# A Maximum Principle for Sums of Subharmonic Functions, and the Convexity of Level Sets

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## Introduction

We show in this paper that certain subharmonic functions satisfy a maximum principle on certain subvarieties of their domains of definition. Its statement (Theorem 1.4) is somewhat reminiscent of the fact that restrictions of plurisubharmonic functions in $\mathbf{C}^n$ to the zero-variety $Z$ of some holomorphic function have no strict local maxima on $Z$. We discovered it in the course of proving the following convexity property of harmonic functions.

THEOREM I. *If* $n \geq 2$ *and*

(a) $W$ *is a bounded convex open subset of* $R^n$,
(b) $K$ *is a compact convex subset of* $W$,
(c) $\Omega = W \setminus K$,
(d) $u: \bar{\Omega} \to R$ *is continuous, harmonic in* $\Omega$, *and* $u = 0$ *on* $\partial K$, $u = 1$ *on* $\partial W$,

*then every level set of* $u$ *in* $\Omega$ *is a strictly convex hypersurface.*

Only after proving this did we realize that we were not the first to do so. In fact, there exist several proofs of this and of related results ([1], [2], [8], [9], [10]). However, our proof is different from these; it is quite elementary, and it gives some new *quantitative* information: Theorem 4.5 shows that all level sets of $u$ are "at least as convex" as are $\partial W$ and $\partial K$. (Similar quantitative information has been found quite recently by Dennis Stowe, again by entirely different methods.) We also hope that our maximum principle may have some further applications.

The term "strictly convex" refers to the Hessian of $u$. This is defined to be the quadratic form

$$H_p(u, \xi) = \sum_{i,j=1}^{n} \frac{\partial^2 u}{\partial x_i \partial x_j}(p) \xi_i \xi_j,$$

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where $p \in \Omega$ and $\xi = (\xi_1, \ldots, \xi_n) \in R^n$. Since the gradient $\nabla u$ of $u$ never vanishes in $\Omega$ (for completeness, we prove this very easy fact in Section 2, although it is not at all new; see [5]), to say that the level set

$$S_p = \{x \in \Omega : u(x) = u(p)\}$$

is strictly convex at $p$ means, by definition, that $H_p(u, \xi) > 0$ for every $\xi$ tangent to $S_p$ at $p$—that is, for every $\xi$ which satisfies $\xi \cdot (\nabla u)(p) = 0$. The dot refers here to the ordinary scalar product in $R^n$.

As usual, $\Delta$ will denote the Laplace operator.

It is an easy corollary of Theorem I that Green's functions of convex regions in $R^n$ have convex level surfaces. This was first proved by Gabriel ([3], [4], [5]).

## 1. A Maximum Principle

1.1. If in some open set of $R^n \times R^n$, where variables are denoted by $(x, y)$, a function $h$ is the sum of two subharmonic functions, of the form $h(x,y) = h_1(x) + h_2(y)$, and if $\Sigma$ is a hyperplane in $R^{2n}$, then the restriction of $h$ to $\Sigma$ cannot have any local strict maximum. Of course such a result is not true for arbitrary subharmonic functions $h$, nor is it true if $\Sigma$ is replaced by a subspace of lower dimension.

In this section we will show that there is a larger class of hypersurfaces on which such a "maximum principle for restrictions" is valid.

1.2. DEFINITIONS. If $h$ is a function defined in an open set $W \subset R^{2n}$, where $R^{2n} = R^n_x \times R^n_y$, and $h$ is locally a sum

(1) $$h(x,y) = h_1(x) + h_2(y)$$

of two real-valued harmonic functions $h_i$, then we say that $h \in \mathcal{H}(W)$.

If $h$ has a local representation (1) in which both $h_1$ and $h_2$ are strictly subharmonic (i.e., they are of class $C^2$ and their Laplacians satisfy $\Delta h_1 > 0$, $\Delta h_2 > 0$), then we say that $h \in \mathcal{H}^+(W)$.

Theorem 1.4 will show that the maximum principle described in Section 1.1 holds if $\Sigma$ is the zero-variety of any function of class $\mathcal{H}$ whose gradient has no zero on this variety. The following lemma describes a special kind of embedding of an open subset of $R^n$ into $\Sigma$ (whose dimension is $2n-1$), through any preassigned point $(x_0, y_0) \in \Sigma$; this will be used in the proofs of Theorems 1.4 and 4.3. For simplicity, we take $(x_0, y_0) = (0, 0)$.

1.3. LEMMA. *Suppose that* $\rho \in \mathcal{H}(W)$, *where* $W$ *is a neighborhood of* $(0,0)$ *in* $R^n \times R^n$ *in which*

$$\rho(x,y) = \rho_1(x) + \rho_2(y), \qquad \rho_1(0) = \rho_2(0) = 0.$$

*Put* $a = (\nabla \rho_1)(0)$, $b = (\nabla \rho_2)(0)$, *and assume* $b \neq 0$; *thus* $b = |b|\xi$, $|b| > 0$, *and* $\xi$ *is a unit vector in* $R^n$. *Then there exist*

(i) *a real number* $c$;
(ii) *a linear isometry* $L: R^n \to R^n$;
(iii) *a neighborhood* $V$ *of* $0$ *in* $R^n$;
(iv) *a harmonic function* $f: V \to R$, $f(s) = O(|s|^2)$; *and*
(v) *a real-analytic function* $\alpha: V \to R$, $\alpha(s) = O(|s|^3)$, *so that the map* $\Psi: V \to R^n \times R^n$ *defined by*

(1) $$\Psi(s) = (s, cLs + f(s)\xi + \alpha(s)\xi)$$

*satisfies*

(2) $$\rho(\Psi(s)) = 0$$

*for every* $s \in V$.

*Proof.* The Taylor expansion of $\rho$, of order 2, is

(3) $$\rho(x, y) = a \cdot x + b \cdot y + Q_1(x) + Q_2(y) + \cdots,$$

where $Q_1$ and $Q_2$ are quadratic forms.

