Order and Lower Order of Composite Meromorphic Functions

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1. Introduction

Let f and g be entire functions and suppose that g is transcendental. Denote the order of f by $\rho(f)$ and the lower order by $\lambda(f)$. Pólya [11] proved the following.

THEOREM A. *If* $\rho(f\circ g)<\infty$, *then* $\rho(f)=0$.

Gross [6, p. 86] pointed out that one can use Pólya's method also to prove the following.

THEOREM B. *If* $\lambda(f\circ g)<\infty$, *then* $\lambda(f)=0$.

Schönhage [12] considered the l-order $\rho_l(f)$ (see also [9, p. 96]) which is defined for a nonnegative integer l by

$$\rho_l(f)=\limsup_{r\to\infty}\frac{\log_{l+1}T(r,f)}{\log r}, \tag{1.1}$$

where $\log_{l+1}x=\log(\log_l x)$ and $\log_0 x=x$, and where $T(r,f)$ is the Nevanlinna characteristic of f. The case $l=0$ is the classical one; that is, $\rho_0(f)=\rho(f)$. Using Pólya's ideas, Schönhage [12, Satz 9] proved the following generalization of Theorem A.

THEOREM C. *If* $\rho_l(f\circ g)<\infty$, *then* $\rho_l(f)=0$.

One can also prove an analogous result for the lower l-order $\lambda_l(f)$ which is defined by taking the limes inferior in (1.1).

Suppose now that f is meromorphic while g is still entire and transcendental. Edrei and Fuchs [5] proved that Theorem A still holds under this more general hypothesis. It is the main purpose of this paper to show that Theorem B also holds for meromorphic f, while Theorem C does not.

We note that the method used by Edrei and Fuchs does not carry over to the case of the lower order. We use a different method which also yields a

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new proof of the theorem of Edrei and Fuchs. While their proof uses the elementary theory of central index and maximum term, our proof is based on Nevanlinna theory. We assume familiarity with the basic definitions and results of the theory (e.g., [7], [9], [10]).

2. Statement of Main Results

THEOREM 1. *Let f be meromorphic and let g be entire and transcendental. If $\lambda(f\circ g)<\infty$, then $\lambda(f)=0$.*

THEOREM 2. *Let $\alpha(r)$ and $\beta(r)$ be two real-valued nondecreasing functions, defined for $r>0$, which tend to infinity as $r\to\infty$.*

(a) *There exist a meromorphic function f and an entire transcendental function g such that $T(r,f\circ g)\le r^{\beta(r)}$ holds for all sufficiently large r while $T(r,f)\ge\alpha(r)$ holds for arbitrarily large values of r.*

(b) *There exist a meromorphic function f and an entire transcendental function g such that $T(r,f\circ g)\le r^{\beta(r)}$ holds for arbitrarily large values of r while $T(r,f)\ge\alpha(r)$ holds for all sufficiently large r.*

COROLLARY. (a) *There exist a meromorphic function f and an entire transcendental function g such that $\rho_l(f\circ g)=0$ and $\rho_l(f)=\infty$ for all $l\ge 1$.*

(b) *There exist a meromorphic function f and an entire transcendental function g such that $\lambda_l(f\circ g)=0$ and $\lambda_l(f)=\infty$ for all $l\ge 1$.*

3. Proof of Theorem 1

The key step in the proof of Theorem 1 is the following lemma. The proof will use an idea due to Clunie [3].

LEMMA 1. *Let g be entire and transcendental and let $K, N, \mu>0$. Then there exist $r_0, R_0>0$ such that if $r\ge r_0$, $T(r,g)\le r^{\mu}$, and $R_0<|\omega|\le r^N$, then $n(r,\omega,g)\ge K$.*

Proof. Suppose that the lemma is false. Then there exist sequences $\{r_k\}$ and $\{\omega_k\}$ satisfying $r_k\to\infty$ and $|\omega_k|\to\infty$ as $k\to\infty$ such that $T(r_k,g)\le r_k^{\mu}$, $|\omega_k|\le r_k^N$, and $n(r_k,\omega_k,g)<K$.

