcal{L}_{2n+1}$ be an isomorphism. By Theorem 2.8, there exists an invertible operator T such that $\Phi(A) = TAT^{-1}$ for all A in Alg $\mathcal{L}_{2n+1}$. Let $A = (a_{ij})$ and $\Phi(A) = (b_{ij})$ be in Alg $\mathcal{L}_{2n+1}$, and let $T = (t_{ij})$. Then $\Phi(A)T = TA$. From this equation we have the following:

(2-1)′ $t_{2i,2j-1} = 0$ for all $i = 1, 2, \ldots, n$ and $j = 1, 2, \ldots, n+1$.

(2-2)′ If $t_{2i,2j} \neq 0$, then

- (1) $a_{2j,2j} = b_{2i,2i}$ for all i and $j = 1, 2, \ldots, n$,

(2) $t_{2i,m}=0$ for all m such that $m\neq 2j$, and
(3) $t_{2k,2j}=0$ for all k such that $k\neq i$.

(2-3)′ If $t_{2i-1,2j-1}\neq 0$, then
(1) $a_{2j-1,2j-1}=b_{2i-2,2i-1}$ for all i and $j=1,2,\dots,n+1$,
(2) $t_{m,2j-1}=0$ for all m such that $m\neq 2i-1$, and
(3) $t_{2i-1,2k-1}=0$ for all k such that $k\neq j$.

If $t_{11}\neq 0$, then with the same proof as (2-4) T belongs to Alg $\mathcal{L}_{2n+1}$. In this case, we can take $S=T$. Let U be a $(2n+1)\times(2n+1)$ matrix whose $(k,2n-k+2)$-component is 1 for $k=1,2,\dots,2n+1$, and all other entries are 0. Then the mapping Φ_1: Alg $\mathcal{L}_{2n+1}\to$ Alg $\mathcal{L}_{2n+1}$ defined by $\Phi_1(A)=UAU^{-1}$ is an isomorphism. So $\Phi_1\circ\Phi(A)=(UT)A(UT)^{-1}$ and $t_{2n+1,1}$ is the $(1,1)$-component of UT. If $t_{2n+1,1}\neq 0$, then with the same proof as (2-4) UT belongs to Alg $\mathcal{L}_{2n+1}$. In this case, we can take $S=TU$. Since $U^2=I$, $S=U(UT)U$ and so S belongs to Alg $\mathcal{L}_{2n+1}$ and $T=SU$. Hence $\Phi(A)=TAT^{-1}=SUAUS^{-1}$ for all A in Alg $\mathcal{L}_{2n+1}$.

Finally, we show that t_{11} and $t_{2n+1,1}$ cannot both be zero. Suppose that $t_{11}=0$ and $t_{2n+1,1}=0$. Then $t_{2i-1,1}\neq 0$ for some i $(i=2,3,\dots,n)$. Thus, either $t_{2i-2,2}\neq 0$ or $t_{2i,2}\neq 0$. By (2-5)(1), $t_{2i-1,1}$ and $t_{2i-2,2}$ cannot be nonzero at the same time. If $t_{2i-1,1}\neq 0$ and $t_{2i,2}\neq 0$, then $t_{2i+m,m+2}\neq 0$ for all $m=1,2,\dots,2n-2i+1$. Comparing the $(2n+1,\ 2n-2i+4)$-component of $\Phi(A)T$ with that of TA, we have

$$t_{2n+1,2n-2i+4}(a_{2n-2i+3,2n-2i+3}-a_{2n-2i+4,2n-2i+4}) = t_{2n+1,2n-2i+3}a_{2n-2i+3,2n-2i+4},$$

which contradicts. Thus either $t_{11}\neq 0$ or $t_{2n+1,1}\neq 0$. □

Let $\mathcal{L}_1$ and $\mathcal{L}_2$ be commutative subspace lattices. An isomorphism φ from Alg $\mathcal{L}_1$ onto Alg $\mathcal{L}_2$ is said to be quasi-spatial if there exists a one-to-one operator T with a dense domain $\mathfrak{D}$, which is an invariant linear manifold for Alg $\mathcal{L}_1$, such that $\varphi(A)Tx=TAx$ for all A in Alg $\mathcal{L}_1$ and x in $\mathfrak{D}$. Isomorphisms Φ: Alg $\mathcal{L}_\infty\to$ Alg $\mathcal{L}_\infty$ need not be spatially implemented.