Put $c = |a|/|b|$. If $a = 0$, put $Ls = s$. If $a \neq 0$, choose $L$ so that $La = -cb$; in that case

$$a \cdot s = La \cdot Ls = -cb \cdot Ls = -b \cdot (cLs).$$

Hence, in either case,

(4) $$a \cdot s + b \cdot (cLs) = 0$$

for all $s \in R^n$, and (3) becomes

(5) $$\rho(s, cLs) = Q_1(s) + c^2 Q_2(Ls) + O(|s|^3).$$

Next, define $f(s) = -\rho(s, cLs)/|b|$. Since $\rho \in \mathcal{H}$ and $L$ is an isometry, $f$ is harmonic in some neighborhood of $s = 0$. Hence (iv) is satisfied, because of (5). Since (4) holds, and $b \cdot f(s)\xi = -\rho(s, cLs)$, it follows from (3) and (5) that

(6) $$\rho(s, cLs + f(s)\xi) = O(|s|^3).$$

To find $\alpha$, set

(7) $$\tau(s, t) = (s, \rho(s, cLs + f(s)\xi + t\xi)).$$

Then $\tau$ is a real-analytic diffeomorphism from a neighborhood of the origin in $R^n \times R$ into $R^n \times R$; note that its Jacobian at the origin is

$$\frac{\partial \tau}{\partial t} = \xi \cdot (\nabla \rho_2)(0) = \xi \cdot b = |b| > 0.$$

Hence there is a function $\alpha$, real-analytic in some neighborhood of 0 in $R^n$, which satisfies

(8) $$(s, \alpha(s)) = \tau^{-1}(s, 0).$$

If $\Psi$ is now defined by (1), it follows from (7) and (8) that (2) is satisfied.

Finally, if we let $M$ be an upper bound for $|\nabla(\tau^{-1})|$ near 0, we obtain the estimate

$$\begin{aligned}|\alpha(s)| &= |(s,0)-(s,\alpha(s))| \le M|\tau(s,0)-\tau(s,\alpha(s))| \\ &= M|\rho(s,cLs+f(s)\xi)| = O(|s|^3)\end{aligned}$$

by (6). This completes the proof. □

1.4. THEOREM. *If $\rho$ is as in the lemma, $\Sigma=\{\rho=0\}$, and $h\in\mathcal{H}^+(W)$, then the restriction of $h$ to $\Sigma$ does not have a local maximum at* $(0,0)$.

*Proof.* Choose $\Psi$ as in Lemma 1.3. Note that $\Psi$ depends only on $\rho$, not on $h$, and that $\Psi(V)$ is an $n$-dimensional submanifold of $\Sigma$, while the dimension of $\Sigma$ is $2n-1$. We shall prove a somewhat more precise result than the theorem asserts—namely, *the restriction of $h$ to $\Psi(V)$ does not have a local maximum at* $(0,0)$—by showing that

(1) $$\Delta(h\circ\Psi)(0)>0.$$

We have $h(x,y)=h_1(x)+h_2(y)$. Thus

(2) $$(h\circ\Psi)(s)=h_1(s)+\beta(s),$$

where

(3) $$\beta(s)=h_2(cLs+f(s)\xi+\alpha(s)\xi).$$

The quadratic term in the Taylor expansion of $\beta(s)-h_2(cLs)$ about $s=0$ is harmonic, because it is $f(s)((\nabla h_2)(0)\cdot\xi)$. Since $h_2$ is subharmonic, it follows that $(\Delta\beta)(0)\ge 0$.

Also, $(\Delta h_1)(0)>0$, by assumption. Hence (1) follows from (2). □

## 2. Starshaped Regions

2.1. DEFINITION. A set $E\subset R^n$ is *starshaped* (relative to the origin) if $tx\in E$ for all $x\in E$ and all $t\in[0,1]$.

2.2. THEOREM. *Suppose $0\in K\subset W\subset R^n$, $K$ is compact, $W$ is bounded and open, and both are starshaped. Put $\Omega=W\backslash K$.*

*If $u:\bar{\Omega}\to R$ is continuous, harmonic in $\Omega$, and $u=0$ on $\partial K$, $u=1$ on $\partial W$, then the radial derivative of $u$ is strictly positive at all points of $\Omega$. Each level set of $u$ is therefore the boundary of a starshaped region.*

*Proof.* If $t>1$, so close to 1 that $tK\subset W$ (where $tK$ is the set of all vectors $tx$, $x\in K$), and if $1-\epsilon=1/t$, then the differences

$$D_\epsilon(x)=u(x)-u((1-\epsilon)x)$$

are harmonic in $W\backslash tK$.

If $x\in\partial W$, then $D_\epsilon(x)=1-u((1-\epsilon)x)\ge 0$. If $x\in\partial(tK)$, then $D_\epsilon(x)=u(x)\ge 0$. Hence $D_\epsilon(x)\ge 0$ for all $x\in W\backslash tK$.

Setting $r=|x|$ it follows that

$$r\frac{\partial u}{\partial r}(x)=\lim_{\epsilon\to 0}\frac{D_\epsilon(x)}{\epsilon}$$

is harmonic in $\Omega$ and is $\geq 0$. It is obviously not $\equiv 0$. Hence $\partial u/\partial r>0$ at all points of $\Omega$. □

## 3. Two Notions of Convexity

3.1. The construction described in Lemma 1.3 will be used in Section 4 to prove a version of Theorem I which is more precise (i.e., more quantitative) than the statement that we gave in the Introduction. In order to do this, we shall describe strict convexity not in terms of Hessians, as in the Introduction, but in terms of an inequality which has a more geometric flavor. The *local* relation between these two notions of convexity is described by Proposition 3.2.

Let $V$ be a convex open set in $R^n$, and suppose that $u: V\to R$ is of class $C^2$ and that $\nabla u$ vanishes at no point of $V$. Consider the following two statements about $u$:

(A) there is a constant $c_0>0$ so that

$$u\left(\frac{x+y}{2}\right)\leq\frac{u(x)+u(y)}{2}-c_0|x-y|^2$$

for all $(x,y)\in V\times V$ for which $u(x)=u(y)$;

(B) there is a constant $c_1>0$ so that

$$H_p(u,\xi)\geq c_1$$

for all $p\in V$ and all unit vectors $\xi$ perpendicular to $(\nabla u)(p)$.

Two comments about (A) seem called for.

(i) Even though $u(x)=u(y)$ is assumed, it seems appropriate, in a statement that describes convexity, to have their average appear in (A).