For sufficiently large k we have $n(1,\omega_k,g)=0$ and therefore $N(r_k,\omega_k,g)<K\log r_k$. This implies that

$$N\left(\frac{r_k}{2},\infty,\frac{g'}{g-\omega_k}\right)\le N\left(\frac{r_k}{2},\omega_k,g\right)\le K\log r_k.$$

Using Hayman's estimate of the logarithmic derivative [7, Lemma 2.3] for $r=r_k/2$, $R=r_k$, and $f(z)=g(z)-\omega_k$, we obtain

$$m\left(\frac{r_k}{2}, \infty, \frac{g'}{g-\omega_k}\right) < 4 \overset{+}{\log} T(r_k, g-\omega_k) + 4 \overset{+}{\log} \overset{+}{\log} \left| \frac{1}{g(0)-\omega_k} \right|$$
$$+5 \overset{+}{\log} r_k + 6 \overset{+}{\log} \frac{2}{r_k} + \overset{+}{\log} \frac{2}{r_k} + 14$$
$$\leq 4 \overset{+}{\log} T(r_k, g) + 4 \overset{+}{\log} \overset{+}{\log} |\omega_k| + 6 \overset{+}{\log} r_k$$
$$\leq (4\mu+7) \overset{+}{\log} r_k$$

for sufficiently large k. It follows that

$$T\left(\frac{r_k}{2}, \frac{g'}{g-\omega_k}\right) \leq (4\mu+K+7) \overset{+}{\log} r_k.$$

If we assume for simplicity that $g'(0) \neq 0$, then Nevanlinna's first fundamental theorem [7, equation (1.10)] yields

$$T\left(\frac{r_k}{2}, \frac{g-\omega_k}{g'}\right) = T\left(\frac{r_k}{2}, \frac{g'}{g-\omega_k}\right) - \log \left| \frac{g'(0)}{g(0)-\omega_k} \right|$$
$$\leq (4\mu+K+7) \log r_k - \log|g'(0)| + \overset{+}{\log}|g(0)| \tag{3.1}$$
$$+ \overset{+}{\log}|\omega_k| + \log 2$$
$$\leq (4\mu+K+N+8) \log r_k$$

for sufficiently large k. In particular, we have

$$N\left(\frac{r_k}{2}, \infty, \frac{g-\omega_k}{g'}\right) \leq (4\mu+K+N+8) \log r_k,$$

and therefore

$$N\left(\frac{r_k}{2}, 0, g'\right) \leq N\left(\frac{r_k}{2}, \infty, \frac{g-\omega_k}{g'}\right) + N\left(\frac{r_k}{2}, \omega_k, g\right)$$
$$\leq (4\mu+2K+N+8) \log r_k.$$

This implies that g' has only finitely many zeros. Since $T(r_k, g) \leq r_k^{\mu}$ we have $\lambda(g) \leq \mu < \infty$, so that $g'(z) = P(z) \exp Q(z)$, where P and Q are polynomials. Let n be the degree of Q. Then there exist $\tau > 0$ and an interval I in $[0, 2\pi]$ such that $|g'(re^{i\theta})| \leq \exp(-\tau r^n)$, if $\theta \in I$ and if r is large enough. Hence $|g(re^{i\theta})| = O(1)$ as $r \to \infty$ uniformly in $\theta \in I$. It follows that

$$m\left(\frac{r_k}{2}, \infty, \frac{g-\omega_k}{g'}\right) \geq c r_k^n$$

for some $c > 0$, if k is sufficiently large. This contradicts (3.1) and the lemma is proved. □

Proof of Theorem 1. Let $\mu > 2\lambda(f \circ g)$. Then there exist arbitrarily large r such that

$$T(r^2, f \circ g) \leq \tfrac{1}{2} r^{\mu}. \tag{3.2}$$

For large values of r we have (cf. [9, p. 147])

$$T(r,g)\le 2T(2r,f\circ g)\le 2T(r^2,f\circ g),$$

so that (3.2) implies that $T(r,g)\le r^{\mu}$.