EXAMPLE 2.11. Consider the mapping Φ: Alg $\mathcal{L}_\infty\to$ Alg $\mathcal{L}_\infty$ defined by $\Phi(A)=TAT^{-1}$ for all A in Alg $\mathcal{L}_\infty$, where T is the infinite diagonal matrix whose (k,k)-component is k for all positive integers k. It is straightforward to show that Φ is an isomorphism and that no bounded operator can implement Φ.

THEOREM 2.12. *Let Φ: Alg $\mathcal{L}_\infty\to$ Alg $\mathcal{L}_\infty$ be an isomorphism. Then there exists an invertible matrix T all of whose entries are 0 except for the (i,i)-component, the $(2i-1,2i)$-component, and the $(2i+1,2i)$-component, for all positive integers i such that $\Phi(A)=TAT^{-1}$ for all A in* Alg $\mathcal{L}_\infty$.

Proof. Let Φ: Alg $\mathcal{L}_\infty\to$ Alg $\mathcal{L}_\infty$ be an isomorphism. By Theorem 2.8 there exists an invertible operator T from $\mathcal{H}$ onto $\mathcal{H}$ such that $\Phi(A)=TAT^{-1}$ for all A in Alg $\mathcal{L}_\infty$. Let $T=(t_{ij})$ and let $A=(a_{ij})$ and $\Phi(A)=(b_{ij})$ be in Alg $\mathcal{L}_\infty$. Then $\Phi(A)T=TA$. From this equation we have the following:

(2-1)″ $t_{2i,2j-1}=0$ for all positive integers i and j.
(2-2)″ If $t_{2i,2j}\neq 0$, then
(1) $a_{2j,2j}=b_{2i,2i}$ for all positive integers i and j,
(2) $t_{2i,m}=0$ for all positive integers m such that $m\neq 2j$, and
(3) $t_{2k,2j}=0$ for all positive integers k such that $k\neq i$.
(2-3)″ If $t_{2i-1,2j-1}\neq 0$, then
(1) $a_{2j-1,2j-1}=b_{2i-1,2i-1}$ for all positive integers i and j,
(2) $t_{2i-1,2k-1}=0$ for all positive integers k such that $k\neq j$, and
(3) $t_{m,2j-1}=0$ for all positive integers m such that $m\neq 2i-1$.

If $t_{11}\neq 0$, then with the same proof as that of (2.4) we have:

(2-6) $T=(t_{ij})$ is an infinite matrix all of whose entries are 0 except for the (i,i)-component, the $(2i-1,2i)$-component, and the$(2i+1,2i)$-component, for all positive integers i.

For the proof of this theorem, it is sufficient to show that $t_{11}\neq 0$. Suppose that $t_{11}=0$; then $t_{2i-1,1}\neq 0$ for some i $(i=2,3,\dots)$. So either $t_{2i-2,2}\neq 0$ or $t_{2i,2}\neq 0$. However, by (2-5)(1), $t_{2i-2,1}$ and $t_{2i-2,2}$ cannot become nonzero at the same time. If $t_{2i-1,1}\neq 0$ and $t_{2i,2}\neq 0$, then $t_{2i+m,m+2}\neq 0$ for all positive integers m. Hence $t_{2,2j}=0$ for all positive integers j by (2-2)″. As $t_{2,2j-1}=0$ for all positive integers j by (2-1)″, we have $t_{2,m}=0$ for all positive integers m. Thus T is not invertible, and therefore $t_{11}\neq 0$. □

THEOREM 2.13. *Let* $T=(t_{ij})$ *be an invertible operator of the form* (2-6). *Then* TAT^{-1} *is in* Alg $\mathcal{L}_\infty$ *for all* A *in* Alg $\mathcal{L}_\infty$ *if and only if*

$$\sup\{|t_{2k,2k}^{-1}t_{2k-1,2k-1}|, |t_{2k,2k}^{-1}t_{2k+1,2k+1}|, |t_{2k,2k}^{-1}t_{2k-1,2k}|, |t_{2k,2k}^{-1}t_{2k+1,2k}| : k=1,2,\dots\}<\infty.$$

