Restriction to Transverse Curves of Some Spaces of Functions in the Unit Ball

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Introduction

Let B denote the unit ball in C^n, and S its boundary. Let $h^\infty(B)$ denote the space of bounded pluriharmonic functions in B, and let $H^\infty(B)$ be the subspace of $h^\infty(B)$ of holomorphic functions. Also, for $\alpha < 1$, we consider the algebra $\mathrm{Lip}_\alpha(B)$ of holomorphic functions in B, satisfying a Lipschitz condition of order α with respect to the Euclidean metric.

In this paper we deal with restrictions of these spaces to closed curves $\Gamma \subset S$. Here we will summarize the main results of the paper and introduce at the same time some of the required notations.

We will work with a simple (without intersections) periodic transverse curve, $\gamma: R \to S$ of class C^1. Recall that a curve is transverse if, for every t in R, $\gamma'(t)$ does not lie in the complex-tangent space $P_{\gamma(t)}$ at the same point. Analytically this condition is equivalent to the relation $\mathrm{Im}\, \gamma'(t)\overline{\gamma(t)} \neq 0$ (whereas $\mathrm{Re}\, \gamma'(t)\overline{\gamma(t)} = 0$, simply because γ is on S). By choosing the reparametrization $s(t) = \int_a^t |\mathrm{Im}\, \gamma'(x)\overline{\gamma(x)}|\, dx$, $a \le t \le b$, where a and b satisfy $\gamma(a) = \gamma(b)$, we obtain a parametrization such that $\gamma'(t)\overline{\gamma(t)} = i$. With an appropriate dilation, we will suppose from now on that the curve is 2π-periodic, and there exists $\lambda > 0$ such that, for all t, $\gamma'(t)\overline{\gamma(t)} = \lambda i$. In the following we will write I for $[-\pi, \pi]$, and $\Gamma = \gamma([-\pi, \pi])$.

We also consider the Koranyi pseudodistance $d(z, w) = |1 - \bar{z}w|$, where $zw = \sum_i z_i w_i$. This defines a pseudodistance only on S, but we will consider it defined as well when one of the two variables is not in S.

In one complex variable, Fatou's theorem gives sense to the space $h^\infty{}_{|T}$ of boundary values of bounded harmonic functions, and the use of the Poisson transform shows that this space equals $L^\infty(T)$.

In several complex variables, a result of Nagel, Rudin, and Wainger (see [5] and [6]) states a Fatou type theorem implying the existence at almost every point of a C^1 transverse curve of the radial limit of a bounded holomorphic function; in fact, it proves the existence of a stronger kind of limit, the restricted K-limit. It is easy to see that the Nagel–Rudin–Wainger theorem holds for bounded pluriharmonic functions so that the space $h^\infty{}_{|\Gamma}$ is

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well defined. On the other hand, in [1] a pluriharmonic Poisson kernel related to transverse curves is introduced. This kernel has in a certain sense the same behavior as the classical Poisson kernel in one complex variable. These two facts will be used in Section 1 to prove the following theorem.

THEOREM A. *The space* $h^\infty{}_{|\Gamma}$ *has finite codimension in the space of bounded functions in* Γ.

In Section 1 we introduce the auxiliary results needed for the following sections. Most of the results of that section are in [1], but the curves considered there are more regular than the curves we are dealing with, so it will be necessary to prove the assertions for our more general class of curves.

In Section 2 we study spaces of holomorphic functions. In one complex variable it is well known that the space $\operatorname{Re} H^\infty{}_{|T}$, consisting of boundary values of real parts of bounded holomorphic functions in D, is the space of real bounded functions on T with bounded conjugate. We will see that, in general, we have Theorem B.

THEOREM B. *The space* $\operatorname{Re} H^\infty{}_{|\Gamma}$ *has finite codimension in the space of real bounded functions in* Γ *with bounded conjugate.*

We will also establish a similar characterization of the space of restrictions to the curve of real parts of Lipschitz functions, showing (provided γ is of class C^2) the next theorem.

THEOREM C. *The space* $\operatorname{Re}\operatorname{Lip}_{\alpha|\Gamma}$, *where* $\alpha < 1$, *has finite codimension in the space of real functions in* Γ *that satisfy a Lipschitz condition of order* α.

As final remarks on notation, we denote by C an arbitrary constant that may change from one occurrence to another, and we write $x \ll y$ or $x = O(y)$ if there exists $M > 0$ such that $x \leq My$ and $x \cong y$ if $x \ll y$ and $y \ll x$. The conjugate of a function f on $[-\pi, \pi]$ will be denoted by $\tilde{f}$.

The theorem of Nagel, Rudin, and Wainger [6] holds not only for curves of class C^1, but for curves which are only continuous and rectifiable. A natural question that remains to be studied is whether or not our results extend to this more general class of curves.

1. Preliminary results. The space $h^\infty{}_{|\Gamma}$

As we have said in the summary, in this section we will collect the auxiliary results and prove Theorem A. First we give another proof of the Nagel–Rudin–Wainger theorem in order to obtain more information about the boundary values of a bounded holomorphic function on a transverse curve. We will need some definitions that can be found in [7].

DEFINITION. For $\xi \in S$, a ξ-curve is a continuous map $\sigma\colon [0,1) \to B$ such that $\lim_{t\to 1} \sigma(t) = \xi$. A ξ-curve is *special* if it satisfies

$$\lim_{t\to 1} \frac{|\sigma(t)-(\sigma(t)\bar{\xi})\xi|^2}{1-|\sigma(t)\bar{\xi}|^2}=0,$$

and it is *restricted* if it satisfies

$$\frac{|\sigma(t)\bar{\xi}-1|}{1-|\sigma(t)\bar{\xi}|}=O(1) \quad \text{if } 0\le t<1.$$

A function $f: B\to C$ has *restricted* K-limit L at ξ if $\lim_{t\to 1} f(\sigma(t))=L$ for every restricted ξ-curve.