Now let $b\in\mathbf{C}$ and let $a_1,a_2,\ldots$ be the b-points of f. Then

$$n(r,b,f\circ g)=\sum_{|a_j|\le M(r,g)} n(r,a_j,g)$$

and Lemma 1 implies (for $K=1$) that

$$\begin{aligned} n(r,b,f\circ g) &\ge \sum_{R_0<|a_j|\le r^N} n(r,a_j,g)\\ &\ge n(r^N,b,f)-n(R_0,b,f). \end{aligned}$$

Setting (cf. [4, p. 341])

$$s(r,f)=\frac{1}{2\pi}\int_0^{2\pi} n(r,e^{i\theta},f)\,d\theta,$$

we find

$$s(r,f\circ g)\ge s(r^N,f)-s(R_0,f).$$

By Cartan's theorem we have

$$\begin{aligned} T(r^2,f\circ g) &=\int_0^{r^2}\frac{s(t,f\circ g)}{t}\,dt+O(1)\\ &\ge \int_r^{r^2}\frac{s(t,f\circ g)}{t}\,dt+O(1)\\ &\ge s(r,f\circ g)\log r+O(1) \end{aligned}$$

and

$$T(r^N,f)\le s(r^N,f)N\log r+O(1).$$

Altogether we have

$$\begin{aligned} T(r^N,f) &\le NT(r^2,f\circ g)+O(\log r)\\ &\le \frac{N}{2}r^{\mu}+O(\log r)\le Nr^{\mu}. \end{aligned} \tag{3.3}$$

This holds for arbitrarily large values of r. It follows that $\lambda(f)\le\mu/N$ and this proves the theorem, since N can be chosen arbitrarily large. □

REMARK. If $\rho(f\circ g)<\infty$ and if $\mu>2\rho(f\circ g)$, then (3.2) and hence (3.3) hold for all large r. It follows that $\rho(f)=0$. Hence our method also yields a new proof of the theorem of Edrei and Fuchs.

4. A Theorem of Valiron and an Example of Hayman

The key step in the proof of Pólya's theorem for meromorphic f by Edrei and Fuchs was the following result due to Valiron ([14], [15]).

THEOREM D. *Let g be an entire transcendental function satisfying $\rho(g)<\infty$. Let $\xi>0$. For $|\omega|>|g(0)|$ define $t=t(|\omega|)$ by $|\omega|=M(t,g)$. Then the equation $g(z)=\omega$ has a solution in $|z|<t^{1+\xi}$, provided $|\omega|>K(g,\xi)$.*

Hayman [8] has given examples of functions which do not satisfy the conclusion of Theorem D. Of course, these functions have infinite order. We construct similar examples which are, however, subject to certain additional growth restrictions.

THEOREM 3. *Let $\alpha(r)$ and $\beta(r)$ be as in Theorem 2. Then there exist real sequences $\{R_n\}$ and $\{T_n\}$, a complex sequence $\{\omega_n\}$, and an entire transcendental function g with the following properties:*

$$\lim_{n\to\infty}|\omega_n|=\lim_{n\to\infty}R_n=\lim_{n\to\infty}T_n=\infty, \tag{4.1}$$

$$g(z)\neq\omega_n \quad \textit{if } |z|\le R_n, \tag{4.2}$$

$$R_n\ge\alpha(|\omega_n|) \quad \textit{and} \quad R_n\ge|\omega_n|^2, \tag{4.3}$$

$$\log M(T_n,g)\le\beta(T_n)\log T_n, \tag{4.4}$$

$$\log M(r,g)\le r^{\beta(r)} \quad \textit{for all sufficiently large } r. \tag{4.5}$$

Proof. We will define the sequences $\{R_n\}$, $\{T_n\}$, $\{\omega_n\}$ and a sequence $\{p_n\}$ of polynomials by recursion. The function g will be defined by

$$g(z)=p_1(z)+\sum_{n=1}^{\infty}(p_{n+1}(z)-p_n(z)), \tag{4.6}$$

where the series will converge uniformly on compact sets.

For abbreviation let $\gamma(r)=r^{\beta(r)-1}$. Choose $\omega_1\in\mathbf{C}$ and R_1 such that

$$R_1\ge 1,\quad R_1\ge\alpha(|\omega_1|),\quad R_1\ge|\omega_1|^2,\quad \gamma(R_1)>1, \tag{4.7}$$

and let

$$p_1(z)=\omega_1+\sum_{n=0}^{N_1}\frac{z^n}{n!}, \tag{4.8}$$

where $N_1\ge 1$ is chosen such that $p_1(z)\neq\omega_1$ for $|z|\le R_1$. It follows that

$$\mu_1=\min_{|z|\le R_1}|p_1(z)-\omega_1|>0.$$

Finally, choose $T_1>R_1$ such that $4\gamma(T_1)^{-1}\le\mu_1$, $\gamma(T_1)+\mu_1/3\le T_1\gamma(T_1)$, and $M(r,p_1)\le\gamma(r)$ for $r\ge T_1$.

