Farrell and Mergelyan Sets for H^p Spaces $(0<p<1)$

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Dedicated to professor Ernst Selmer on the occasion of his 70th birthday

1. Introduction

Let F be a relatively closed subset of the open unit disc $\mathbf{D}$ of the complex plane $\mathbf{C}$, and let A be a space of analytic functions on $\mathbf{D}$ endowed with a topology τ such that the polynomials are τ-dense in A. The set F is said to be a *Farrell set for* (A,τ) if for each function $f\in A$ whose restriction $f|_F$ is bounded there exists a sequence $(p_\nu)_{\nu=1}^{\infty}$ of polynomials satisfying:

(1) $p_\nu\to f$ in the τ-topology, as $\nu\to\infty$,

(2) $p_\nu\to f$ pointwise on F, as $\nu\to\infty$, and

(3) $\|p_\nu\|_F\to\|f\|_F$, as $\nu\to\infty$.

As usual, $\|g\|_B$ denotes $\sup\{|g(z)|: z\in B\}$. Similarly, F is said to be a *Mergelyan set for* (A,τ) if, for each function $f\in A$ whose restriction to F is uniformly continuous, there exists a sequence $(p_\nu)_{\nu=1}^{\infty}$ of polynomials such that

(α) $p_\nu\to f$ in the τ-topology, as $\nu\to\infty$, and

(β) $p_\nu\to f$ uniformly on F, as $\nu\to\infty$.

Farrell and Mergelyan sets have been described for several cases: (a) A is the space $H^\infty(\mathbf{D})$ of all bounded analytic functions and τ the topology of pointwise convergence on $\mathbf{D}$ [9]; (b) A is the Hardy space H^p $(1\le p<\infty)$ and τ is the weak topology [8] or the norm topology [7]; (c) A is the space $H(\mathbf{D})$ of all analytic functions on $\mathbf{D}$ with the topology of uniform convergence on compact subsets of $\mathbf{D}$.

A holomorphic function in $\mathbf{D}$ is said to belong to the Nevanlinna class N if its characteristic function

$$T(r,f)=\frac{1}{2\pi}\int_0^{2\pi}\log(1+|f(re^{i\theta})|)\,d\theta$$

is bounded for $0\le r<1$. In this case $N(f)=\operatorname{Sup}_{0\le r<1}T(r,f)$. A function $f\in N$ is said to belong to the Smirnov class N^+ if there hold

$$\lim_{r\to1}\int_0^{2\pi}\log(1+|f(re^{i\theta})|)\,d\theta=\int_0^{2\pi}\log(1+|f(e^{i\theta})|)\,d\theta.$$

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For $0<p<q<\infty$ it is well known that $H^\infty \subset H^q \subset H^p \subset N^+ \subset N$ with all inclusion relations being proper (see, e.g., [3]). The class H^p, $0<p<1$, does not form a Banach space but is a complete metric topological vector space with the metric

$$\|f-g\|_p^p = \frac{1}{2\pi}\int_0^{2\pi} |f(e^{i\theta})-g(e^{i\theta})|^p \, d\theta, \quad f,g\in H^p, \tag{1}$$

and there are bounded linear functionals on H^p enough to separate functions in H^p. ($(H^p)^*$ can be identified with a Lipschitz space of order α, $0<\alpha<1$, α depending on p; cf. [4].) An analogous affirmation can be made for the Smirnov class N^+, which is a complete metric topological vector algebra with the distance functions

$$\rho(f,g) = \lim_{r\to 1} \frac{1}{2\pi}\int_0^{2\pi} \log(1+|f(re^{i\vartheta})-g(re^{i\vartheta})|)\, d\vartheta, \tag{2}$$

$f,g\in N^+$. N^+ also has a separating dual space $(N^+)^*$ which can be identified with the space of all the functions f in the disc algebra whose Taylor coefficients satisfy $\hat{g}(n) = O(\exp[-c\sqrt{n}])$ for a positive constant c [13]. Hence we can consider the weak topologies on H^p $(0<p<1)$ and N^+.

In this paper we prove the following theorem, extending the main result in [8] but using different methods.