THEOREM 1 [5; 6]. *Let $\gamma:[-\pi,\pi]\to C$ be a simple periodic transverse curve of class C^1. Then:* (a) *If $f\in H^\infty(B)$ then the restricted K-limit of f at $\gamma(t)$ exists a.e. $t\in I$. Denoting this restricted K-limit by f^*, $\widetilde{\operatorname{Re} f^*}$ is bounded* in I. (b) *If $f\in h^\infty(B)$ then the restricted K-limit of f at $\gamma(t)$ exists a.e. t in I.*

Proof of Theorem 1. For part (a), given $f\in H^\infty(B)$, let us consider (as in [5]) the composition $F=f\circ\Phi$, where Φ is the "quasianalytic" disc constructed there. That is, $\Phi: D\to B$ is a C^1 function in D satisfying:

(i) $\Phi(e^{it})=\gamma(t)$ for $t\in I$;
(ii) $\Phi(z)\in B$ if $z\in D$; and
(iii) for every $t\in I$, the curve Γ_t defined by $\Gamma_t(r)=\Phi(re^{it})$ is a "special" approach curve to the point $\gamma(t)\in S$.

The function F satisfies:

(iv) $|\bar{\partial}F(z)|=O((1-|z|)^{-1/2})$.

What we are going to do now is to "correct" F, in order to obtain a bounded holomorphic function in D. We need the following.

THEOREM 2. *Let f be a C^1 function on D such that $|f(z)|=O((1-|z|)^{\alpha-1})$, where $0<\alpha<1$. Then there exists a function $u\in \operatorname{Lip}_\alpha(D)$ such that $\bar{\partial}u(z)=f(z)$ for all z in D.*

Proof of Theorem 2. Let us introduce the operator given by

$$T_1 f(z)=\frac{1}{2\pi i}\int_D \frac{1-|\xi|^2}{1-\bar{\xi}z}\,\frac{f(\xi)}{\xi-z}\,dm(\xi).$$

As is well known, this operator also solves the $\bar{\partial}$-equation, and we will prove that

(1) $$u(z)=T_1 f(z)$$

is in $\operatorname{Lip}_\alpha(D)$.

Let z_1, z_2 be in D and let $\delta=|z_1-z_2|$. From the equality

$$\left\{\frac{1}{(1-\bar{\xi}z_1)(\xi-z_1)}-\frac{1}{(1-\bar{\xi}z_2)(\xi-z_2)}\right\}=\frac{(z_1-z_2)(1+|\xi|^2-\bar{\xi}(z_1+z_2))}{(1-\bar{\xi}z_1)(\xi-z_2)(1-\bar{\xi}z_2)(\xi-z_2)}$$

and from the definition of T_1, we deduce that it is enough to show that

$$(2)\qquad \left|\int_D \frac{(1-|\xi|^2)(1+|\xi|^2-\bar\xi(z_1+z_2))}{(1-\bar\xi z_1)(\xi-z_1)(1-\bar\xi z_2)(\xi-z_2)} f(\xi)\,dm(\xi)\right| = O(\delta^{\alpha-1}).$$

Since

$$1+|\xi|^2-\bar\xi(z_1+z_2) = 1-|\xi|^2+\bar\xi(\xi-z_1)+\bar\xi(\xi-z_2),$$

(2) will be deduced from

$$(3)\qquad \int_D \frac{(1-|\xi|^2)^2|f(\xi)|}{|1-\bar\xi z_1||\xi-z_1||1-\bar\xi z_2||\xi-z_2|}\,dm(\xi) = O(\delta^{\alpha-1}),$$

$$(4)\qquad \int_D \frac{(1-|\xi|^2)|f(\xi)|}{|1-\bar\xi z_1||1-\bar\xi z_2||\xi-z_2|}\,dm(\xi) = O(\delta^{\alpha-1}),$$

and

$$(5)\qquad \int_D \frac{(1-|\xi|^2)|f(\xi)|}{|1-\bar\xi z_1||\xi-z_1||1-\bar\xi z_2|}\,dm(\xi) = O(\delta^{\alpha-1}).$$

Before proving these estimates, we need a technical lemma about the Koranyi pseudodistance.

LEMMA 1. *Let $z, \xi \in D$. Then*

$$(6)\qquad |1-\bar\xi z| \cong (1-|\xi|)+|z-\xi|.$$

Proof of Lemma 1. The upper estimate is obvious. For the estimate from below, since $|1-\bar\xi z| \ge 1-|\xi|$ the lemma will follow if we prove (6) in the case $|z-\xi| \ge C(1-|\xi|)$ for some C. From the inequality

$$|1-\bar\xi z| \ge |\xi-z||\xi|-(1-|\xi|^2),$$

we have that if $C \ge \frac12$ and $|\xi| \ge \frac12$, then $|1-\bar\xi z| \ge \frac12|\xi-z|-(2/C)|\xi-z| \ge \frac14|\xi-z|+(1-|\xi|)$. Finally, if $|\xi| \le \frac12$, then both quantities of (6) do not vanish, and (6) is then obvious. □

Following with the proof of Theorem 2, we will prove (3) first. Lemma 1 and the hypothesis on the function f give that the numerator is bounded by

$$(1-|\xi|^2)^2|f(\xi)| \ll (1-|\xi|)^{\alpha+1} \ll |1-\bar\xi z_1|^{(\alpha+1)/2}|1-\bar\xi z_2|^{(\alpha+1)/2}.$$

Hence (3) will be a consequence of

$$(7)\qquad \int_D \frac{dm(\xi)}{|1-\bar\xi z_1|^{(1-\alpha)/2}|1-\bar\xi z_2|^{(1-\alpha)/2}|\xi-z_1||\xi-z_2|} = O(\delta^{\alpha-1}),$$

and applying Lemma 1, the estimate (7) will follow from

$$(8)\qquad \int_{\mathbf{C}} \frac{dm(\xi)}{|\xi-z_1|^{(3-\alpha)/2}|\xi-z_2|^{(3-\alpha)/2}} = O(\delta^{\alpha-1}).$$

Now we decompose the integral of (8) in the three regions that appear when we consider the discs $D(z_i, \delta/2)$, $i=1,2$, and we will see that each of the resulting integrals satisfies an estimate like (8).