Assume now that $n\ge 2$ and that ω_k, R_k, T_k, and p_k have been specified for $k=1,\ldots,n-1$. Choose $\omega_n\in\mathbf{C}$ such that

$$|\omega_n|=\gamma(T_{n-1})^3 \tag{4.9}$$

and R_n so large that

$$R_n\ge n,\quad R_n\ge\alpha(|\omega_n|),\quad R_n\ge|\omega_n|^2,\quad R_n>T_{n-1}, \tag{4.10}$$

and

$$\mu_n \le \tfrac{1}{4}\mu_{n-1}, \tag{4.11}$$

where $\mu_k = \frac{1}{2}\exp(-\gamma(R_k))$ for $k \ge 2$. Now put

$$S_n(z) = -\frac{p_{n-1}(z)}{\omega_n} \quad \text{and} \quad q_n(z) = \omega_n(1 - \exp(S_n(z))).$$

Let

$$\eta_n = \frac{1}{2} \min_{|z| \le R_n} |\exp(S_n(z))|,$$

and define

$$p_n(z) = -\omega_n\left(\sum_{k=1}^{l_n} \frac{1}{k!}(S_n(z))^k\right),$$

where l_n is chosen so that

$$|p_n(z) - q_n(z)| = \left|\omega_n \sum_{k=l_n+1}^{\infty} \frac{1}{k!}(S_n(z))^k\right| \le \eta_n \tag{4.12}$$

for $|z| \le R_n$. Finally, choose $T_n > R_n$ such that

$$4\gamma(T_n)^{-1} \le \mu_n, \tag{4.13}$$

$$\gamma(T_n) + \frac{4^{1-n}}{3}\mu_1 \le T_n\gamma(T_n), \tag{4.14}$$

and

$$M(r, p_n) \le \gamma(r) \tag{4.15}$$

for $r \ge T_n$.

Now let $\{R_n\}$, $\{T_n\}$, $\{\omega_n\}$, and $\{p_n\}$ be sequences as defined above and let μ_n, S_n, q_n, and η_n be the corresponding auxiliary quantities. Then (4.9)-(4.15) hold for all $n \ge 2$ and (4.13)-(4.15) also hold for $n = 1$. Moreover, we have $T_n > R_n \ge n$ for all n. This and (4.9) imply that (4.1) holds. Furthermore, (4.3) follows from (4.7) and (4.10).

It remains to show that (4.6) defines an entire transcendental function g which satisfies (4.2), (4.4), and (4.5). It follows from (4.9) and (4.15) that

$$|S_n(z)| = \left|\frac{p_{n-1}(z)}{\omega_n}\right| \le \frac{M(T_{n-1}, p_{n-1})}{|\omega_n|} \le \frac{\gamma(T_{n-1})}{|\omega_n|} = \gamma(T_{n-1})^{-2}$$

for $n \ge 2$ and $|z| \le T_{n-1}$. This, together with (4.9) and (4.13), implies that

$$\begin{aligned} |p_n(z) - p_{n-1}(z)| &= \left|\omega_n \sum_{k=2}^{l_n} \frac{1}{k!}(S_n(z))^k\right| \\ &\le \gamma(T_{n-1})^3 \sum_{k=2}^{l_n} \frac{1}{k!}\gamma(T_{n-1})^{-2k} \\ &\le \gamma(T_{n-1})^{-1} \sum_{k=2}^{\infty} \frac{1}{k!} \le \gamma(T_{n-1})^{-1} \le \frac{\mu_{n-1}}{4}. \end{aligned}$$

Since $\mu_j \le \mu_n 4^{n-j}$ for $j > n$ by (4.11), we have

$$\left|\sum_{k=n+1}^{\infty}(p_k(z)-p_{k-1}(z))\right| \le \sum_{k=n+1}^{\infty}\frac{\mu_{k-1}}{4} \tag{4.16}$$
$$\le \mu_n \sum_{k=n+1}^{\infty} 4^{n-k} = \frac{1}{3}\mu_n \le \frac{4^{1-n}}{3}\mu_1$$

if $n \ge 1$ and $|z| \le T_n$. It follows that the series (4.6) converges uniformly on every compact set, and hence it defines an entire function g.

