Affine Variation Formulas and Affine Minimal Surfaces

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1. Introduction

In Section 2, we state the necessary preliminaries concerning affine hypersurfaces M^n in the $(n+1)$-dimensional standard affine space $\mathbf{A}^{n+1}$. In Section 3, we give examples of affine minimal surfaces in $\mathbf{A}^3$. In particular, we completely classify all affine minimal translation surfaces. Also, by giving a counterexample, we show that the affine Berstein problem (which is solved affirmatively for convex surfaces by Calabi [2]) has a negative solution in the nonconvex case. In Section 4 we obtain, also in the nonconvex case, that the affine minimal hypersurfaces M^n in $\mathbf{A}^{n+1}$ are those which have an extremal volume under variations in the direction of the affine normal. Furthermore, we find that for nonconvex affine minimal surfaces, in contrast to the convex case, the second variation of the affine area does not necessarily have a sign.

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2. Preliminaries

Let $\mathbf{A}^{n+1}$ be the standard $(n+1)$-dimensional real affine space, that is, $\mathbf{R}^{n+1}$ endowed with the standard linear connection D and the volume element Ω given by the determinant. Then $D\Omega=0$, and so (D,Ω) defines an *equi-affine structure* on $\mathbf{R}^{n+1}$ [4; 5].

Let M^n be an oriented hypersurface in $\mathbf{A}^{n+1}$. Then the natural problem of how to induce an equi-affine structure (∇,θ) on M^n starting from (D,Ω) on $\mathbf{A}^{n+1}$ was solved by Nomizu in the following way [4]. Let ξ be any *transversal vector field* on M^n such that, $\forall x\in M^n$, $T_x\mathbf{A}^{n+1}=T_xM^n\oplus\operatorname{span}\{\xi_x\}$. Consider the corresponding formulas of Gauss and Weingarten:

$$D_XY=\nabla_XY+h(X,Y)\xi,\qquad D_X\xi=-SX+\tau(X)\xi,$$

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which decompose $D_X Y$ and $D_X \xi$ into a tangential and a transversal component, where X and Y are tangent vector fields on M. Then ∇ is a *torsion-free affine connection*, h a *symmetric bilinear form*, S a $(1,1)$-*tensor* and τ a 1-*form* on M. All these notions depend on the choice of transversal vector field ξ. The natural way to define a *volume element* θ on M, using ξ, is

$$\theta(X_1, X_2, \ldots, X_n) = \Omega(X_1, X_2, \ldots, X_n, \xi),$$

where $X_1, \ldots, X_n$ are any tangent vector fields to M. Then one has $\nabla_X \theta = \tau(X)\theta$. So in order that (∇, θ) should determine an equi-affine structure on M we must demand that

$$\tau = 0. \tag{2.1}$$

Then, one further condition is imposed to determine ξ, ∇, and θ uniquely. To obtain this condition, we first note that if one looks at all possible choices of transversal vector fields ξ, one observes that the fact that *h is nondegenerate* is independent of the choice of affine transversal vector field. From now on, we will always assume nondegeneracy. Hence we can consider ν, the metric volume element corresponding with the semi-Riemannian metric h, with respect to the given orientation. The second condition then states that this metric volume element coincides with the induced affine volume element:

$$\nu = \theta, \tag{2.2}$$

or equivalently

$$|H| = 1, \quad H = \det[h(Y_k, Y_l)],$$

($\{Y_1, \ldots, Y_n\}$ being a tangent basis for M such that $\theta(Y_1, Y_2, \ldots, Y_n) = 1$.) The corresponding ξ is called the affine normal. One then has the following equations of Gauss, Codazzi, and Ricci:

$$R(X,Y)Z = h(Z,Y)SX - h(X,Z)SY, \tag{2.3}$$

$$(\nabla h)(X,Y,Z) = (\nabla h)(Y,X,Z), \tag{2.4}$$

$$(\nabla_X S)(Y) = (\nabla_Y S)(X), \tag{2.5}$$

$$h(X,SY) = h(SX,Y). \tag{2.6}$$

M is *flat* ($R = 0$) if $S = 0$. M is *convex* if h is positive or negative definite [2]. M is said to be an affine *minimal* hypersurface when $\operatorname{Tr} S = 0$.