If $\xi \in D(z_i, \delta/2)$, then $|\xi - z_j| \geq \delta/2$ for $j \neq i$ and $i = 1, 2$, and consequently a polar change of coordinates gives

$$(9)\qquad \int_{D(z_i,\delta/2)} \frac{dm(\xi)}{|\xi - z_1|^{(3-\alpha)/2}|\xi - z_2|^{(3-\alpha)/2}} \ll \delta^{(\alpha-3)/2}\int_0^\delta r^{(\alpha-1)/2}\,dr \ll \delta^{\alpha-1}, \quad i = 1, 2.$$

If $\xi \notin D(z_1, \delta/2) \cup D(z_2, \delta/2)$, then $|\zeta - z_1| \cong |\zeta - z_2|$, and a polar change of coordinates gives

$$(10)\qquad \int_{\mathbf{C}\setminus D(z_1,\delta/2)\cup D(z_2,\delta/2)} \frac{dm(\xi)}{|\xi - z_1|^{(3-\alpha)/2}|\xi - z_2|^{(3-\alpha)/2}} \ll \int_\delta^{+\infty} r^{\alpha-2}\,dr \ll \delta^{\alpha-1}.$$

Finally, from (9) and (10) we get (8), and hence (3). The other estimates follow in a similar way. □

Applying Theorem 2 to the function $\bar{\partial}F$, we conclude that there exists a function u in $\mathrm{Lip}_{1/2}(D)$ such that $F - u$ is a bounded holomorphic function in D. By Fatou's theorem the $\lim_{t\to 1}(F(re^{it}) - u(re^{it}))$ exists a.e. t in I. Hence, $\lim_{r\to 1} F(re^{it}) = \lim_{r\to 1} f \circ \Phi(re^{it})$ exists a.e. t in I. Using property (iii) that is satisfied by the function Φ and Cirka's theorem (see [7, p. 171]) we get the existence a.e. t in I of the K-limit of the function f.

Since u is in $\mathrm{Lip}_{1/2}(D)$, $\tilde{u}$ verifies a Lipschitz estimate with exponent 1/2. Since $F - u$ has a bounded conjugate it follows that F has also a bounded conjugate.

For part (b), it is enough to prove the statement for real bounded pluriharmonic functions. Given such a function f, let g be such that $f + ig \in H(B)$. Then (b) will follow by applying (a) to the bounded holomorphic functions $\exp(f + ig)$ and $\exp[-(f + ig)]$. □

Before proceeding we need to introduce two pluriharmonic kernels. For γ a 2π-periodic simple transverse C^1 curve, let $P(z, t)$ be the pluriharmonic "Poisson" kernel (introduced by Bruna and Ortega [1]) given by

$$(11)\qquad P(z,t) = 2\,\mathrm{Re}\,\frac{1}{1 - \overline{\gamma(t)}z} - 1 = \frac{1 - |\overline{\gamma(t)}z|^2}{|1 - \overline{\gamma(t)}z|^2}, \quad z \in \bar{B}\setminus\Gamma,\ t \in I.$$

Similarly, we introduce the conjugate of P, $Q(z, t)$, given by

$$(12)\qquad Q(z,t) = 2\,\mathrm{Im}\,\frac{1}{1 - \overline{\gamma(t)}z} = \frac{2\,\mathrm{Im}\,\overline{\gamma(t)}z}{|1 - \overline{\gamma(t)}z|^2}, \quad z \in \bar{B}\setminus\Gamma,\ t \in I.$$

The following lemma of Chaumat–Chollet [2] is used in [1] to give an estimate of P. We state it without proof.

LEMMA 2 [2]. *Let Γ be a curve as above. There exists a neighbourhood U of Γ in $\mathbf{C}^n$, and a mapping $p: U \to \Gamma$ of class C^1, such that:*

(i) *$p(z)=z$ for $z\in\Gamma$, and $\operatorname{Im} p(z)\bar{z}=0$, provided $z\in\bar{B}\cap U$.*

(ii) *For $z\in\bar{B}\cap U$, let t_z be such that $p(z)=\gamma(t_z)$. Then, for t in I,*

$$|1-\overline{\gamma(t)}z|\cong|t-t_z|+|1-\overline{\gamma(t_z)}z|.$$

The following estimates, in case the curve is more regular, are in [1]. We state and prove them for C^1 curves.

LEMMA 3 [1]. *For $z\in U\cap B$ and $|t-t_z|\leq\pi$,*

(i)
$$P(z,t)=O\left(\frac{r_z+|t-t_z|^2}{r_z^2+|t-t_z|^2}\right),$$

where $r_z=d(p(z),z)$. If $I_z=\{t\,|\,|t-t_z|\leq d_z\}$, where $d_z=r_z^{1/2}$ and $t\in I_z$, then

$$P(z,t)\cong\frac{r_z}{r_z^2+|t-t_z|^2}.$$

(ii)
$$|Q(z,t)|=O\left(\frac{|t-t_z|}{r_z^2+|t-t_z|^2}\right).$$

Proof of Lemma 3. We will see that for fixed z in $U\cap B$, the function $f(t)=1-|\overline{\gamma(t)}z|^2$ is $O(r_z+|t-t_z|^2)$. We have that $f(t_z)\leq 2r_z$, and if we differentiate f and evaluate it in a point ξ between t and t_z, we get