(ii) If we think of (A) locally, we can take $V$ so small that $|\nabla u|$ is bounded and bounded from 0 in $V$. Then (A) holds if and only if

$$\text{(A}'\text{)}\qquad u\left(\frac{x+y}{2}\right)\leq\frac{u(x)+u(y)}{2}-\gamma_0|x-y|^2\left|\nabla u\left(\frac{x+y}{2}\right)\right|$$

for some $\gamma_0>0$. One advantage of (A′) over (A) is that if (A′) holds for $u$ then it also holds for every positive multiple of $u$, *with the same* $\gamma_0$. The size of $\gamma_0$ may thus be viewed as a measure of convexity: If (A′) holds for some other function (say, $u'$) with $\gamma_0'>\gamma_0$, then the level sets of $u'$ in $V$ are "more convex" or "more curved" than those of $u$.

We shall work with (A) rather than (A′), because (A) is easier to handle, but it is the above-mentioned feature of (A′) which justifies calling Theorem 4.5 a quantitative version of Theorem I.

3.2. PROPOSITION. (i) *If* (A) *holds, then* (B) *holds with* $c_1 = 8c_0$.

(ii) *If* (B) *holds and* $8c_0 < c_1$ *then* (A) *holds, provided that* $|x-y|$ *is sufficiently small.*

*Proof of* (i). Fix $p \in V$, put $\eta = (\nabla u)(p)$, let $\xi$ be a unit vector so that $\xi \cdot \eta = 0$, and let

$$S_p = \{x \in V : u(x) = u(p)\}.$$

Since $\xi$ is tangent to $S_p$ at $p$, there is a $\delta > 0$ and a real function $\beta$ of the form $\beta(t) = \beta_0 t^2 + o(t^2)$, so that the points $x = x(t) = p + t\xi + \beta(t)\eta$ lie in $S_p$ for all $t \in (-\delta, \delta)$. Since $\xi \cdot \eta = 0$,

$$u(x) = u(p) + \beta(t)|\eta|^2 + \frac{t^2}{2} H_p(u, \xi) + o(t^2).$$

Similarly, if $y = x(-t)$ then

$$u(y) = u(p) + \beta(-t)|\eta|^2 + \frac{t^2}{2} H_p(u, \xi) + o(t^2)$$

and

$$u\left(\frac{x+y}{2}\right) = u(p) + \frac{\beta(t) + \beta(-t)}{2} |\eta|^2 + o(t^2).$$

These three Taylor expansions yield

$$\frac{u(x)+u(y)}{2} - u\left(\frac{x+y}{2}\right) = \frac{t^2}{2} H_p(u, \xi) + o(t^2).$$

By (A), the left side is at least $c_0|x-y|^2 = 4c_0 t^2$, and this gives (letting $t \to 0$) that $H_p(u, \xi) \geq 8c_0$.

*Proof of* (ii). Set $c_1 - 8c_0 = \epsilon$. There is a $\delta > 0$ with the following property: If $p \in V$, $x \in V$, and $|x-p| < \delta$, then the error term in the Taylor expansion of order 2, of $u(x)$ about $p$, is less than $\epsilon|x-p|^2$.

Suppose now that $x \in V$, $y \in V$, $u(x) = u(y)$, and $|x-y| < \delta$. Put $\xi = (x-y)/|x-y|$. Let $p \in [x, y]$ $(p \neq x, p \neq y)$ be so chosen that the restriction of $u$ to the interval $[x, y]$ has derivative 0 at $p$. (Note that $p$ exists because $u(x) = u(y)$.) Then $x = p + t\xi$ and $y = p - s\xi$ for some $t > 0$ and $s > 0$, and

$$\left| u(x) - u(p) - \frac{t^2}{2} H_p(u, \xi) \right| < \epsilon t^2,$$

$$\left| u(y) - u(p) - \frac{s^2}{2} H_p(u, \xi) \right| < \epsilon s^2,$$

$$\left| u\left(\frac{x+y}{2}\right) - u(p) - \frac{(t-s)^2}{8} H_p(u, \xi) \right| < \frac{\epsilon}{4}(t-s)^2.$$

Hence

$$\left| \frac{u(x)+u(y)}{2} - u\left(\frac{x+y}{2}\right) - \frac{(t+s)^2}{8} H_p(u, \xi) \right| < \epsilon(t+s)^2.$$

Since $t+s=|x-y|$, we conclude that

$$\frac{u(x)+u(y)}{2}-u\left(\frac{x+y}{2}\right)\geq\left(\frac{c_1}{8}-\epsilon\right)|x-y|^2=c_0|x-y|^2. \qquad \square$$

## 4. Convexity of Level Sets

Theorem 1.4 cannot be used directly to prove Theorem I because, in addition to points $x$ and $y$ in $\Omega$, their midpoint also plays a role. This forces us to split the proof into two cases, depending on the parity of the dimension of the space in which we work. The following simple fact from linear algebra explains why we need to do this. (Recall that harmonicity is preserved by multiples of isometries, and that the linear isometries are precisely the orthogonal transformations.)

4.1. LEMMA. (a) *If $\xi$ and $\eta$ are unit vectors in $R^{2k}$ then there is a linear isometry $T$ of $R^{2k}$ so that* (i) $T\eta=\xi$ *and* (ii) $\lambda I+T$ *is a multiple of an isometry of $R^{2k}$, for every $\lambda\in R$.*

(b) *If $S$, $T$, and $\alpha S+\beta T$ are linear isometries of $R^{2k+1}$, then $\alpha=0$ or $\beta=0$ or $T=S$ or $T=-S$.*

*Proof.* (a) Choose an orthonormal basis $\{e_1,\dots,e_{2k}\}$ so that $\eta=e_1$ and $\xi=\alpha e_1+\beta e_2$, and define $T$ by

$$Te_i=\begin{cases}\alpha e_i+\beta e_{i+1} & \text{if } i=1,3,\dots,2k-1,\\ -\beta e_{i-1}+\alpha e_i & \text{if } i=2,4,\dots,2k.\end{cases}$$

Then $T\eta=\xi$, $T$ is an orthogonal transformation, and if $c^2=(\alpha+\lambda)^2+\beta^2$ then $\lambda I+T=cU$ and $U$ is orthogonal. This proves (a). $\square$

We shall not use (b), and leave its proof as an exercise.