Proof. Suppose that T is an invertible operator of the form (2-6). Let A_1 be an invertible matrix whose $(2k-1,2k)$-component is 1 for all positive integers k and all other entries are 0. Then A_1 is in Alg $\mathcal{L}_\infty$ and so TA_1T^{-1} belongs to Alg $\mathcal{L}_\infty$. Since TA_1T^{-1} is a matrix whose $(2k-1,2k)$-component is $t_{2k,2k}^{-1}t_{2k-1,2k-1}$ for all positive integers k and all other entries are 0, we have

$$\sup\{|t_{2k,2k}^{-1}t_{2k-1,2k-1}| : k=1,2,\dots\}<\infty.$$

Let A_2 be an infinite matrix whose $(2k+1,2k)$-component is 1 for all positive integers k and all other entries are 0. Then A_2 is in Alg $\mathcal{L}_\infty$ and so TA_2T^{-1} belongs to Alg $\mathcal{L}_\infty$. Since TA_2T^{-1} is a matrix whose $(2k+1,2k)$-component is $t_{2k,2k}^{-1}t_{2k+1,2k+1}$ for all positive integers k and all other entries are 0, we have

$$\sup\{|t_{2k,2k}^{-1}t_{2k+1,2k+1}| : k=1,2,\dots\}<\infty.$$

Let A_3 be a diagonal operator whose $(2k-1,2k-1)$-component is 1 and $(2k,2k)$-component is 2 for all positive integers k. Then A_3 is in Alg $\mathcal{L}_\infty$ and so TA_3T^{-1} belongs to Alg $\mathcal{L}_\infty$. Since TA_3T^{-1} is the matrix whose

(1) $(2k,2k)$-component is 2,
(2) $(2k-1,2k-1)$-component is 1,

(3) $(2k-1, 2k)$-component is $t_{2k,2k}^{-1} t_{2k-1,2k}$,
(4) $(2k+1, 2k)$-component is $t_{2k,2k}^{-1} t_{2k+1,2k}$, and
(5) all other entries are 0 for all positive integers k.

We have

$$\sup\{|t_{2k,2k}^{-1} t_{2k-1,2k}|, |t_{2k,2k}^{-1} t_{2k+1,2k}| : k = 1, 2, \ldots\} < \infty.$$

Thus,

$$\sup\{|t_{2k,2k}^{-1} t_{2k-1,2k-1}|, |t_{2k,2k}^{-1} t_{2k+1,2k+1}|, |t_{2k,2k}^{-1} t_{2k-1,2k}|, |t_{2k,2k}^{-1} t_{2k+1,2k}| : k = 1, 2, \ldots\} < \infty.$$

Conversely, let $A = (a_{ij})$ be in Alg $\mathcal{L}_\infty$. TAT^{-1} is the matrix whose

(1) (k, k)-component is a_{kk},
(2) $(2k-1, 2k)$-component is
$t_{2k,2k}^{-1} t_{2k-1,2k}(a_{2k,2k} - a_{2k-1,2k-1}) + t_{2k,2k}^{-1} t_{2k-1,2k-1} a_{2k-1,2k}$,
(3) $(2k+1, 2k)$-component is
$t_{2k,2k}^{-1} t_{2k+1,2k}(a_{2k,2k} - a_{2k+1,2k+1}) + t_{2k,2k}^{-1} t_{2k+1,2k+1} a_{2k+1,2k}$,
and
(4) all other components are 0 for all positive integers k.

Let B_1 be the diagonal operator whose (k, k)-component is a_{kk} for all positive integers k. Let B_2 be the matrix whose $(2k-1, 2k)$-component is

$$t_{2k,2k}^{-1} t_{2k-1,2k}(a_{2k,2k} - a_{2k-1,2k-1})$$

for all positive integers k and all other entries are 0. Let B_3 be the matrix whose $(2k-1, 2k)$-component is

$$t_{2k,2k}^{-1} t_{2k-1,2k-1} a_{2k-1,2k}$$

for all positive integers k and all other entries are 0. Let B_4 be the matrix whose $(2k+1, 2k)$-component is

$$t_{2k,2k}^{-1} t_{2k+1,2k}(a_{2k,2k} - a_{2k+1,2k+1})$$

for all positive integers k and all other entries are 0. Let B_5 be the matrix whose $(2k+1, 2k)$-component is

$$t_{2k,2k}^{-1} t_{2k+1,2k+1} a_{2k+1,2k}$$

for all positive integers k and all other entries are 0. Then $TAT^{-1} = B_1 + B_2 + B_3 + B_4 + B_5$.