$$\begin{aligned}-f'(\xi)&=2\operatorname{Re}[(\overline{\gamma(\xi)}z)(\gamma(\xi)\bar{z})]\\&=2[\operatorname{Re}(\overline{\gamma'(\xi)}z)\operatorname{Re}(\gamma(\xi)\bar{z})-\operatorname{Im}(\overline{\gamma'(\xi)}z)\operatorname{Im}(\gamma(\xi)\bar{z})]\\&=O(|z-\gamma(t_z)|+|\gamma(t_z)-\gamma(\xi)|+|\zeta-t_z|)=O(|z-\gamma(t_z)|+|t-t_z|),\end{aligned}$$

since, by construction, $\operatorname{Im}\gamma'(t_z)\bar{z}=0$ and $\operatorname{Re}\gamma'(\xi)\overline{\gamma(\xi)}=0$. Applying the mean value theorem we have $|f(t)-f(t_z)|\leq\sup_{\xi\in[t,t_z]}|f'(\xi)||t-t_z|$, and since $|z-\gamma(t_z)|\ll r_z^{1/2}$, we deduce that $|f(t)-f(t_z)|=O(r_z+|t-t_z|^2)$, and we get part (i).

For part (ii) it is enough to notice that $\operatorname{Im}\overline{\gamma(t)}z=\operatorname{Im}(\overline{\gamma(t)}-\overline{\gamma(t_z)})z=O(|t-t_z|)$. □

Note that P has in I_z the same growth as the classical Poisson kernel. On the other hand, $P(z,t)$ is bounded for $t\notin I_z$. So $P(z,t)$ is like an approximation of the identity, concentrated in I_z, plus a bounded perturbation. The following weak version of a lemma in [1] makes this assertion precise.

LEMMA 4 [1]. *Shrinking U if necessary, one has for $z\in U$*

$$\lim_{z\to\Gamma,\,z\in U\cap\bar{B}\setminus\Gamma}\frac{1}{2\pi}\int_{I_z}P(z,t)\,dt=\frac{1}{\lambda}.$$

Proof of Lemma 4. Since $P(z,t)=2\operatorname{Re}(1-\gamma(t)\bar{z})^{-1}-1$ and

$$\operatorname{Re}\lambda(1-\gamma(t)\bar{z})^{-1}=\operatorname{Im}[(\lambda i)(1-\gamma(t)\bar{z})^{-1}],$$

it is enough to show that

$$\lim_{z\to\Gamma,\, z\in U\cap\bar{B}\setminus\Gamma}\int_{I_z}\frac{\lambda i}{1-\gamma(t)\bar{z}}=\pi i.$$

From the fact that $\gamma'(t)\overline{\gamma(t)}=\lambda i$, we deduce that this last integral is equal to

$$\int_{I_z}\frac{\lambda i}{1-\gamma(t)\bar{z}}\,dt=\int_{I_z}\frac{\gamma'(t)\bar{z}}{1-\gamma(t)\bar{z}}\,dt+\int_{I_z}\frac{\gamma'(t)(\overline{\gamma(t)}-\bar{z})}{1-\gamma(t)\bar{z}}\,dt=\mathrm{I}+\mathrm{II},$$

and what we will see is that I has the limit we are looking for and that II converges to zero. Let us show the first of these assertions. We integrate by parts and, denoting by "log" the principal branch of the logarithm,

$$\int_{I_z}\frac{\gamma'(t)\bar{z}}{1-\gamma(t)\bar{z}}\,dt=\log(1-\gamma(t_z-d_z)\bar{z})-\log(1-\gamma(t_z+d_z)\bar{z}).$$

Writing the expression appearing on the right in the form

$$\log\frac{1-\gamma(t_z-d_z)\bar{z}}{d_z}-\log\frac{1-\gamma(t_z+d_z)\bar{z}}{d_z}$$

and using the Taylor development, we obtain

$$1-\gamma(t_z-d_z)\bar{z}=1-\gamma(t_z)\bar{z}+d_z\gamma'(t_z)\bar{z}+o(d_z)$$

and

$$1-\gamma(t_z+d_z)\bar{z}=1-\gamma(t_z)\bar{z}-d_z\gamma'(t_z)\bar{z}+o(d_z).$$

By (ii) of Lemma 2, $|1-\overline{\gamma(t_2\pm d_z)}z|\cong r_z=d_z^2$, and $\gamma'(t_z)\bar{z}\to i$ as z approaches Γ. Hence

$$\log\frac{1-\gamma(t_z-d_z)\bar{z}}{d_z}=\log\gamma'(t_z)\bar{z}+o(1),$$

$$\log\frac{1-\gamma(t_z+d_z)\bar{z}}{d_z}=\log[-\gamma'(t_z)\bar{z}]+o(1),$$

and

$$\log\frac{1-\gamma(t_z-d_z)\bar{z}}{d_z}-\log\frac{1-\gamma(t_z+d_z)\bar{z}}{d_z}-\pi i=o(1),$$

an estimate that shows that the limit of I is πi.

For integral II we use that $|\gamma(t)-z|\ll|1-\overline{\gamma(t)}z|^{1/2}$ and Lemma 2 to obtain

$$\left|\int_{I_z}\frac{\gamma(t)(\overline{\gamma(t)}-\bar{z})}{1-\gamma(t)\bar{z}}\,dt\right|\ll\int_{I_z}\frac{dt}{|t-t_z|^{1/2}},$$

an expression that converges to zero as $z\to\Gamma$. □

An immediate corollary to Lemmas 3 and 4 is the following.

LEMMA 5. *For* $z\in B\setminus\Gamma$,

$$\int_I P(z,t)\,dt=O(1).$$

For f a 2π-periodic function in $L^1(I)$, let us consider the pluriharmonic function in B given by

$$Pf(z)=\int_I P(z,t)f(t)\,dt, \quad z\in\bar{B}\backslash\Gamma.$$

Also, as in [1], we consider for $t\neq s$ the kernel

$$K(t,s)=P(\gamma(t),s)=\frac{1-|\overline{\gamma(s)}\gamma(t)|^2}{|1-\overline{\gamma(s)}\gamma(t)|^2},$$

which is bounded and of class C^1 outside the diagonal. So it makes sense to consider, for f in $L^1(I)$ and 2π-periodic, the function $T_K f$ given by

$$T_K f(t)=\frac{1}{2\pi}\int_I K(\gamma(t),s)f(s)\,ds.$$

Then the following result holds.