To prove (4.2), we note first that $|\omega_n| = \gamma(T_{n-1})^3 > \gamma(R_1)^3 > 1$. This, together with (4.12), implies that for $n \ge 2$ and $|z| \le R_n$,

$$|p_n(z)-\omega_n| \ge |q_n(z)-\omega_n| - |p_n(z)-q_n(z)|$$
$$\ge |\omega_n \exp(S_n(z))| - \eta_n \ge 2\eta_n - \eta_n = \eta_n.$$

Using $R_n \ge T_{n-1}$ and (4.15) we obtain

$$\eta_n \ge \tfrac{1}{2}\exp(-M(R_n, S_n)) \ge \tfrac{1}{2}\exp(-M(R_n, p_{n-1}))$$
$$\ge \tfrac{1}{2}\exp(-\gamma(R_n)) = \mu_n.$$

Altogether we have

$$|p_n(z)-\omega_n| \ge \mu_n \tag{4.17}$$

for $|z| \le R_n$ and $n \ge 2$. By definition of μ_1, however, (4.17) holds also for $n=1$. It follows from (4.16) and (4.17) that, for $n \ge 1$ and $|z| \le R_n < T_n$,

$$|g(z)-\omega_n| \ge |p_n(z)-\omega_n| - \left|\sum_{k=n+1}^{\infty}(p_k(z)-p_{k-1}(z))\right|$$
$$\ge \mu_n - \frac{1}{3}\mu_n = \frac{2}{3}\mu_n > 0.$$

This proves (4.2).

Furthermore, (4.4) follows since, for $|z| = T_n$,

$$|g(z)| \le |p_n(z)| + \left|\sum_{k=n+1}^{\infty}(p_k(z)-p_{k-1}(z))\right|$$
$$\le \gamma(T_n) + \frac{4^{1-n}}{3}\mu_1 \le T_n\gamma(T_n) = T_n^{\beta(T_n)}$$

by (4.14), (4.15), and (4.16).

To prove (4.5), let $T_{n-1} \le |z| = r \le T_n$. Then

$$|p_n(z)| \le |\omega_n| \sum_{k=1}^{l_n}\frac{1}{k!}|S_n(z)|^k$$
$$\le |\omega_n| \exp|S_n(z)|$$
$$\le \gamma(T_{n-1})^3 \exp|p_{n-1}(z)|$$
$$\le \gamma(r)^3 \exp\gamma(r)$$

by (4.9) and (4.15). This, together with (4.16), implies that

$$|g(z)|\le|p_n(z)|+\left|\sum_{k=n+1}^{\infty}(p_k(z)-p_{k-1}(z))\right|$$
$$\le\gamma(r)^3\exp\gamma(r)+\frac{\mu_1}{3}.$$

It follows that

$$M(r,g)\le\exp 2\gamma(r)\le\exp r^{\beta(r)}$$

for sufficiently large r, which is (4.5).

It remains to show that g is transcendental. It follows from (4.6) and (4.16) that $|g(z)-p_1(z)|\le\mu_1/3$ if $|z|\le T_1$. Since $T_1>R_1\ge 1$, Cauchy's inequality yields $|g'(0)-p_1'(0)|\le\mu_1/3$. Since $p_1'(0)=1$ and $\mu_1\le|p_1(0)-\omega_1|=1$, we have $|g'(0)-1|\le 1/3$ and it follows that g is not constant.

Suppose that g is a polynomial of degree m—say, $g(z)=a_m z^m+\cdots+a_0$. If

$$|\omega|\le\frac{|a_m|}{2}R_n^m$$

and if n is sufficiently large, then the equation $g(z)=\omega$ has a solution in $|z|\le R_n$. It follows that

$$|\omega_n|>\frac{|a_m|}{2}R_n^m.$$

This is a contradiction for large n, since $|\omega_n|^2\le R_n$ and $m\ge 1$. Hence g is transcendental and the proof is complete. □

REMARKS. Hayman [8] essentially constructed entire functions g which satisfy (4.1), (4.2), and (4.3). It is not difficult to show that this implies that g does not satisfy the conclusion of Theorem D (cf. [8]). In Theorem 3 we have the additional growth restrictions (4.4) and (4.5).