THEOREM. *Let F be a relatively closed subset of $\mathbf{D}$ and $0<p<\infty$. The following assertions are equivalent:*

(i) *F is a Farrell set for $(H^p, \|\;\|_p)$.*
(ii) *F is a Mergelyan set for $(H^p, \|\;\|_p)$.*
(iii) *There exists a set $E\subset \mathbf{T}\cap\bar{F}$ with $m(E)=0$, such that if $\zeta\in\bar{F}\cap\mathbf{T}\setminus E$ then there is a sequence $(\zeta_\nu)\subset F$ converging nontangentially to ζ.*
(iv) *Suppose g is a uniformly continuous function on F and that $f\in H^p$ is bounded on F. Then there exist polynomials p_ν $(\nu=1,2,\dots)$ such that $p_\nu\to f$ in H^p metric and $\lim_{\nu\to\infty}\|g-p_\nu\|_F = \|g-f\|_F$.*

2. Proof of the Theorem

We prefer to begin by showing the following proposition.

PROPOSITION. *Let Y be any of the metric spaces $H^p(0<p<1)$ or N^+ and let F be a relatively closed subset of $\mathbf{D}$. If almost every point of $\bar{F}\cap\mathbf{D}$ is a nontangential limit of a sequence of points of F, then F is a Farrell set for Y.*

Proof. Let f be any function belonging to Y whose restriction $f|_F$ to F is bounded. Let $f=B\cdot G$ be the inner–outer factorization of f, G being its outer part

$$G(z) = \exp\left(\frac{1}{2\pi}\int_0^{2\pi} \frac{e^{it}+z}{e^{it}-z}u(t)\,dt\right), \quad z\in\mathbf{D}, \tag{3}$$

where $u(t)=\log|f(e^{it})|\in L^1(\mathbf{T})$. We cut off the function $u(t)$ defining

(4) $$u_n(t)=\min\{u(t),K_n\},\quad n=1,2,\ldots,$$

where K_n are positive constants and $K_n\to\infty$, and let G_n be an outer function whose modulus on $\mathbf{T}$ is $u_n(t)$; that is,

$$G_n(z)=e^{i\alpha_n}\exp\left(\frac{1}{2\pi}\int_0^{2\pi}\frac{e^{it}+z}{e^{it}-z}u_n(t)\,dt\right),$$

$z\in\mathbf{D}$. It is not hard to show that $\{G_n\}_{n=1}^{\infty}$ is a sequence in $H^\infty(\mathbf{D})$ satisfying

(5) $$\left|\frac{G_n(z)}{G(z)}\right|\le 1\quad\text{for any } z\in\mathbf{D};\qquad \frac{G_n(\theta)}{G(\theta)}\to 1\quad\text{a.e. } d\theta\text{ on }\mathbf{T}.$$

Furthermore, if we choose the constants α_n and K_n conveniently, it can be shown that

(6) $$\frac{G_n}{G}\to 1\quad\text{in } H^2\text{-norm, as } n\to\infty$$

(see [6, p. 85] for details). As $|G_n(z)|\le|G(z)|$, we can suppose

(7) $$|G_n(e^{i\theta})|\le|G(e^{i\theta})|,\quad\text{a.e. } d\theta.$$

Now, the inequality $|G_n(\theta)-G(\theta)|=|G(\theta)||1-G_n(\theta)/G(\theta)|\le 2|G(\theta)|$ and the dominated convergence theorem prove that $G_n\to G$ in the H^p metric if $Y=H^p$ (any p), while if $Y=N^+$ the inequality

$$\begin{aligned}\mathrm{Log}(1+|G_n(\theta)-G(\theta)|)&\le\mathrm{Log}(1+|G_n(\theta)|)+\mathrm{Log}(1+|G(\theta)|)\\&\le 2\,\mathrm{Log}(1+|G(\theta)|)\end{aligned}$$

leads to $G_n\to G$ in the ρ-metric. Next, we define the functions $f_n=B\cdot G_n$, $n=1,2,\ldots$. According to the properties above for the G_n's, it is clear that there exists $f_n\in H^\infty(\mathbf{D})$, $\|f_n\|_F\le\|f\|_F$, $|f_n(z)|\le|f(z)|$, $z\in\mathbf{D}$, $n=1,2,\ldots$, such that f_n converge to f in the Y-metric, and, in particular, pointwise on $\mathbf{D}$. Moreover, since f is bounded on F, we get $\sup_n\|f_n\|_F=\|f\|_F$. In the next step, we shall use the geometric condition imposed on F in the hypothesis and, in order to be more clear, we regard two cases: $Y=H^p$ and $Y=N^+$. First, we suppose $f\in H^p$. Since F is a Farrell set for $H^\infty(\mathbf{D})$ with the boundedly pointwise convergence topology [9], each f_n can be approximated by a sequence $\{q_\nu^n\}_{\nu=1}^{\infty}$ of polynomials satisfying