THEOREM 3 [1]. *Let f be a bounded 2π-periodic function on Γ, and let Pf and $T_K f$ be as before. Then:*

(i) *Pf is a bounded pluriharmonic function in B.*

(ii) *If f is continuous at t_0,*

$$\lim_{z\to\gamma(t_0),\, z\in\bar{B}\backslash\Gamma} Pf(z)=\frac{1}{\lambda}f(t_0)+T_K f(t_0).$$

(iii) *If, for f 2π-periodic in $L^1(I)$ and $\alpha>1$ we define*

$$M_\alpha Pf(t)=\sup\{|Pf(z)|,\, d(\gamma(t),z)\le\alpha d(\gamma(t_z),z)\},$$

then $M_\alpha Pf(t)\ll Mf(t)$, where Mf is the Hardy–Littlewood maximal function, and (ii) *also holds for a.e. t_0 provided the limit is taken within the admissible regions*

$$D_\alpha(t)=\{z\in U\cap B,\, d(\gamma(t),z)\le\alpha d(\gamma(t_z),z)\}.$$

Proof of Theorem 3. The first of the assertions is a consequence of the pluriharmonicity of the kernel P, and of Lemma 5. (ii) follows as in [1], substituting Lemma 4 for Lemma 3.3. For the maximal inequality in (iii), let z be a point in $D_\alpha(t)$. Then

$$|Pf(z)|\le\int_I P(z,s)|f(s)|\,ds=\int_{I_z} P(z,s)|f(s)|\,ds+\int_{I\backslash I_z} P(z,s)|f(s)|\,ds$$
$$=\mathrm{I}+\mathrm{II},$$

where I_z is the interval considered in Lemma 3. In $I\backslash I_z$, $P(z,s)$ is bounded (see (i) of Lemma 3) and hence II is bounded by the Hardy–Littlewood maximal function at t. In order to estimate I, we write $J_z=\{s\,|\,|s-t_z|\le r_z\}$ and $J_k=2^kJ_z$ with $k\le N$, where N is the least integer verifying $|J_N|>r_z^{1/2}$. Then

$$\mathrm{I}\ll\frac{1}{r_z}\int_{J_z}|f(s)|\,ds+\sum_{k=1}^{N}\frac{1}{2^{2k}r_z}\int_{J_k}|f(s)|\,ds\ll Mf(t),$$

since, by Lemma 4 and the definition of $D_\alpha(t)$, we have that $|s-t_z| \ll 2^k r_z$ in J_k. This maximal inequality gives, with a standard method, the confirmation of (iii). □

Let us state a lemma concerning the operator T_K.

LEMMA 6.

(i) *$T_K: L^\infty(I) \to L^\infty(I)$ is a compact operator.*
(ii) *If $f \in L^\infty(I)$ then $T_K f \in \bigcap_{\alpha<1} \mathrm{Lip}_\alpha(I)$.*

Proof of Lemma 6. Part (i) follows from (ii). For part (ii), let $t_1, t_2 \in I$, $\delta = |t_1 - t_2|$, and let t_0 be the middle point. With a constant a that will be chosen in a convenient way, we write

$$|T_K f(t_1) - T_K f(t_2)| \le \frac{1}{2\pi} \int_{|t_0-s|\le a\delta} |K(t_1,s) - K(t_2,s)|\,|f(s)|\,ds$$

$$+ \frac{1}{2\pi} \int_{|t_0-s|\ge a\delta} |K(t_1,s) - K(t_2,s)|\,|f(s)|\,ds = \mathrm{I} + \mathrm{II}.$$

Clearly $I \le C\delta$, using the boundedness of K. For the estimate II, we apply the mean value theorem to get

$$|K(t_1,s) - K(t_2,s)| \le \sup_{t\in[t_1,t_2]} \frac{|2\,\mathrm{Re}[\overline{\gamma'(t)}\gamma(s)(1-\gamma(t)\overline{\gamma(s)})^2]|}{|1-\overline{\gamma(t)}\gamma(s)|^4} |t_1 - t_2|.$$

Calling $H_s(t)$ the function that appears in the numerator of the last fraction,

$$\begin{aligned} H_s(t) &= \mathrm{Re}(\overline{\gamma'(t)}\gamma(s))\,\mathrm{Re}(1-\gamma(t)\overline{\gamma(s)})^2 - \mathrm{Im}(\overline{\gamma'(t)}\gamma(s))\,\mathrm{Im}(1-\gamma(t)\overline{\gamma(s)})^2 \\ &= \mathrm{Re}(\overline{\gamma'(t)}\gamma(s))[\mathrm{Re}(1-\gamma(t)\overline{\gamma(s)})^2 - (\mathrm{Im}(1-\gamma(t)\overline{\gamma(s)}))^2] \\ &\quad + 2\,\mathrm{Im}(\overline{\gamma'(t)}\gamma(s))\,\mathrm{Re}(1-\gamma(t)\overline{\gamma(s)})\,\mathrm{Im}\,\gamma(t)\overline{\gamma(s)}. \end{aligned}$$

The first of the summands in the last equality is $O(|s-t|^3)$ and so is the second, as $2\,\mathrm{Re}(1-\overline{\gamma(t)}\gamma(s)) = |\gamma(t)-\gamma(s)|^2 = O((t-s)^2)$ and $\mathrm{Im}\,\overline{\gamma(t)}\gamma(s) = O(|t-s|)$. Hence $H_s(t) = O(|s-t|^3)$, and choosing a conveniently we see that

$$|K(t_1,s) - K(t_2,s)| \le C \frac{|t_1-t_2|}{|t_0-s|}.$$

Then

$$\mathrm{II} \ll \delta \int_{|t_0-s|\ge a\delta} \frac{1}{|t_0-s|}\,ds \ll \delta \log \frac{1}{\delta},$$

an estimate that, with the estimate obtained for I, gives part (ii). □

As an immediate corollary we have the following.