4.2. DEFINITIONS. We assume now, and throughout this section, that the hypotheses of Theorem I hold: $W$ is convex, bounded, and open in $R^n$; $K$ is compact and convex, $K\subset W$, $\Omega=W\backslash K$; and $u:\bar{\Omega}\to R$ is continuous, harmonic in $\Omega$, 0 on $\partial K$, and 1 on $\partial W$.

We define $\Sigma$ to be the set of all points $(x,y)\in\bar{\Omega}\times\bar{\Omega}$ such that $u(x)=u(y)$ and $(x+y)/2\in\bar{\Omega}$. We define $b\Sigma$ (the "boundary" of $\Sigma$) to be the set of those $(x,y)\in\Sigma$ for which either $x$ and $y$ are in $\partial W$ or $(x+y)/2\in\partial K$.

The following theorem shows that $b\Sigma$ shares some of the properties of a Silov boundary. This is the reason for the notation $b\Sigma$.

4.3. THEOREM. *In the above setting, let $h_1,\dots,h_M$ be continuous real-valued functions on $\bar{\Omega}$ which are harmonic in $\Omega$, and define $F:\Sigma\to R$ by*

$$F(x,y)=u\left(\frac{x+y}{2}\right)-\frac{u(x)+u(y)}{2}+\sum_{i=1}^{M}|h_i(x)-h_i(y)|^2. \tag{1}$$

*The maximum of $F$ over $\Sigma$ is then the same as its maximum over $b\Sigma$.*

*Proof.* Let $m = \max F$ on $b\Sigma$ and assume, to reach a contradiction, that $F(x,y) > m$ at some point of $\Sigma \setminus b\Sigma$. For sufficiently small $\epsilon > 0$, the max of $F(x,y) + \epsilon|x|^2$ relative to $\Sigma$ will then be attained only at points of $\Sigma \setminus b\Sigma$. Let $(x_0, y_0)$ be such a point.

We now split the proof.

(a) *Assume n is even.* We apply Lemma 1.3, with $u(x) - u(y)$ in place of $\rho(x,y)$, $\Omega$ in place of $W$, and $(x_0, y_0)$ in place of $(0,0)$. Recall that $\nabla u$ has no zero, by Theorem 2.2. The construction that proved Lemma 1.3 shows, when combined with Lemma 4.1(a), that there is a constant $c > 0$ and a linear isometry $L$ of $R^n$ such that

(2) $$L(\nabla u)(x_0) = c(\nabla u)(y_0)$$

*and* such that $I + cL$ is a multiple of an isometry.

Moreover, there is a unit vector $\xi$ [in the direction of $(\nabla u)(y_0)$], a harmonic function $f$ of the form

(3) $$f(s) = \{u(y_0 + cLs) - u(x_0 + s)\} / |(\nabla u)(y_0)|$$

which satisfies $f(s) = O(|s|^2)$, and a remainder $\alpha$ with $\alpha(s) = O(|s|^3)$, so that by setting

(4) $$x(s) = x_0 + s, \qquad y(s) = y_0 + cLs + f(s)\xi + \alpha(s)\xi$$

we have

(5) $$(x(s), y(s)) \in \Sigma$$

for all $s$ in some neighborhood $V$ of the origin of $R^n$.

Define

(6) $$\varphi(s) = F(x(s), y(s)) \quad (s \in V).$$

Since $\varphi(s) + \epsilon|x_0 + s|^2$ has a local maximum at $s = 0$, its Laplacian cannot be positive at $s = 0$. To get our contradiction, it is therefore enough to show that

(7) $$(\Delta\varphi)(0) \geq 0.$$

Since $u(y) = u(x)$ on $\Sigma$, the definition of $F$ shows that $\varphi = \varphi_1 - \varphi_2 + \varphi_3$, where

$$\varphi_1(s) = u\left(\frac{x(s) + y(s)}{2}\right), \qquad \varphi_2(s) = u(x_0 + s),$$

$$\varphi_3(s) = \sum_{i=1}^{M} |h_i(x(s)) - h_i(y(s))|^2.$$

The Taylor expansions of $\varphi_1$ and of

(8) $$\psi_1(s) = u\left(\frac{x_0 + y_0}{2} + \frac{1}{2}(I + cL)s\right)$$

about $s = 0$ show, by (4), that

(9) $$\varphi_1(s)-\psi_1(s)=\frac{1}{2}f(s)(\nabla u)\left(\frac{x_0+y_0}{2}\right)\cdot\xi+O(|s|^3).$$

Since $f$ is harmonic, $(\Delta\varphi_1)(0)=(\Delta\psi_1)(0)$. Since $I+cL$ is a multiple of an isometry, $\psi_1$ is harmonic. Consequently, $(\Delta\varphi_1)(0)=0$.

Next, $\varphi_2$ is harmonic, so $(\Delta\varphi_2)(0)=0$.

We claim that $(\Delta\varphi_3)(0)\geq 0$. Set $g(s)=h_i(x(s))-h_i(y(s))$, for some $i$. Note that $s\to h_i(x(s))$ is harmonic, and that

$$h_i(y(s))-h_i(y_0+cLs)=f(s)(\nabla h)(y_0)\cdot\xi+O(|s|^3).$$

It follows that $(\Delta g)(0)=0$. Hence $\Delta(g^2)=2|\nabla g|^2+2g\Delta g\geq 0$ at $s=0$. Thus $(\Delta\varphi_3)(0)\geq 0$. This proves (7), and completes the even-dimensional case.

(b) *Assume n is odd.* We shall use the following fact: If

(i) $L:R^{n+1}\to R^{n+1}$ is a linear isometry,

(ii) $g:R^n\to R$ is harmonic, and

(iii) $\pi(s_1,\dots,s_n,s_{n+1})=(s_1,\dots,s_n)$ projects $R^{n+1}$ to $R^n$,

then $s\to g(\pi Ls)$ is harmonic in $R^{n+1}$. To see this, put $G(s_1,\dots,s_n,s_{n+1})=g(s_1,\dots,s_n)$. Then $G$ is harmonic and $g(\pi Ls)=G(Ls)$.

Lemma 4.1(a) gives us now a linear isometry $L$ of $R^{n+1}$ which satisfies

(2′) $$L((\nabla u)(x_0),0)=(c(\nabla u)(y_0),0)$$

in place of (2), so that $I+cL$ is a multiple of an isometry of $R^{n+1}$.