(8) $$q_\nu^n(z)\to f_n(z),\quad \nu\to\infty,\quad\text{any } z\in\mathbf{D};\qquad \|q_\nu^n\|_\infty\le\|f_n\|_\infty\quad(\nu=1,2,\ldots);\qquad \|q_\nu^n\|_F\le\|f_n\|_F\quad(\nu=1,2,\ldots);$$

and taking a subsequence if necessary, we can even suppose

(9) $$\sup_\nu\|q_\nu^n\|_F=\lim_{\nu\to\infty}\|q_\nu^n\|_F=\|f_n\|_F,$$

and $q_\nu^n(e^{i\theta})\to f_n(e^{i\theta})$ a.e. $d\theta$ on $\mathbf{T}$, as $\nu\to\infty$. So, the second estimation in (8) and dominated convergence theorem lead to $q_\nu^n\to f_n$, as $\nu\to\infty$, in the H^p

metric. Now, if we fix an arbitrary positive integer n and f_n is such that $\|f-f_n\|_F^p \le 1/2n$, and for this such f_n we choose a polynomial $q_\nu^n = p_n$ satisfying $\|f_n - p_n\|_p^p < 1/2n$, we can then find a sequence of polynomials converging to f in H^p metric, with $\lim_{n\to\infty}\|p_n\|_F = \|f\|_F$ by (8) and (9).

If $Y = N^+$, we can do exactly the same we did in the case before, even being able to approximate each f_n by polynomials (and boundedly on F) in any H^p metric $(0<p<\infty)$. Here the hypothesis on F and results in [8] and [7] have been used again. The proof is now complete. □

REMARK. With the notation as in the above proof, assume f admits continuous extension to $\mathbf{T}\setminus\bar{F}$. Let V denote a compact annulus centered at a point of $\mathbf{T}$ and assume $V\cap\mathbf{T}\subset(\mathbf{T}\setminus\bar{F})\cup F_0$, where F_0 denotes the interior of $\bar{F}\cap\mathbf{T}$ relative to $\mathbf{T}$.

In this special situation we observe that $\|f\|_{V\cap\mathbf{D}}<\infty$, and we claim that the approximant polynomials p_n can be chosen to satisfy $\|p_n\|_{V\cap\mathbf{D}}\le K$, with K independent of n. This additional property of $\{p_n\}$ will be crucial at a later stage.

To justify the claim, we first remark that the functions $f_n = B\cdot G_n$ constructed above will extend continuously to $(\mathbf{T}\setminus\bar{F})\cap V$ if n is large enough. Also, $\|f_n\|_{V\cap\mathbf{D}}\le\|f\|_{V\cap\mathbf{D}}$ by construction. So fix n large, and put $H=f_n$. Given $\epsilon>0$, we choose a smooth function φ with $0\le\varphi\le1$, $\varphi\equiv1$ in a neighbourhood of $\bar{F}$, $\varphi\equiv0$ near $\mathbf{T}\cap V\setminus\bar{F}$, and

$$|(\varphi H)(z)|\le\|H\|_F+\epsilon \quad \text{if } z\in(V\cap\mathbf{D})\setminus\bar{F}.$$

Since H is continuous on $(V\cap\bar{\mathbf{D}})\setminus\bar{F}$, this is possible. Since the implication (iii) ⇒ (iv) in the Theorem holds if $p=\infty$ (see [11]), we may apply it to the relatively closed subset $F\cup(V\cap\mathbf{D})$ with the function u defined as $u\equiv0$ on $\bar{F}$ and $u=(1-\varphi)H$ on $(V\cap\bar{\mathbf{D}})\setminus\bar{F}$. Then we can find polynomials p_υ converging to H pointwise boundedly in $\mathbf{D}$ and in addition satisfying

$$\|p_\upsilon-u\|_{F\cup(V\cap\mathbf{D})}\to\|H-u\|_{F\cup(V\cap\mathbf{D})} \quad \text{as } \nu\to\infty.$$

But $\|H-u\|_{F\cup(V\cap\mathbf{D})}\le\|H\|_F+\epsilon\le\|f\|_F+\epsilon$, and since $|u|\le|H|$ on $V\cap\mathbf{D}$, it follows that $|p_\upsilon|\le\|f\|_F+\epsilon+\|f\|_{V\cap\mathbf{D}}$ on $V\cap\mathbf{D}$, independently of υ and the number n used to define H as $B\cdot G_n$.