COROLLARY 1. *The range of* $\mathrm{Id} + \lambda T_K$, $\mathbf{R}(\mathrm{Id} + \lambda T_K)$, *has finite codimension in the space of bounded functions on the curve.*

We now have the necessary tools to prove Theorem A.

THEOREM A. *The space $h^\infty{}_{|\Gamma}$ has finite codimension in $L^\infty(\Gamma)$.*

Proof of Theorem A. Given a bounded function f on the curve Γ, the function $u=\lambda Pf$ is, using (i) of Theorem 3, a bounded pluriharmonic function in B that, by (iii) of the same theorem, converges to $f+\lambda T_K f$ a.e. t in I. Hence $h^\infty{}_{|\Gamma}$ contains the range of $\mathrm{Id}+\lambda T_K$, which by Corollary 1 has finite codimension. □

If the transverse curve is a slice, $\gamma(t)=e^{it}\xi$, $|\xi|=1$, then this codimension is zero, but in general it can be seen that the codimension is different from zero. It would be interesting to characterize the codimension in terms of some properties of Γ.

2. The Spaces $\mathrm{Re}\, H^\infty{}_{|\Gamma}$ and $\mathrm{Re}\,\mathrm{Lip}_{\alpha|\Gamma}$

In this section we give a characterization of the space

$$\mathrm{Re}\, H^\infty{}_{|\Gamma}=\{f\in L^\infty_R(I)\,|\,\exists F\in H^\infty(B),\ \mathrm{Re}\, F^*(t)=f(t) \text{ a.e. } t\in I\},$$

where F^* denotes as usual the restricted K-limit of F.

The following result gives a way to construct bounded holomorphic functions from functions defined on a transverse curve.

PROPOSITION 1. *Let f be a real bounded function on Γ such that its conjugate is also a bounded function. Then the function*

$$F(z)=\frac{1}{2\pi}\int_I \frac{1+\overline{\gamma(t)}z}{1-\overline{\gamma(t)}z} f(t)\,dt$$

is in $H^\infty(B)$.

Proof of Proposition 1. From Lemma 5 we deduce that $\mathrm{Re}\, F$ is a bounded function. We will see that $Qf(z)=\mathrm{Im}\, F(z)$ is also a bounded function in B. We need three technical lemmas. The first generalizes a classical result on one complex variable (see, e.g., [3, p. 103]).

LEMMA 7. *Let $2<p\le+\infty$. Then there exists $a>0$ and $C>0$ such that, for every real 2π-periodic function f in $L^p(I)$ and z in $U\cap B$,*

$$\left|Qf(z)-\frac{1}{2\pi}\int_{|t-t_z|\ge ar_z} Q(\gamma(t_z),t)f(t)\,dt\right|\le C(Mf(t_z)+\|f\|_p),$$

where U is as in Lemma 2 and Mf is the Hardy–Littlewood maximal function.

Proof of Lemma 7. An upper bound of the left term above is

$$\left|\int_{|t-t_z|\le ar_z} Q(z,t)f(t)\,dt\right| + \left|\int_{|t-t_z|\ge ar_z}\{Q(z,t)-Q(\gamma(t_z),t)\}f(t)\,dt\right| = \mathrm{I}+\mathrm{II},$$

and we will see that both I and II satisfy the estimate. For I, it suffices to use (ii) of Lemma 3 to get

$$\left|\int_{|t-t_z|\le ar_z} Q(z,t)f(t)\,dt\right| \ll \int_{|t-t_z|\le ar_z} \frac{|t-t_z|}{|t-t_z|^2+r_z^2}|f(t)|\,dt \ll Mf(t_z).$$

In order to obtain the estimate of II, note that every $z\in U\cap B$ splits into the sum of two orthogonal vectors (one in the direction of $\gamma(t_z)$), writing $z=\lambda\gamma(t_z)+w$ where $\lambda=\overline{\gamma(t_z)}z=1-r_z$. Now we decompose the expression II introducing the factor $\int_{|t-t_z|\ge ar_z} Q(\lambda\gamma(t_z),t)f(t)\,dt$, and we obtain

$$\int_{|t-t_z|\ge ar_z}\{Q(z,t)-Q(\lambda\gamma(t_z),t)\}f(t)\,dt$$
$$+\int_{|t-t_z|\ge ar_z}\{Q(\lambda\gamma(t_z),t)-Q(\gamma(t_z),t)\}f(t)\,dt=\text{III}+\text{IV}.$$

For III we apply the mean value theorem to the function $Q(\cdot,t)$ in the segment joining $\lambda\gamma(t_z)+w$ and $\lambda\gamma(t_z)$, and use the orthogonality between w and $\gamma(t_z)$ to get

$$|Q(\lambda\gamma(t_z)+w,t)-Q(\lambda\gamma(t_z),t)|\le \sup_{\tau\in[0,1]}\left[\frac{|\mathrm{Im}(\overline{\gamma(t)}-\overline{\gamma(t_z)})w)|}{|1-\overline{\gamma(t)}(\lambda\gamma(t_z)+\tau w|^2}+\right.$$
$$\left.+\frac{|\mathrm{Im}(\overline{\gamma(t)}(\lambda\gamma(t_z)+\tau w))\,2\,\mathrm{Re}(\overline{\gamma(t)}-\overline{\gamma(t_z)})w)(1-\overline{\gamma(t)}(\lambda\gamma(t_z)+\tau w)))|}{|1-\overline{\gamma(t)}(\lambda\gamma(t_z)+\tau w)|^4}\right]$$
$$\ll\frac{|t-t_z||w|}{r_z^2+|t-t_z|^2},$$

where the last inequality comes from Lemma 2, provided a is chosen large enough to have $|1-\overline{\gamma(t)}(\lambda\gamma(t_z)+\tau w)|\gg|t-t_z|+r_z$. Since $|w|\ll r_z^{1/2}$, an application of Hölder's inequality shows that III is bounded by

$$\left|\frac{1}{2\pi}\int_{|t-t_z|\ge ar_z}\{Q(z,t)-Q(\lambda\gamma(t_z),t)\}f(t)\,dt\right|$$
$$\ll\int_{|t-t_z|\ge ar_z}\frac{|t-t_z||w|}{r_z^2+|t-t_z|^2}|f(t)|\,dt\ll\int_{|t-t_z|\ge ar_z}\frac{|f(t)|}{|t-t_z|^{1/2}}\,dt\ll\|f\|_p.$$