Condition (4.5) shows that the hypothesis $\rho(g)<\infty$ in Theorem D is best possible in a certain sense. We will need (4.5) for the proof of Theorem 2.

Condition (4.4) shows that g may have finite lower order. In fact, if $\beta(r)=\log r$ then $\lambda(g)=0$. This shows that Theorem D does not hold with $\rho(g)<\infty$ replaced by $\lambda(g)<\infty$. This is, however, claimed by Song and Yang [13, Lemma 1] and used in their proof of Theorem 1 [13, Cor. 1]. The fact that the conclusion of Theorem D fails to be true for certain functions of finite lower order shows that the method of Edrei and Fuchs does not carry over to the case of lower order.

If one assumes that g is of finite order, then of course Theorem D can be used. With this additional hypothesis, even Theorem C and the corresponding result for the lower l-order hold for meromorphic f ([2, Satz 5.2], [9, Satz 16.7, Satz 16.9]).

5. Proof of Theorem 2

Clunie [3] has constructed a meromorphic function f and an entire function g such that

$$\liminf_{r\to\infty} \frac{T(r, f\circ g)}{T(r, f)} = 0.$$

To construct g, he used Hayman's result [8]. Instead of Hayman's result we will use the more precise Theorem 3, since we need the growth restriction (4.5). We also will use an infinite series to define f, instead of the infinite product used by Clunie. Although this is not essential, it does simplify the computations.

Proof of Theorem 2. First of all, let $\alpha^*(r)=\alpha(er)$ and $\beta^*(r)=\beta(r)-2$. Let g be a transcendental entire function with corresponding sequences $\{\omega_n\}$ and $\{R_n\}$ such that the conclusion of Theorem 3 is satisfied with $\alpha^*(r)$ and $\beta^*(r)$ instead of $\alpha(r)$ and $\beta(r)$. (We do not need the sequence $\{T_n\}$ and (4.4) here.) We may assume that

$$R_n \geq n+1; \tag{5.1}$$

otherwise, we may consider a suitable subsequence.

Let $\{k_n\}$ be a sequence of positive integers (to be specified later) which satisfies

$$\sum_{j=1}^{n-1} k_j \leq k_n \tag{5.2}$$

for $n \geq 2$. Let

$$\gamma_n = \min_{|z|\leq R_n} |g(z)-\omega_n|. \tag{5.3}$$

There exists a sequence $\{\delta_n\}$ of positive real numbers such that

$$\sum_{n=1}^{\infty} \delta_n \leq 1 \tag{5.4}$$

and

$$\sum_{n=1}^{\infty} \delta_n(\gamma_n)^{-k_n} \leq 1. \tag{5.5}$$

It follows from (4.1) and (5.4) that

$$f(z) = \sum_{n=1}^{\infty} \frac{\delta_n}{(z-\omega_n)^{k_n}}$$

defines a meromorphic function f.

For $r \leq R_{n+1}$ we have

$$N(r, \infty, f\circ g) = \sum_{j=1}^{n} k_j N(r, \omega_j, g). \tag{5.6}$$

The first fundamental theorem implies that

$$m(r, \omega_j, g)+N(r, \omega_j, g) \leq T(r, g)+O(1) \leq 2T(r, g) \tag{5.7}$$

for sufficiently large r and for all $j \geq 1$, since by (4.1) $g(0)$ is not a limit point of $\{\omega_j\}$. It follows from (5.2), (5.6), and (5.7) that

$$N(r, \infty, f\circ g) \leq 2T(r, g) \sum_{j=1}^{n} k_j \leq 4k_n T(r, g), \tag{5.8}$$

if $r \le R_{n+1}$ and if r is sufficiently large. Moreover, (5.1)-(5.5) and (5.7) imply that, for $R_n \le r \le R_{n+1}$ with r sufficiently large,