In place of (3), we define

(3′) $$f(s)=\{u(y_0+c\pi Ls)-u(x_0+\pi s)\}/|(\nabla u)(y_0)|$$

for $s$ in some neighborhood $V$ of the origin of $R^{n+1}$. Then $f$ is harmonic in $V$, and the proof of Lemma 1.3 shows that if we replace (4) by

(4′) $$x(s)=x_0+\pi s,\qquad y(s)=y_0+c\pi Ls+f(s)\xi+\alpha(s)\xi,$$

where $\alpha(s)=O(|s|^3)$, then $(x(s),y(s))\in\Sigma$ for all $s\in V$.

Defining $\varphi$ as in (6), but with $s$ in $R^{n+1}$, we have to prove (7). As before, $\varphi=\varphi_1-\varphi_2+\varphi_3$. We now have

$$\varphi_1(s)=u\left(\frac{x_0+y_0}{2}+\frac{1}{2}\pi(I+cL)s+\frac{1}{2}f(s)\xi+\frac{1}{2}\alpha(s)\xi\right).$$

Using Taylor expansions, as before, together with the fact that $I+cL$ is a multiple of an isometry of $R^{n+1}$, we find that $(\Delta\varphi_1)(0)=0$.

Next, $\Delta\varphi_2=0$ because $\varphi_2$ is harmonic. The proof that $(\Delta\varphi_3)(0)\geq 0$ is as before, using (4′) in place of (4). This completes the odd-dimensional case.

□

Having proved Theorem 4.3, let us now look at some of its consequences. We begin with a weak form of Theorem I, which will be used in Section 4.6 to prove the theorem in its full strength.

4.4. THEOREM. *If the hypotheses of Theorem I hold, then every level set of $u$ in $\Omega$ is a convex hypersurface.*

*Proof.* The inequality

$$(1) \qquad u\left(\frac{x+y}{2}\right) \le \frac{u(x)+u(y)}{2}$$

is completely trivial for $(x,y) \in b\Sigma$, simply because $u=0$ on $\partial K$, $u=1$ on $\partial W$, and therefore $0<u<1$ in $\Omega$. Theorem 4.3 shows therefore that (1) holds also for all $(x,y) \in \Sigma$.

In other words, if $0<t<1$, $u(x)=u(y)=t$, and $u$ is extended to all of $W$ by setting $u \equiv 0$ on $K$, then $\frac{1}{2}(x+y)$ lies in the set where $u \le t$. This set is therefore convex. □

4.5. THEOREM. *Assume that the hypotheses of Theorem I hold, that C is a positive constant, and that*

$$(1) \qquad u\left(\frac{x+y}{2}\right) \le \frac{u(x)+u(y)}{2} - C|x-y|^2$$

*for all* $(x,y) \in b\Sigma$. *Then* (1) *holds also for all* $(x,y) \in \Sigma$.

*Proof.* Take $h_i(x) = C^{1/2}x_i$ in Theorem 4.3, for $i=1,\dots,n$. □

4.6. PROOF OF THEOREM I. We must now strengthen the conclusion of Theorem 4.4 from "convex" to "strictly convex" without using the extra assumption made in Theorem 4.5.

If we replace $\Omega$ by the set in which $\epsilon<u<1-\epsilon$, for arbitrarily small $\epsilon>0$, we may (and shall) assume without loss of generality that $\partial W$ and $\partial K$ are real-analytic hypersurfaces—which are *convex* by Theorem 4.4—and that $u$ is harmonic in some open set containing $\bar{\Omega}$.

There is an open ball $V \subset R^n$, centered at some point of $\partial W$, such that $H_p(u,\xi)>0$ if $p \in V$ and $\xi \cdot (\nabla u)(p)=0$. (This is so, simply because every compact hypersurface of class $C^2$ is strictly convex at every point whose distance from the origin is maximal, and hence is strictly convex in some open set.) Moreover, we take $V$ so small that Proposition 3.2(ii) can be applied, and we set $V_0 = V \cap \partial W$. Now fix $t$, $0<t<1$, and put $S(t)=\{x \in \Omega : u(x)=t\}$. We shall prove that this level set is strictly convex.

There is a compact set $E \subset V_0$, and there are functions $h_i : \bar{\Omega} \to [0,\infty)$ ($1 \le i \le M$) which are harmonic in $\Omega$, smooth on $\bar{\Omega}$ (Lipschitz is all we really need), 0 on $\partial K$, and 0 on $\partial W \backslash E$, so that the set $\{h_1,\dots,h_M\}$ separates points on $S(t)$ in the strong sense that

$$(1) \qquad |x-y|^2 \le \sum_{i=1}^{M} (h_i(x)-h_i(y))^2$$

for all $x \in S(t)$ and $y \in S(t)$. (See the postscript.)

We claim: *There is a* $c_0>0$ *such that*

$$(2) \qquad c_0 \sum_{i=1}^{M} (h_i(x)-h_i(y))^2 \le \frac{u(x)+u(y)}{2} - u\left(\frac{x+y}{2}\right)$$

*for all* $(x,y) \in b\Sigma$.

Once we have this, Theorem 4.3 shows that (2) holds for all $(x,y)\in\Sigma$, and (1) then implies that

$$c_0|x-y|^2\le\frac{u(x)+u(y)}{2}-u\left(\frac{x+y}{2}\right) \tag{3}$$

for all $(x,y)\in S(t)\times S(t)$, and hence also (with $c_0/2$ in place of $c_0$) in some neighborhood of this set. Proposition 3.2(i) shows then that $H_p(u,\xi)>0$ if $p\in S(t)$ and $\xi\cdot(\nabla u)(p)=0$, which is precisely what had to be proved.

We turn to the proof of (2). The definition of $b\Sigma$ shows that several cases must be considered.

(i) The harmonic functions $u$ and $h_i$ are positive in $\Omega$, 0 on $\partial K$, and their gradients are bounded and $\neq 0$ on $\partial K$. Hence there is a $\beta<\infty$ such that $0<h_i<\beta u$ in $\Omega$ for $1\le i\le M$. Set $c_1=1/\beta^2M$.