Finally, in order to estimate IV, we apply the mean value theorem once more, getting in a similar way (choosing a greater if necessary) that

$$|Q(\lambda\gamma(t_z),t)-Q(\gamma(t_z),t)|\ll r_z/|t-t_z|^2.$$

Hence IV is bounded by

$$\left|\frac{1}{2\pi}\int_{|t-t_z|\ge ar_z}\{Q(\lambda(\gamma(t_z),t)-Q(\gamma(t_z),t)\}f(t)\,dt\right|$$
$$\ll\int_{|t-t_z|\ge ar_z}\frac{r_z}{|t-t_z|^2}|f(t)|\,dt\ll Mf(t_z),$$

where the last inequality is obtained with a standard "doubling method". □

The statement of the next lemma corresponds to the well-known fact that in one complex variable the difference $2/t-\cot(t/2)$ is a bounded function.

LEMMA 8.

$$\left|Q(\gamma(t),s)+\frac{2}{\lambda(t-s)}\right|=O(1),$$

provided $|s|\le\pi$ *and* $|t-s|\le\pi$.

Proof of Lemma 8. Recall that we have chosen a suitable parametrization such that $\overline{\gamma(t)}=\lambda i$, where λ is a strictly positive constant. Hence, a Taylor formula development gives

$$1-\overline{\gamma(t)}\gamma(s)=\overline{\gamma'(s)}\gamma(s)(t-s)+o(|t-s|)=-\lambda i(t-s)+o(|t-s|),$$

and then

$$|1-\overline{\gamma(t)}\gamma(s)|^2=(t-s)^2(\lambda^2+o(1))$$

and

$$\operatorname{Im}\overline{\gamma(t)}\gamma(s)=-\lambda(t-s)+o(|t-s|).$$

As a consequence, provided $|t-s|$ is small enough,

$$\frac{1}{2}\left(Q(\gamma(t),s)+\frac{2}{\lambda(t-s)}\right)=\frac{\lambda(t-s)\operatorname{Im}\overline{\gamma(t)}\gamma(s)+|1-\overline{\gamma(t)}\gamma(s)|^2}{|1-\overline{\gamma(t)}\gamma(s)|^2\lambda(t-s)}.$$

Calling $h_s(t)$ the numerator of the last expression, we will see that $h_s(t)=O(|t-s|^3)$. In the first place, $h_s(s)=0$, and by differentiating we have

$$\begin{aligned}
&\lambda\operatorname{Im}\overline{\gamma(t)}\gamma(s)+\lambda(t-s)\operatorname{Im}\overline{\gamma'(t)}\gamma(s)-2\operatorname{Re}(\overline{\gamma'(t)}\gamma(s)(1-\gamma(t)\overline{\gamma(s)}))\\
&=\lambda\operatorname{Im}\overline{\gamma(t)}\gamma(s)+\lambda(t-s)\operatorname{Im}\overline{\gamma'(t)}\gamma(s)-2\operatorname{Re}(\overline{\gamma'(t)}\gamma(s))\operatorname{Re}(1-\gamma(t)\overline{\gamma(s)})\\
&\quad-2\operatorname{Im}(\overline{\gamma'(t)}\gamma(s))\operatorname{Im}(\gamma(t)\overline{\gamma(s)})\\
&=-2\operatorname{Re}(\overline{\gamma'(t)}\gamma(s))\operatorname{Re}(1-\gamma(t)\overline{\gamma(s)})-\operatorname{Im}(\overline{\gamma'(t)}(\gamma(s)-\gamma(t)))\operatorname{Im}(\gamma(t)\overline{\gamma(s)})\\
&\quad+\lambda(t-s)\operatorname{Im}\overline{\gamma'(t)}\gamma(s)-\operatorname{Im}(\overline{\gamma'(t)}\gamma(s))\operatorname{Im}(\gamma(t)\overline{\gamma(s)}).
\end{aligned}$$

The first two terms obtained in the last equality are, using the properties of γ, $O((s-t)^2)$. We shall see that

$$\lambda(t-s)\operatorname{Im}\overline{\gamma'(t)}\gamma(s)-\operatorname{Im}\overline{\gamma'(t)}\gamma(s)\operatorname{Im}\gamma(t)\overline{\gamma(s)}$$

is also of this type. We apply the mean value theorem to obtain

$$\begin{aligned}
&\lambda(t-s)\operatorname{Im}\overline{\gamma'(t)}\gamma(s)-\operatorname{Im}\overline{\gamma'(t)}\gamma(s)\operatorname{Im}\gamma(t)\overline{\gamma(s)}\\
&\quad=\lambda(t-s)\operatorname{Im}\overline{\gamma'(t)}\gamma(s)-\operatorname{Im}\overline{\gamma'(t)}\gamma(s)(t-s)\operatorname{Im}\gamma'(\xi)\overline{\gamma(s)}\\
&\quad=(t-s)\operatorname{Im}\overline{\gamma'(t)}\gamma(s)[\operatorname{Im}(\gamma'(\xi)(\overline{\gamma(\xi)}-\overline{\gamma(s)}))],
\end{aligned}$$

where ξ is a point in the segment joining t and s. Therefore we get that $h_s(t)=O(|t-s|^3)$, and the lemma follows. □