The proposition above shows that (iii) ⇒ (i). Trivially (iv) ⇒ (i) (take $g\equiv0$) and (iv) ⇒ (ii) (take $g=f|_F$). To conclude the proof of the theorem, we shall develop the following steps. First, we show that (ii) ⇒ (iii), then (iii) ⇒ (iv), and finally (i) ⇒ (ii).

(ii) ⇒ (iii): Suppose (iii) fails. Using Detraz's construction [2], there exists a function $f\in H^\infty$ not identically zero such that $f|_F$ is uniformly continuous and such that $f(z)\to0$ as $z\to\zeta\in B$, $z\in F$, where $B\subset\bar{F}\cap\mathbf{T}$ and $m(B)>0$. Let $\{p_\nu\}$ be a sequence of polynomials converging to f in H^p metric, and uniformly on F. We can assume that $f(0)\ne0$. Since $p_\nu(0)\to f(0)$, Jensen inequality leads to a contradiction as follows:

$$\operatorname{Log}|p_\nu(0)|\le\int_{\mathbf{T}}\operatorname{Log}|p_\nu|\,dm\le\int_B+\int_{\mathbf{T}\setminus B} \quad (\nu=1,2,\dots).$$

Since $p_\nu \to f$ in the N^+ metric, p_ν converge to f in measure, and by passing to a subsequence if necessary, $p_\nu \to f$ almost everywhere with respect to m on $\mathbf{T}$. Then

$$\int_{\mathbf{T}\setminus B} \operatorname{Log}|p_\nu|\,dm \le \int_{\mathbf{T}\setminus B} \operatorname{Log}^+|p_\nu|\,dm$$
$$\le \int_{\mathbf{T}\setminus B} \operatorname{Log}^+|p_\nu - f| + \int_{\mathbf{T}\setminus B} \operatorname{Log}^+|f| + \int_{\mathbf{T}\setminus B} \operatorname{Log} 2$$

and $\lim_{\nu\to\infty} \int_{\mathbf{T}\setminus B} \operatorname{Log}|p_\nu|\,dm < \infty$. On the other hand, $\int_B \operatorname{Log}|p_\nu| \to -\infty$, as $\nu \to \infty$, and $f(0)$ should vanish.

(iii) ⇒ (iv): We use an idea from [12]. Suppose now F satisfies the geometrical condition (iii). By the Proposition, F must be a Farrell set. To show that F even satisfies the formally stronger condition in (iv), we apply the simple but useful observation that $F \cap \Delta$ is a Farrell set whenever Δ is a disc and $F \cap \Delta \ne \emptyset$.

Given f and g as in (iv), we choose a finite covering of $\bar{F}$ by discs $\Delta_1, \Delta_2, \ldots, \Delta_N$ such that

(a) $z, z' \in \Delta_j \Rightarrow |g(z) - g(z')| < \epsilon,\ 1 \le j \le N$.

(b) If $\zeta \in \mathbf{T} \cap \partial\Delta_j$ for some j, then f has a nontangential limit at ζ.

(c) If $\zeta \in \mathbf{T} \cap \partial\Delta_j$ for some j, then either $\zeta \notin \bar{F}$ or ζ belongs to the interior F_0 of $\bar{F} \cap \mathbf{T}$ relative to $\mathbf{T}$.

Such a choice is possible by Fatou's theorem. Let $E = \bigcup_{j=1}^N \partial\Delta_j$. In the proof of (iii) ⇒ (iv) we shall assume that f is bounded in $V \cap \mathbf{D}$ for some neighbourhood V of E. We shall justify this assumption at the end of the proof.