3. Examples of Minimal Surfaces in $\mathbf{A}^3$

1. SURFACES OF REVOLUTION. Locally, every surface of revolution can be written in the following way:

$$x(u,v) = (g(u), u\cos v, u\sin v).$$

For this surface to be minimal, we obtain (after a straightforward computation) the following differential equation for g:

$$\left(-\frac{1}{u}g'g''+g''^2+g'g'''\right)^4=cg''\left(\frac{1}{u}g'g''\right)^3,$$

where c is an arbitrary constant. For more details see [1].

2. SURFACES OF TRANSLATION. A surface of translation generated by plane curves is a surface which can be written in the following form:

$$x(u,v)=(u,v,f(u)+\epsilon g(v)),$$

where f and g are arbitrary functions and $\epsilon\in\{-1,1\}$. The nondegeneracy of the surface implies that $f_{uu}\cdot g_{vv}\neq 0$. Therefore, by choosing ϵ, we may assume that $f_{uu}>0$ and $g_{vv}>0$. By demanding that this surface be affine minimal, we obtain (after a straightforward computation) the following differential equations for f and g:

$$f_{uuuu}f_{uu}-\tfrac{7}{4}f_{uuu}^2=C, \tag{3.1}$$

$$g_{vvvv}g_{vv}-\tfrac{7}{4}g_{vvv}^2=\epsilon C, \tag{3.2}$$

where C is an arbitrary constant.

Let us then define a function F by

$$F(y)=\int_{C_1}^{y}\frac{-\frac{3}{4}\,dx}{x^{7/4}\sqrt{\frac{9}{4}\alpha x^{-1/2}+D}},$$

where C_1, α and D are constants satisfying $C_1>0$, $y>0$, $\frac{9}{4}\alpha C_1^{-1/2}+D>0$, and $\frac{9}{4}\alpha y^{-1/2}+D>0$. Then the inverse function h of F satisfies the differential equation $-hh''+\frac{7}{4}h'^2=\alpha h^3$. Therefore, by taking $\alpha=-C$ or $\alpha=-\epsilon C$ and integrating two more times, we obtain the functions f and g. In the special case where $\alpha=0$, we have the following examples:

$$f_1(u)=D_1(u+C_1)^{2/3}+E_1u+F_1,$$
$$g_1(v)=D_2(v+C_2)^{2/3}+E_2v+F_2;$$

$$f_2(u)=D_1u^2+E_1u+F_1,$$
$$g_2(v)=D_2(v+C_2)^{2/3}+E_2v+F_2;$$

$$f_3(u)=D_1(u+C_1)^{2/3}+E_1u+F_1,$$
$$g_3(v)=D_2v^2+E_2v+F_2;$$

$$f_4(u)=D_1u^2+E_1u+F_1,$$
$$g_4(v)=D_2v^2+E_2v+F_2,$$

where due to regularity conditions f_1 and f_3 are defined on $\mathbf{R}\setminus\{C_1\}$ and g_1 and g_2 are defined on $\mathbf{R}\setminus\{C_2\}$. Furthermore, the nondegeneracy implies that D_1 and D_2 are different from zero. Pictures of all these affine analogues of Scherk's surface in the Euclidean space $\mathbf{E}^3$ are given in Figures 1–6.