$$\begin{aligned} m(r, f\circ g) &= \frac{1}{2\pi}\int_0^{2\pi} \overset{+}{\log}\left|\sum_{j=1}^{\infty} \frac{\delta_j}{(g(re^{i\theta})-\omega_j)^{k_j}}\right| d\theta \\ &\le \frac{1}{2\pi}\int_0^{2\pi} \sum_{j=1}^{n} \overset{+}{\log}\left|\frac{\delta_j}{(g(re^{i\theta})-\omega_j)^{k_j}}\right| d\theta \\ &\quad + \overset{+}{\log} \sum_{j=n+1}^{\infty} \delta_j(\gamma_j)^{-k_j} + \log(n+1) \\ &\le \sum_{j=1}^{n} (k_j m(r, \omega_j, g) + \overset{+}{\log}\, \delta_j) + \log R_n \\ &\le 2T(r,g)\sum_{j=1}^{n} k_j + \log r \\ &\le 5k_n T(r,g). \end{aligned}$$

This, together with (5.8), yields that

$$T(r, f\circ g) \le 9k_n T(r,g) \tag{5.9}$$

if $R_n \le r \le R_{n+1}$ and if r is sufficiently large.

To prove (a), we choose k_n such that

$$\alpha^*(|\omega_n|) \le k_n < \alpha^*(|\omega_n|) + 1.$$

We may assume that $k_n \ge 1$, that (5.2) holds, and that $k_n \le 2\alpha^*(|\omega_n|)$; otherwise, we consider a suitable subsequence. It follows from (4.3), (4.5), and (5.9) that

$$\begin{aligned} T(r, f\circ g) &\le 18\alpha^*(|\omega_n|)T(r,g) \le 18R_n \log M(r,g) \le 18rr^{\beta^*(r)} \\ &\le r^2 r^{\beta^*(r)} = r^{\beta(r)} \end{aligned}$$

if $R_n \le r \le R_{n+1}$ and if r is large enough. On the other hand, we have

$$\begin{aligned} T(e|\omega_n|, f) &\ge N(e|\omega_n|, \infty, f) \ge \int_{|\omega_n|}^{e|\omega_n|} \frac{n(t,\infty,f)}{t}\,dt \ge n(|\omega_n|, \infty, f) \\ &= \sum_{j=1}^{n} k_j \ge k_n \ge \alpha^*(|\omega_n|) = \alpha(e|\omega_n|). \end{aligned}$$

This proves (a). To prove (b), we choose

$$\alpha^*(|\omega_{n+1}|) \le k_n < \alpha^*(|\omega_{n+1}|) + 1.$$

Again we may assume that $k_n \ge 1$, that (5.2) holds, and that $k_n \le 2\alpha^*(|\omega_{n+1}|)$. Analogously, we find

$$\begin{aligned} T(R_{n+1}, f\circ g) &\le 18\alpha^*(|\omega_{n+1}|)T(R_{n+1}, g) \le 18R_{n+1}R_{n+1}^{\beta^*(R_{n+1})} \\ &\le R_{n+1}^{\beta(R_{n+1})} \end{aligned}$$

for sufficiently large n; for $|\omega_n| \le R < |\omega_{n+1}|$,

$$T(eR,f) \geq N(eR,\infty,f) \geq n(R,\infty,f) \geq n(|\omega_n|,\infty,f)$$
$$\geq k_n \geq \alpha^*(|\omega_{n+1}|) \geq \alpha^*(R) = \alpha(eR).$$

This proves (b). □

Proof of the Corollary. The conclusion follows from elementary computations, if we choose $\beta(r) = \log r$ and $\alpha(r) = \exp_{[r]} r$, the $[r]$-times iterated exponential function. Here $[r]$ is the greatest integer function.

REMARKS. (1) It is not essential in Theorem 2 and 3 that $\alpha(r)$ and $\beta(r)$ be nondecreasing; this was assumed only to simplify the proofs. It is essential, however, that $\beta(r) \to \infty$ as $r \to \infty$. If $\beta(r)$ is bounded, then the conclusion of Theorem 3 does not hold because of Theorem D. The theorem of Edrei and Fuchs and Theorem 1 show that Theorem 2(a) and (b) do not hold for a bounded function $\beta(r)$.

(2) Schönhage obtained Theorem C as a consequence of a more general result which he called "Multiplikationssatz" ([12, Satz 6], [9, Satz 16.1]). Jank and Volkmann [9, Bemerkung 16.10] asked whether this result holds for meromorphic f. This is not the case: as shown by the corollary, Theorem C does not hold for meromorphic f.

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