Let us select constants $u_j = g(z_j)$ for some fixed points $z_j \in \Delta_j$, $1 \le j < N$. Since $F \cap \Delta_j$ is a Farrell set for H^p we choose polynomials p_n^j such that

$$\|(f - u_j) - p_n^j\|_p^p < \frac{1}{n} \quad \text{and} \quad \|p_n^j\|_{F\cap\Delta_j} \le \|f - u_j\|_F + \frac{1}{n}$$

for $n = 1, 2, \ldots$ and $1 \le j \le N$. Then we define

$$q_n = \begin{cases} p_n^1 + u_1 & \text{in } \Delta_1, \\ p_n^j + u_j & \text{in } \Delta_j \setminus \bigcup_{k=1}^{j-1} \Delta_k, \quad j = 2, 3, \ldots, N, \\ f_n & \text{in } \mathbf{D} \setminus \bigcup_{k=1}^N \Delta_k, \end{cases} \tag{10}$$

where $f_n(z) = f((1 - 1/n)z)$, $n = 1, 2, \ldots$. Since f is locally bounded near E, we can arrange it so that q_n has the same property with a bound independent of n. This is possible in light of the remark following the Proposition above. We observe that

$$\frac{1}{2\pi} \int_{\mathbf{T}} |q_n - f|^p\,d\theta \le \frac{N+1}{n} \tag{11}$$

and

$$\|q_n - g\|_F \le \|f - g\|_F + 2\epsilon + \frac{1}{n}, \tag{12}$$

since $|g - u_j| < \epsilon$ on $F \cap \Delta_j$. The functions q_n may be discontinuous on E, but as n increases the situation gets better, since $q_n \to f$ uniformly on compact subsets of $\mathbf{D}$.

To modify q_n, we use a well-known idea from rational approximation. If h is a locally integrable function and φ is a continuously differentiable function with compact support, the Vitushkin operator is defined by

$$T_\varphi(h)(\zeta)=\frac{1}{\pi}\iint_{\mathbf{C}}\frac{h(z)-h(\zeta)}{z-\zeta}\frac{\partial\varphi}{\partial\bar{z}}\,dx\,dy,\quad z=x+iy,$$

and by the Green formula we also have

$$T_\varphi(h)(\zeta)=\varphi(\zeta)h(\zeta)-\frac{1}{\pi}\iint_{\mathbf{C}}\frac{h(z)}{z-\zeta}\frac{\partial\varphi}{\partial\bar{z}}\,dx\,dy.$$

We refer to [5] for more details and a proof of the following useful identity:

$$\frac{\partial}{\partial\bar{z}}(T_\varphi(h))=\varphi\frac{\partial h}{\partial\bar{z}},\tag{13}$$

which is also valid when $\partial/\partial\bar{z}$ is taken in the sense of distributions. Now we select a finite covering of E by discs $\{D_k\}$ and functions $\varphi_k\in C_0^1(D_k)$ such that

$$\sum_k\varphi_k\equiv 1\quad\text{near } E.\tag{14}$$

Let us numerate $\{D_k\}$ such that $D_k\subset\mathbf{D}$ for $1\le k\le k_0$. If $k>k_0$, $D_k\cap\mathbf{T}\neq\emptyset$ and we assume then that D_k is centered at some $\zeta_k\in E$. The radius δ_k of D_k will be chosen so small that q_n is bounded on D_k, independently of n and $k>k_0$, and we also require that

$$\left\|\frac{\partial\varphi_k}{\partial\bar{z}}\right\|_\infty\le\frac{4}{\delta_k}\quad\text{if } k>k_0.\tag{15}$$

If g is any of the functions q_n in (10) or f, we assume g has been extended to $\mathbf{C}$ by the equation $g(1/\bar{z})=g(z)$, $z\in\mathbf{D}$. We can finally define

$$\tilde{q}_n=q_n-\sum_{k=1}^{k_0}T_{\varphi_k}(q_n-f)-\sum_{k>k_0}T_{\varphi_k}(q_n-\lambda_k),\tag{16}$$

where λ_k is the nontangential limit of f at the center ζ_k of D_k. This limit exists inside any cone Γ_k terminating at ζ_k. Hence the area of $D_k\cap\mathbf{D}\backslash\Gamma_k$ compared to $D_k\cap\Gamma_k$ can be made as small as we please, simply by selecting Γ_k properly.

Since $f\approx\lambda_k$ in $\Gamma_k\cap D_k$ and $q_n\to f$ as $n\to\infty$, we can choose the discs D_k so small that

$$\left|\sum_{k>k_0}T_{\varphi_k}(q_n-\lambda_k)\right|\le\left|\sum\varphi_k(q_n-\lambda_k)\right|+\epsilon\tag{17}$$

if n is sufficiently large. This follows from the uniform boundedness of $\{q_n\}$ in D_k and estimating the integrals

$$\frac{1}{\pi}\iint_{\mathbf{C}}\frac{q_n-\lambda_k}{z-\zeta}\frac{\partial\varphi_k}{\partial\bar{z}}\,dx\,dy\tag{18}$$

using Hölder inequality and the above considerations about f, λ_k, and Γ_k.