If $\frac{1}{2}(x+y)\in\partial K$ and $u(x)=u(y)=r$ for some $r\in(0,1)$, it follows that $|h_i(x)-h_i(y)|<\beta r$ and hence

$$c_1\sum_{i=1}^{M}(h_i(x)-h_i(y))^2\le c_1\beta^2Mr^2<r. \tag{4}$$

Note that $r$ is here equal to the right-hand side of (2).

(ii) If both $x$ and $y$ are in $\partial W\backslash E$, then $h_i(x)=h_i(y)=0$, so (2) holds no matter what $c_0$ is.

(iii) Proposition 3.2(ii), together with our choice of $V$, shows that for some $c_2>0$ we have

$$c_2|x-y|^2\le\frac{u(x)+u(y)}{2}-u\left(\frac{x+y}{2}\right) \tag{5}$$

for all $(x,y)\in V_0\times V_0$. Since each $h_i$ satisfies a Lipschitz condition, we see that (2) holds (with some $c_3>0$ in place of $c_0$) for all $(x,y)\in V_0\times V_0$.

(iv) There is a $\delta>0$ such that $|x-y|\ge\delta$ if $x\in E$ and $y\in\partial W\backslash V_0$. Since $\partial W$ is a compact level surface of the real-analytic function $u$, it contains no straight line interval. The convexity of $W$ implies therefore that $\frac{1}{2}(x+y)$ is not in $\partial W$. It follows that there is a $\gamma>0$ such that

$$u\left(\frac{x+y}{2}\right)\le 1-\gamma \tag{6}$$

whenever $x\in E$ and $y\in\partial W\backslash V_0$.

Under these conditions, the right side of (2) is thus $\le -\gamma<0$, and the boundedness of the sum in (2) shows that (2) holds with some $c_4>0$ in place of $c_0$.

If we now take $c_0=\min(c_1,c_3,c_4)$, we obtain (2) for all $(x,y)\in b\Sigma$ and the proof is complete. □

## 5. Two Counterexamples

5.1. One may attempt to find generalizations of Theorem I in which the Laplacian is replaced by the Laplace–Beltrami operator associated to some

Riemannian metric that is different from the Euclidean one. Of course, the notion of convexity must then be adapted to the operator (i.e., to the metric).

In the Euclidean case, strict convexity of a hypersurface $S$ in $R^n$ is a local property which can be defined in terms of

(a) the Hessian of a defining function of $S$ (as we did in the present paper),
(b) intersections with straight lines, or
(c) the intrinsic Gaussian curvature of $S$ (when $n \geq 3$).

Of course, (b) and (c) make sense relative to any Riemannian metric, if straight lines are replaced by geodesics in (b). The following examples show merely that neither of these possible definitions of convexity leads to a generalization of Theorem I. Of course, other generalizations may well exist.

We shall give two examples, one in $R^3$ and one in $R^2$, of Riemannian metrics $d\tau$ on annular regions $\Omega$, and of functions $f$ that are harmonic with respect to their associated Laplace–Beltrami operators $\Delta_\tau$, with the following properties:

(i) $d\tau$ differs from the Euclidean metric only on some compact subset of $\Omega$;
(ii) $f$ is constant on each component of $\partial\Omega$;
(iii) in the $R^3$ example, some level surface $S$ of $f$ has negative Gaussian curvature at some point, relative to the metric induced on $S$ by $d\tau$; in the $R^2$ example, some level curve $\Gamma$ of $f$ is nonconvex (in the usual sense) and $d\tau$ coincides with the Euclidean metric in a neighborhood of $\Gamma$.

This shows that Theorem I does not extend to these settings.

We recall that to each Riemannian metric

$$d\tau^2 = \sum_{i,j=1}^{n} g_{ij}\,dx_i\,dx_j \tag{1}$$

(in some coordinate system $\{x_1, \ldots, x_n\}$) corresponds its so-called Laplace–Beltrami operator

$$\Delta_\tau = \frac{1}{\sqrt{g}} \sum_{j=1}^{n} \frac{\partial}{\partial x_j}\left( \sum_{i=1}^{n} g^{ij}\sqrt{g}\,\frac{\partial}{\partial x_i}\right), \tag{2}$$

where $(g^{ij})$ is the inverse of the matrix $(g_{ij})$ and $g = \det(g_{ij})$; see [7, p. 245]. In the special case where $(g_{ij})$ is diagonal and $f$ depends only on $x_1$, (2) becomes

$$\Delta_\tau f = \frac{1}{\sqrt{g}}\,\frac{\partial}{\partial x_1}\left( g^{11}\sqrt{g}\,\frac{\partial f}{\partial x_1}\right). \tag{3}$$

We shall use the fact that if

$$d\tau^2 = \alpha(dx_1^2 + \cdots + dx_n^2), \tag{4}$$

where $\alpha$ is a smooth positive function (i.e., if $d\tau$ is "conformal"), then an easy calculation leads from (2) to

(5) $$\Delta_\tau f = \frac{\Delta f}{\alpha} + \frac{n-2}{2\alpha^2}(\nabla\alpha)\cdot(\nabla f),$$

where $\Delta$ is the ordinary Laplacian. The dimension 2 thus plays a special role here: $\Delta f = 0$ implies $\Delta_\tau f = 0$ if (4) holds and $n = 2$.

5.2. AN EXAMPLE IN $R^3$. Pick $a, b$, $0 < a < 1 < b < \infty$, let

(1) $$\Omega = \{x \in R^3 : a < |x| < b\},$$

and denote the unit sphere $\{x : |x| = 1\}$ by $S$.

We shall construct a Riemannian metric $d\tau$ on $\bar{\Omega}$ so that

(i) $d\tau$ coincides with the standard Euclidean metric outside a small neighborhood of $(0,0,1)$ (the north pole of $S$) whose closure does not intersect $\partial\Omega$;

(ii) the function $f(x) = 1/|x|$ satisfies $\Delta_\tau f = 0$ on $\bar{\Omega}$; but

(iii) the restriction of $d\tau$ to $S$ (a level surface of $f$) imposes *negative* curvature on $S$ in a neighborhood of $(0,0,1)$.

In fact, this restriction of $d\tau$ will be the Poincaré metric, transplanted from the unit disc in **C**.