By the hypothesis,

$$\sup\{|t_{2k,2k}^{-1} t_{2k-1,2k-1}|, |t_{2k,2k}^{-1} t_{2k+1,2k+1}|, |t_{2k,2k}^{-1} t_{2k-1,2k}|, |t_{2k,2k}^{-1} t_{2k+1,2k}| : k = 1, 2, \ldots\} < \infty.$$

Because

$$\sup\{|a_{2k,2k} - a_{2k-1,2k-1}|, |a_{2k,2k} - a_{2k+1,2k+1}| : k = 1, 2, \ldots\} < \infty,$$

we have that B_1, B_2, B_3, B_4, and B_5 belong to Alg $\mathcal{L}_\infty$. Thus TAT^{-1} belongs to Alg $\mathcal{L}_\infty$. □

THEOREM 2.14. *A map* Φ: Alg $\mathcal{L}_\infty \to$ Alg $\mathcal{L}_\infty$ *is an isomorphism if and only if there exists an invertible operator (not necessarily bounded)* $T=(t_{ij})$ *of the form* (2-6) *satisfying*

$$\sup\{|t_{ii}^{-1}t_{jk}|: |i-j|\le 1, |j-k|\le 1, |k-i|\le 1 \text{ for all positive integers } i, j, \text{ and } k\}<\infty$$

such that $\Phi(A)=TAT^{-1}$ *for all* A *in* Alg $\mathcal{L}_\infty$.

Proof. Let Φ: Alg $\mathcal{L}_\infty \to$ Alg $\mathcal{L}_\infty$ be an isomorphism. Then, by Theorem 2.12, there exists an invertible operator $T=(t_{ij})$ of the form (2-6) such that $\Phi(A)=TAT^{-1}$ for all A in Alg $\mathcal{L}_\infty$. By Theorem 2.13,

$$\sup\{|{}^{-1}_{2k,2k}t_{2k-1,2k-1}|, |t_{2k,2k}^{-1}t_{2k+1,2k+1}|, |t_{2k,2k}^{-1}t_{2k-1,2k}|, |t_{2k,2k}^{-1}t_{2k+1,2k}|: k=1,2,\ldots\}<\infty.$$

Since Φ is surjective, $T^{-1}AT$ is in Alg $\mathcal{L}_\infty$ for all A in Alg $\mathcal{L}_\infty$. Since T^{-1} is the matrix whose

(1) (k,k)-component is t_{kk}^{-1},
(2) $(2k-1,2k)$-component is $-(t_{2k-1,2k}/t_{2k-1,2k-1}t_{2k,2k})$,
(3) $(2k+1,2k)$-component is $-(t_{2k+1,2k}/t_{2k+1,2k+1}t_{2k,2k})$, and
(4) all other components are 0 for all positive integers k,

by Theorem 2.13 we have

$$\sup\{|t_{2k+1,2k+1}^{-1}t_{2k+2,2k+2}|, |t_{2k+1,2k+1}^{-1}t_{2k+1,2k+2}|, |t_{2k+1,2k+1}^{-1}t_{2k+1,2k}|, |t_{2k+1,2k+1}^{-1}t_{2k,2k}|: k=1,2,\ldots\}<\infty.$$

Conversely, suppose that $T=(t_{ij})$ has the form (2-6) and that

$$\sup\{|t_{ii}^{-1}t_{jk}|: |i-j|\le 1, |j-k|\le 1 \text{ and } |k-i|\le 1 \text{ for all positive integers } i, j, \text{ and } k\}<\infty.$$

Define Φ: Alg $\mathcal{L}_\infty \to$ Alg $\mathcal{L}_\infty$ by $\Phi(A)=TAT^{-1}$ for all A in Alg $\mathcal{L}_\infty$. Then Φ is well defined and $T^{-1}AT$ is in Alg $\mathcal{L}_\infty$ for all A in Alg $\mathcal{L}_\infty$, by Theorem 2.13. It is clear that Φ is an isomorphism. □

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