Again, due to regularity conditions, these are the only solutions to (3.1) and (3.2). Furthermore, after a straightforward computation, we find that the minimal translation surfaces given by

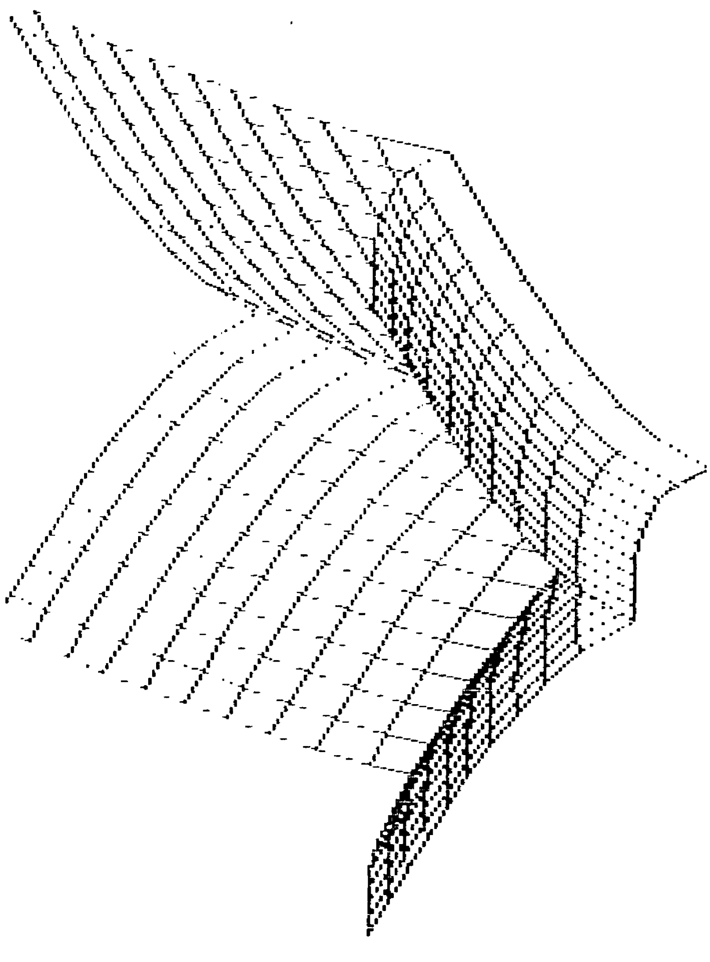

Figure 1 $(u, v, u^{2/3}+v^{2/3})$

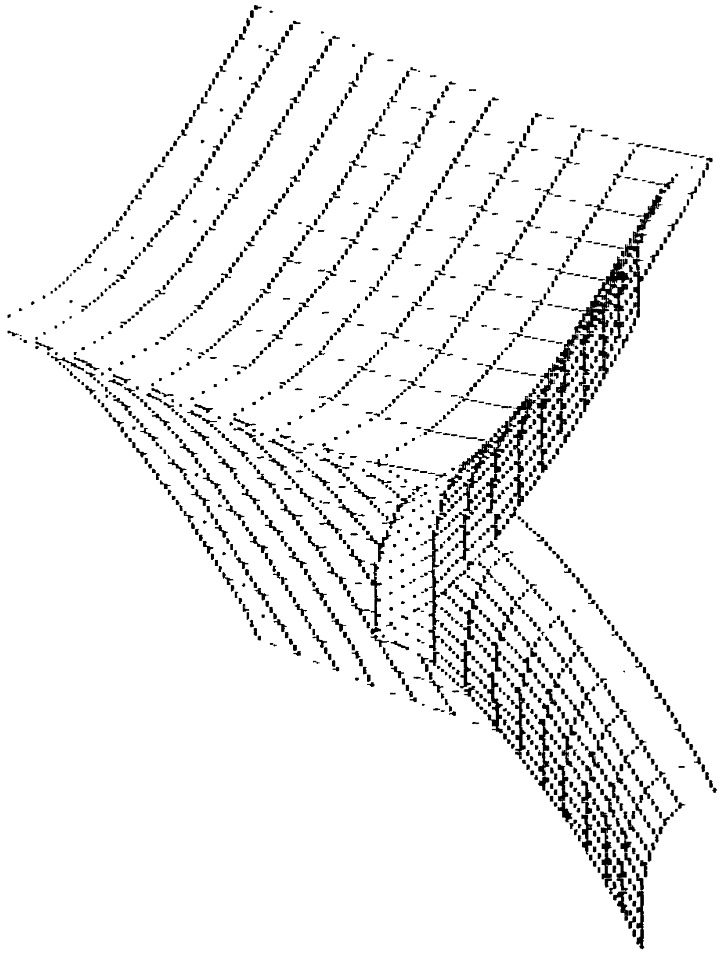

Figure 2 $(u, v, -u^{2/3}+v^{2/3})$

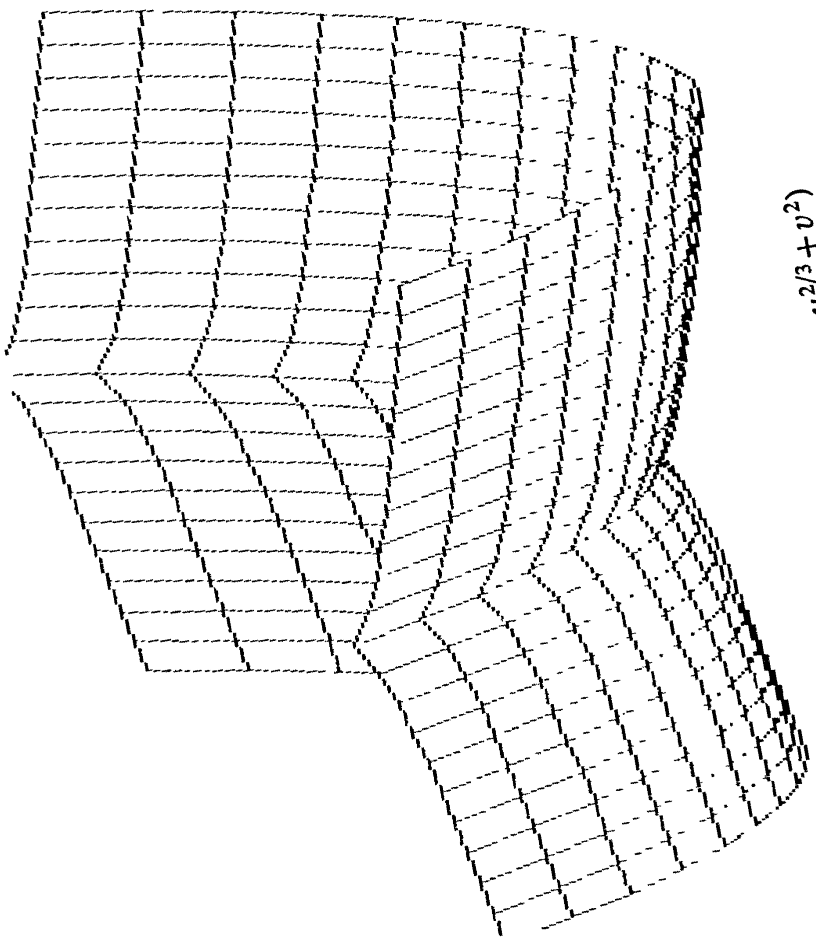

Figure 4 $(u, v, -u^{2/3}+v^2)$

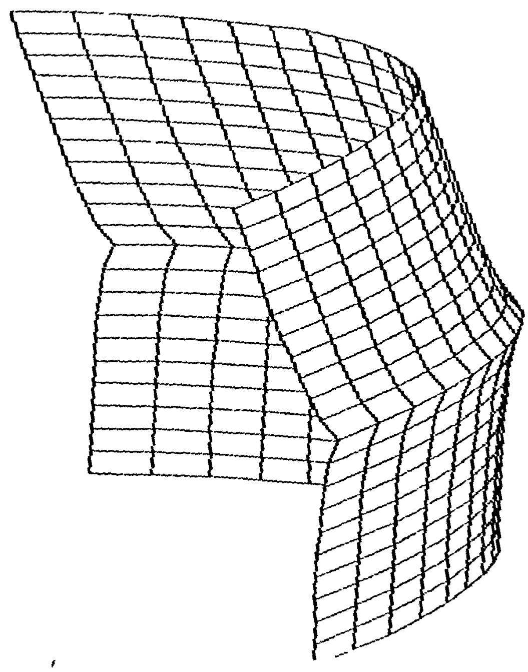

Figure 3 $(u, v, u^{2/3}+v^2)$

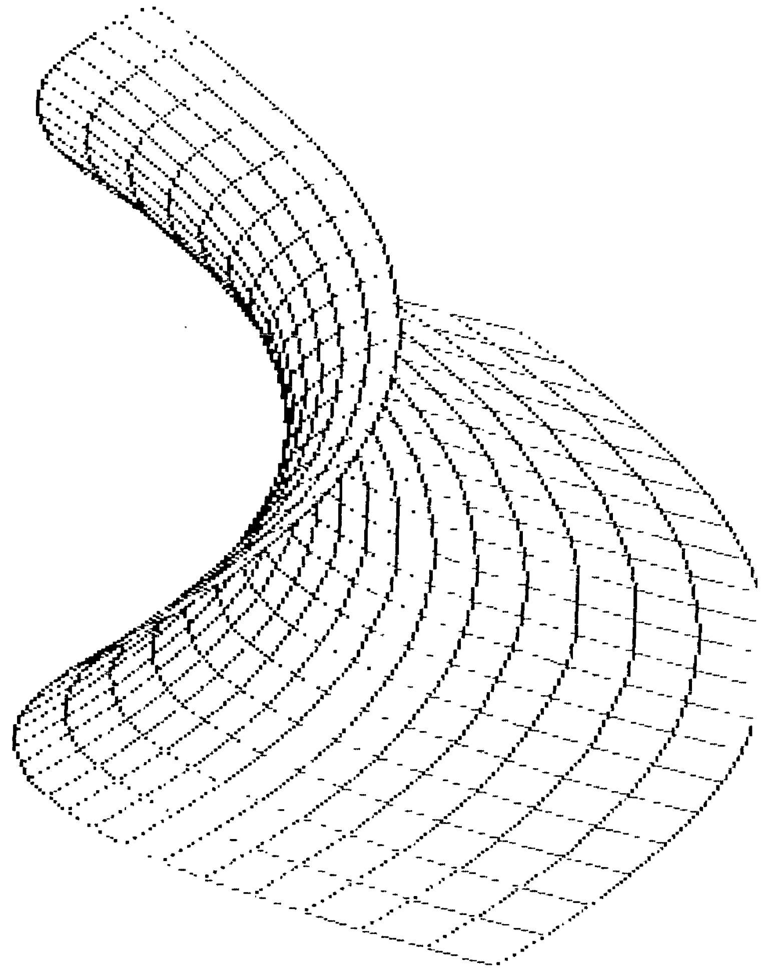

Figure 6 $(u, v, u^2 - v^2)$

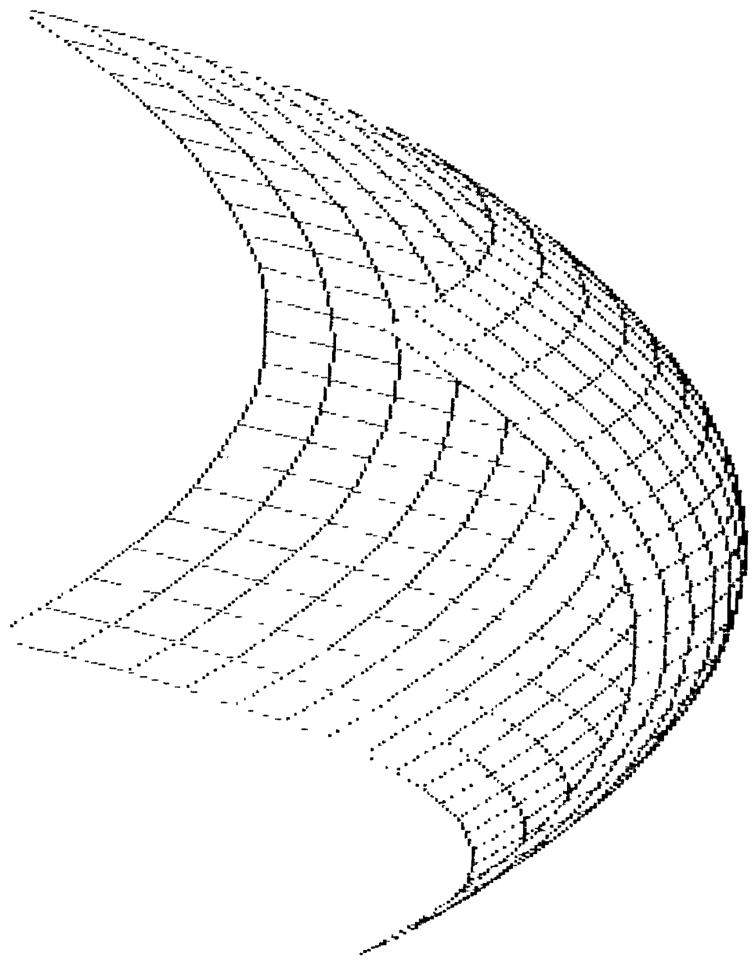

Figure 5 $(u, v, u^2 + v^2)$

$$x_2(u,v)=(u,v,f_2(u)+g_2(v)),$$