Recall that

(2) $$x_1 = r\cos\theta\sin\varphi, \qquad x_2 = r\sin\theta\sin\varphi, \qquad x_3 = r\cos\varphi,$$

where $(r, \theta, \varphi)$ are the usual spherical coordinates, $0 \le \theta \le 2\pi$, and $0 \le \varphi \le \pi$. In these coordinates, the Euclidean metric $ds^2 = dx_1^2 + dx_2^2 + dx_3^2$ becomes

(3) $$ds^2 = dr^2 + r^2(\sin^2\varphi\, d\theta^2 + d\varphi^2).$$

Consider the stereographic projection

(4) $$(1, \theta, \varphi) \to z = \left(\tan\frac{\varphi}{2}\right)e^{i\theta}$$

of $S\backslash\{(0,0,-1)\}$ onto **C**. This allows us to write the restriction of $ds$ to $S\backslash\{(0,0,-1)\}$ in the form

(5) $$ds^2 = \sin^2\varphi\, d\theta^2 + d\varphi^2 = \frac{4|dz|^2}{(1+|z|^2)^2},$$

which is the usual "spherical metric" on **C**.

The Poincaré metric in the unit disc (whose curvature is negative) is

(6) $$d\sigma = \frac{2|dz|}{1-|z|^2} = \frac{1+|z|^2}{1-|z|^2}\, ds.$$

Since $|z| = \tan\varphi/2$, we see that the pull-back of $d\sigma$ to the upper half of $S$ (we denote it again by $d\sigma$) is given by

(7) $$d\sigma = \frac{ds}{\cos\varphi}.$$

The $\sigma$-curvature of the upper half of $S$ is thus negative.

Now pick a cut-off function $\chi \in C^\infty(R^3)$ with compact support in $\Omega \cap \{x_3 > 0\}$, $0 \le \chi \le 1$, $\chi \equiv 1$ in a neighborhood of $(0,0,1)$; set

(8) $$h = 1 - \chi + (\cos\varphi)^{-2}\chi$$

and define the desired metric $d\tau$ by

(9) $$d\tau^2 = h^2 dr^2 + hr^2(\sin^2\varphi \, d\theta^2 + d\varphi^2).$$

Then $d\tau = ds$ wherever $\chi = 0$, but the restriction of $d\tau$ to $S$ coincides with the Poincaré metric in a neighborhood of $(0,0,1)$.

To finish we apply formula 5.1(1), with $(r,\theta,\varphi)$ in place of $(x_1, x_2, x_3)$, to the metric (9) and the function $f = 1/r$. We see that $(g_{ij})$ is diagonal, that

(10) $$g^{11} = h^{-2}, \qquad g = h^4 r^4 \sin^2\varphi, \qquad \frac{\partial f}{\partial r} = -r^{-2},$$

so that $g^{11}\sqrt{g}(\partial f/\partial r) = \sin\varphi$ and therefore

(11) $$\Delta_\tau f = \frac{1}{\sqrt{g}} \frac{\partial}{\partial r}(-\sin\varphi) = 0.$$

5.3. AN EXAMPLE IN $R^2$. We identify $R^2$ with $\mathbf{C}$, put $\Omega = \{\frac{1}{2} < |z| < 2\}$, define $G$ on $\bar{\Omega}$ by $G(z) = \log|z|$, and let $\Psi$ be a $C^\infty$-diffeomorphism of $\bar{\Omega}$ onto $\bar{\Omega}$ so that (see the Postscript):

(a) $\Psi(z) = z$ in a neighborhood of $\partial\Omega$;
(b) $\Psi$ is holomorphic in a neighborhood of the unit circle $T$; and
(c) $\Psi(T)$ is a nonconvex curve $\Gamma$.

Define $\Phi = \Psi^{-1}$ and let $d\sigma$ be the pull-back of the Euclidean metric $ds$ by $\Phi$. (Thus $\Phi$ is an isometry: The $s$-length of a curve $\Phi(L)$ equals the $\sigma$-length of $L$.) Set $f = G \circ \Phi$. Then $\Delta f = 0$ in the neighborhood $W = \Psi(V)$ of $\Gamma$, because $\Phi$ is holomorphic in $W$ and $\Delta G = 0$.

Now pick $\chi \in C^\infty(\mathbf{C})$, $0 \le \chi \le 1$, with support in $W$, so that $\chi \equiv 1$ in some open set $W_0 \supset \Gamma$; define our metric $d\tau$ by

(1) $$d\tau^2 = \chi ds^2 + (1-\chi) d\sigma^2.$$

In $W$, $d\sigma^2 = |\Phi'|^2 ds^2$; hence

(2) $$d\tau^2 = [\chi + (1-\chi)|\Phi'|^2] ds^2$$

so that $d\tau$ is conformal. Thus (see 5.1(5)) $\Delta_\tau f = \Delta f = 0$ in $W$.

In $\Omega \backslash W$, (1) shows that $d\tau = d\sigma$ and thus

(3) $$\Delta_\tau f = \Delta_\sigma(G \circ \Phi) = (\Delta G) \circ \Phi = 0,$$

because Laplace–Beltrami operators commute with isometries.

Finally, $d\tau$ coincides with the Euclidean metric $ds$ near the boundary of $\Omega$ as well as in the neighborhood of the nonconvex curve $\Gamma$. The level curves $\{|z| = \frac{1}{2}\}$ and $\{|z| = 2\}$ of $f$ are thus $\tau$-convex, but $\Gamma = \{f = 0\}$ is not.

5.4. Michael Papadimitrakis has shown quite recently that the analogue of Theorem I does hold relative to the Poincaré metric in the unit disc.

POSTSCRIPT. The referee suggested that two statements made in our paper should be explained in some detail.

*(I).* In Section 4.6 we claim that there are smooth nonnegative functions $h_1, \dots, h_M$ on $\bar{\Omega}$, harmonic in $\Omega$ and 0 on $(\partial\Omega)\backslash E$ (where $E = B \cap \partial\Omega$ for some ball $B$ with center in $\partial\Omega$), so that

$$(1) \qquad |x-y|^2 \le \sum_{i=1}^{M} (h_i(x) - h_i(y))^2$$

for all $x$ and $y$ in $S(t)$. We prove this with an arbitrary compact $K \subset \Omega$ in place of $S(t)$.