If $k \le k_0$ then $q_n \to f$ uniformly on D_k. So by increasing n further we have

$$\left| \sum_{k \le k_0} T_{\varphi_k}(q_n - f)(\zeta) \right| \le \epsilon \tag{19}$$

uniformly in ζ.

Let us now estimate $\|\tilde{q}_n - g\|_F$ for n large. In $F \cap D_k$ $(k > k_0)$ we have

$$\begin{aligned} |\tilde{q}_n - g| &\le |q_n - \varphi_k(q_n - \lambda_k) - g| + 2\epsilon \\ &\le |(1-\varphi_k)(q_n - g)| + |\varphi_k(\lambda_k - g)| + 2\epsilon \\ &\le |1-\varphi_k|(\|f-g\|_F + 2\epsilon + n^{-1}) + |\varphi_k| \|f-g\|_F + 3\epsilon. \end{aligned}$$

In the last inequality we used (12). We also used that $|\lambda_k - g|_F \le \|f-g\|_F + \epsilon$. This last inequality holds since $|f(e^{i\theta}) - g(e^{i\theta})| \le \|f-g\|_F$ almost everywhere on $\bar{F} \cap \mathbf{T}$ with respect to linear measure m. Hence if the interior of $\bar{F} \cap \mathbf{T}$ with respect to $\mathbf{T}$ is nonempty, we may assume $\{\Delta_j\}$ has been selected so that $|g - \lambda_k| \le \|f-g\|_F$ at the center ζ_k of D_k. We conclude that $\|\tilde{q}_n - f\|_F \le \|f-g\|_F + 6\epsilon$ if n is large enough. It is also clear that we can get

$$\int_{\mathbf{T}} |\tilde{q}_n - q_n|^p \, d\theta \tag{20}$$

as small as we please by choosing $\{D_k\}$ carefully.

It remains to prove that $\tilde{q}_n$ can be uniformly approximated on $\mathbf{D}$ by polynomials. Recall that the T_φ operator preserves continuity and analyticity [5]. Therefore $\tilde{q}_n$ is continuous on $\bar{\mathbf{D}} \setminus E$. By (13) it follows that

$$\frac{\partial}{\partial \bar{z}}(\tilde{q}_n) = 0 \quad \text{near } E,$$

and hence $\tilde{q}_n$ is continuous in $\{z : |z| \le 1\}$ and analytic in $\mathbf{D}$. This completes the proof that (iii) $\Rightarrow$ (iv), except that we must justify our assumption about f being bounded near E. We have the following.

LEMMA. *Let F be a relatively closed subset of* $\mathbf{D}$ *and assume $f \in H^p$ is bounded on F. Then there are $f_n \in H^p$ $(n = 1, 2, \ldots)$ such that $f_n \to f$ in H^p, $f_n \to f$ uniformly on F, and each f_n extends to be analytic across* $\mathbf{T} \setminus \bar{F}$.

Proof. It is sufficient to prove this if $p > 1$. The general case can be reduced to this situation since any H^p-function is the sum of two nonvanishing H^p-functions [3, p. 79].

Let f be given as above. Using Lemma 3 of [10], we can find $g_n \in H^p$ such that $|f - g_n| \le 1/n$ on $F \cup \{z : |z| \le 1 - 1/n\}$, g_n is analytic across $\mathbf{T} \setminus \bar{F}$, and $\|g_n\|_p \le C_p \|f\|_p$, where C_p depends only on p.

In particular, $g_n \to f$ weakly in H^p. Since $p > 1$, H^p is reflexive, and hence we can obtain convex combinations $\{f_n\}$ from $\{g_n\}$ such that the Lemma holds.

(i) $\Rightarrow$ (ii): Finally, let f be a function in H^p and let $f|_F$ be uniformly continuous on F, and suppose (i) holds. In order to approximate f in H^p and uniformly on F by polynomials, it is enough to replace g by f in proving the implication above. This completes the proof of the theorem. □

3. Remarks

Looking at the proof of the theorem, one notes that it holds even for the Smirnov class N^+. Hence, we can say that the geometrical condition in (iii) is sufficient in order for F to be a Farrell set or a Mergelyan set for H^p or N^+ with its weak topologies.

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