LEMMA 9. *Let f be a real bounded 2π-periodic function such that $\tilde{f}$ is bounded. Then the function*

$$f_1(\theta) = \sup_{\epsilon>0} \left| \int_{|t-\theta|>\epsilon} \cot \frac{(\theta-t)}{2} f(t)\,dt \right| \in L^\infty(I).$$

Proof of Lemma 9. This lemma is a consequence of [3, p. 103], where it is proved that if f is a 2π-periodic L^1 function then

$$\left| Qf(re^{i\theta}) - \frac{1}{2\pi} \int_{|t-\theta|>1-r} \cot \frac{(\theta-t)}{2} f(t)\,dt \right| \ll Mf(\theta),$$

where Qf is the conjugate function of Pf.

Let f be a real bounded function with bounded conjugate. Then Hf, the Herglotz transform of f, is a function in $H^\infty(D)$, since $P(f+i\tilde{f})$ is in $H^\infty(D)$ and coincides (except for an imaginary constant) with Hf. Applying the result of [3] cited before, we get that f_1 is bounded. □

Now we can prove Proposition 1. From Lemma 9 and Lemma 8, we obtain that the function

$$\sup_{\epsilon>0} \left| \int_{|t-\theta|>\epsilon} Q(\gamma(\theta),t) f(t)\,dt \right|$$

is also a bounded function. Hence, applying Lemma 7, Qf is bounded. □

Now we can prove Theorem B.

THEOREM B. *The space* $\operatorname{Re} H^\infty{}_{|\Gamma}$ *has finite codimension in the space of real bounded functions in* Γ *with bounded conjugate.*

Proof of Theorem B. From Theorem 1, given F in $H^\infty(B)$, we get that $\widetilde{\operatorname{Re} F^*}$ is in $L^\infty(I)$. On the other hand, if $f \in \mathbf{R}(\mathrm{Id}+\lambda T_K)$ and $\tilde{f}$ is bounded then we have $f = g + \lambda T_K g$, where g is a bounded function on Γ. From (ii) of Lemma 6 we deduce that $T_K g$ is in every $\mathrm{Lip}_\alpha(I)$ with $\alpha<1$, and, in particular, that $\tilde{g}$ is bounded. Applying Proposition 1, the function

$$G(z) = \frac{\lambda}{2\pi} \int_I \frac{1+\overline{\gamma(t)}z}{1-\overline{\gamma(t)}z} g(t)\,dt \in H^\infty(B),$$

and, from (iii) of Theorem 3, a.e. t in I, $\operatorname{Re} G_{|\Gamma} = g + \lambda T_K g = f$. Hence the space $\operatorname{Re} H^\infty{}_{|\Gamma}$ contains $\mathbf{R}(\mathrm{Id}+\lambda T_K) \cap \{f \in L^\infty_R(\Gamma) \mid \tilde{f} \in L^\infty_R(\Gamma)\}$, a space that has, by Corollary 1, finite codimension in the space $\{f \in L^\infty_R(\Gamma) \mid \tilde{f} \in L^\infty_R(\Gamma)\}$.
□

We finally give a similar result concerning the space $\mathrm{Lip}_\alpha(B)$, $0<\alpha<1$, of holomorphic Lipschitz functions. For technical reasons (and as stated in the introduction), we will suppose from now on that the curves are of class C^2.

THEOREM C. *The space* $\operatorname{Re} \mathrm{Lip}_{\alpha|\Gamma}$, *where* $\alpha<1$, *has finite codimension in the space of real functions in* Γ *satisfying a Lipschitz condition of order* α.

Proof of Theorem C. Let f be in $\mathrm{Lip}_\alpha(\Gamma)$, and let Hf be the corresponding Herglotz transform of f related to Γ; that is,

$$Hf(z)=\frac{1}{2\pi}\int_I \frac{1+\overline{\gamma(t)}z}{1-\overline{\gamma(t)}z}f(t)\,dt.$$

We will see first that Hf is in $\text{Lip}_\alpha(B)$. Since $(1+\overline{\gamma(t)}z)/(1-\overline{\gamma(t)}z)$ and $2/(1-\overline{\gamma(t)}z)$ differ by a constant, it is enough to prove that

$$H_1f(z)=\frac{1}{2\pi}\int_I \frac{f(t)}{1-\overline{\gamma(t)}z}\,dt\in \text{Lip}_\alpha(B).$$

This last assertion will follow (see [7]) from the estimate

$$|RH_1f(z)|=O((1-|z|)^{\alpha-1}), \tag{13}$$

where

$$RH_1f(z)=\sum_{i=1}^{n}\frac{\partial H_1f}{\partial z_i}(z)z_i$$

is the radial derivative of H_1f. By a result of Nagel [4] the function

$$\int_I \frac{dt}{1-\overline{\gamma(t)}z}\in A^1(B),$$

provided γ is of class C^2. Hence (13) will hold if we prove that

$$\left|\int_I \frac{\overline{\gamma(t)}z}{(1-\overline{\gamma(t)}z)^2}(f(t)-f(t_z))\,dt\right|=O(r_z^{\alpha-1}),\quad z\in U,$$

where t_z, r_z, and U are as in Section 1 (notice that $r_z\gg 1-|z|$). By Lemma 2 and the hypothesis on f, the last estimate is a consequence of the easily verified estimate

$$\left|\int_{\{|t-t_z|\le\pi\}} \frac{|t-t_z|^\alpha}{|t-t_z|^2+r_z^2}\,dt\right|=O(r_z^{\alpha-1}).$$

Now, using the fact that T_K maps $L^\infty(\Gamma)$ in $\bigcap_{\alpha<1}\text{Lip}_\alpha(\Gamma)$, $\alpha<1$, the same argument as for Theorem B finishes the proof. □

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