For $w \in E$, let $P_w$ be the Poisson kernel for $\Omega$ with pole at $w$. Thus $P_w$ is a positive harmonic function in $\Omega$ which vanishes on $\partial\Omega\backslash\{w\}$. Each $P_w$ and its derivatives can be uniformly approximated on $K$ by functions $h_i$ (as above) and their derivatives. Hence it is enough to prove (1) with positive multiples of $P_{w_i}$ in place of $h_i$, where $w_i \in E$. That this can be done is a consequence of the following two facts, combined with the compactness of $K$:

(a) the set $\{P_w : w \in E\}$ separates points in $\Omega$;

(b) if $x \in \Omega$ and $u$ is a unit vector, then $(\partial_u P_w)(x) \ne 0$ for some $w \in E$.

Here $\partial_u$ denotes the directional derivative in the direction $u$. Note that (b) implies positive local lower bounds for $|P_w(x) - P_w(y)|/|x-y|$.

Recall that $P_w(x) = (\partial_\nu G_x)(w)$, where $G_x$ is Green's functions of $\Omega$ with pole at $x$ and $\partial_\nu$ denotes the normal derivative evaluated at $w \in \partial\Omega$. We shall use the following well-known uniqueness theorem.

(*) *If $H \in C^1(\Omega \cup E)$, $H$ is harmonic in $\Omega$, and $H(w) = (\partial_\nu H)(w) = 0$ for all $w \in E$, then $H \equiv 0$ in $\Omega$.*

To prove (a), suppose $P_w(x) = P_w(y)$ for all $w \in E$. Put $H = G_x - G_y$. Then (*), with $\Omega\backslash\{x, y\}$ in place of $\Omega$, shows that $G_x = G_y$, which of course forces $x = y$.

To prove (b), fix $x$ and $u$ and set $H = \partial_u G_x$. If $w_0 = x - \epsilon u$ and $\epsilon > 0$ is sufficiently small, then $(\partial_u G_x)(w_0) \ne 0$. Thus $H \not\equiv 0$ in $\Omega\backslash\{x\}$. Since $H(w) = 0$ for all $w \in \partial\Omega$, (*) implies that $(\partial_\nu H)(w) \ne 0$ for some $w \in E$, and therefore

$$\partial_u P_w(x) = \partial_u \partial_\nu G_x(w) = \partial_\nu \partial_u G_x(w) \ne 0.$$

For the sake of completeness, here is a proof of (*): Extend $H$ to $\Omega \cup B$ so as to be 0 in $B\backslash\Omega$. Let $\psi$ be a test function with support in $B$. By Green's theorem and the fact that $H \in C^1(\Omega \cup B)$,

$$\int_{\Omega \cup B} H\Delta\psi = \int_\Omega (H\Delta\psi - \psi\Delta H) = \int_{\partial\Omega} (H\partial_\nu\psi - \psi\partial_\nu H) = 0$$

because the last integrand is 0 on $\partial\Omega$. Thus $H$ is harmonic in $\Omega \cup B$. Since $H \equiv 0$ in the nonempty open set $B\backslash\Omega$ and since $\Omega$ is connected, we conclude that $H \equiv 0$ in $\Omega$.

*(II).* In Section 5.3 we consider the annulus $\Omega = \{\frac{1}{2} < |z| < 2\}$ and assert that there is a $C^\infty$-diffeomorphism $\Psi$ of $\bar{\Omega}$ onto $\bar{\Omega}$ such that:

(a) $\Psi(z)=z$ near $\partial\Omega$;
(b) $\Psi$ is holomorphic in an annulus $A\supset T$, where $T$ is the unit circle; and
(c) $\Psi(T)$ is a nonconvex curve $\Gamma$.

Here is a sketch of how $\Psi$ can be obtained.

Start with a real-analytic diffeomorphism

$$\varphi(e^{i\theta})=R(\theta)e^{i\theta} \quad (R>0)$$

of $T$ onto a nonconvex curve $\Gamma$ which is close to $T$, in the sense that $\Gamma\subset\Omega$ and $\Gamma$ has no tangent that passes through the origin. Being real-analytic, $\varphi$ extends to a conformal map

$$\varphi(re^{i\theta})=|\varphi(re^{i\theta})|e^{i\alpha(r,\theta)}$$

of an annulus $A_0$ onto a neighborhood $V$ of $\Gamma$; we take $A_0$ so narrow that $\bar{V}\subset\Omega$, that $\partial|\varphi|/\partial n>0$ in $\bar{A}_0$, and $|\alpha(r,\theta)-\theta|<\pi/2$.

In $A_0$ are three disjoint annuli, $A, A^+, A^-$, such that $T\subset A$ and $A^+, A^-$ lie (respectively) along the outer and inner boundary of $A_0$. Let $\psi: A_0\to[0,1]$ be a $C^\infty$ cut-off function, depending only on $r$, so that $\psi=1$ on $A^+\cup A^-$ and $\psi=0$ on $A$. Use $\psi$ to define a $C^\infty$-diffeomorphism $\tau$ from $\bar{A}_0$ onto $\bar{A}_0$:

$$\tau(re^{i\theta})=r\exp\{i[\psi(r)\alpha(r,\theta)+(1-\psi(r))\theta]\}.$$

Then $g=\varphi\circ\tau^{-1}$ is a $C^\infty$-diffeomorphism of $\bar{A}_0$ on $\bar{V}$, $g$ is conformal in $A$ because $g=\varphi$ on $A$, and $g$ has the form

$$g(re^{i\theta})=R(r,\theta)e^{i\theta} \quad \text{on } A^+\cup A^-$$

for some positive $C^\infty$-function $R$, with $\partial R/\partial r>0$.

To obtain $\Psi$ from $g$ we extend $R$ to a $C^\infty$-function $\tilde{R}:\Omega\to\Omega$ (across the gaps between $A^+$ and $\{|z|=2\}$, and between $\{|z|=\frac{1}{2}\}$ and $A^-$) in such a way that $\tilde{R}(r,\theta)=r$ near $r=2$ and near $r=\frac{1}{2}$, and that $\partial\tilde{R}/\partial r>0$. We define

$$\Psi(re^{i\theta})=\begin{cases} g(re^{i\theta}) & \text{in } A_0,\\ \tilde{R}(r,\theta)e^{i\theta} & \text{in } \Omega\backslash A_0.\end{cases}$$

The construction of $\tilde{R}$ involves, for each $\theta$, a smooth extension across an interval on the real axis, done in such a way that the extensions depend smoothly on the parameter $\theta$.

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