$$x_3(u,v)=(u,v,f_3(u)+g_3(v)),$$
$$x_4(u,v)=(u,v,f_4(u)+g_4(v))$$

are also *flat*.

3. RULED SURFACES. Finally, we give some examples of *ruled, nonconvex, affine minimal* surfaces which will play a very important role in the next section. See also [6] and [7].

a. The hyperbolic paraboloid. The hyperbolic paraboloid has a parameterization that is given by $x(u,v)=(u,v,uv)$. The affine normal ξ is given by $\xi(x(u,v))=(0,0,1)$. Thus, we see that the hyperbolic paraboloid is a flat surface. But the hyperbolic paraboloid is not the only nonconvex, flat surface; the following deformation of the hyperbolic paraboloid also remains nonconvex and flat:

$$x(u,v)=(u,f(u),g(u))+v(0,1,u),$$

where f and g are arbitrary functions of u.

b. The helicoid. The helicoid has a parameterization given by $x(u,v)=(u\cos v, u\sin v, v)$, where $u,v\in\mathbf{R}$. The affine normal ξ is given by $\xi(x(u,v))=(\sin v, -\cos v, 0)$. Furthermore,

$$Sx_u=0, \qquad Sx_v=-x_u.$$

Hence, helicoids are affine minimal surfaces.

c. Another affine minimal surface is given by the following parameterization: $x(u,v)=(u,v,ve^u)$; for a picture, see Figure 7. Then:

$$x_u=(1,0,ve^u), \qquad x_v=(0,1,e^u),$$
$$\xi=(0,-\tfrac{1}{2}e^{-(1/2)u},\tfrac{1}{2}e^{(1/2)u});$$
$$Sx_u=-\tfrac{1}{4}e^{-(1/2)u}x_v, \qquad Sx_v=0,$$
$$h(x_u,x_u)=ve^{(1/2)u}, \qquad h(x_u,x_v)=e^{(1/2)u}, \qquad h(x_v,x_v)=0;$$
$$\nabla_{x_u}x_u=\tfrac{1}{2}vx_v, \qquad \nabla_{x_u}x_v=\tfrac{1}{2}x_v=\nabla_{x_v}x_u, \qquad \nabla_{x_v}x_v=0.$$

So we see that this surface is an affine minimal, nonconvex surface which is *globally* the graph of a function. Furthermore this surface is *complete*, corresponding to the induced connection ∇ as well as to the Levi-Civita connection $\tilde{\nabla}$ of h, since its geodesics with respect to ∇ and $\tilde{\nabla}$ are (respectively) given by

$$\gamma_1(t)=x(at+b, e^{-at/2}(c_1\cos(at/2)+c_2\sin(at/2))$$

and

$$\gamma_2(t)=x(at+b, e^{-(1/2)at}(c_1t+c_2)),$$

where $a,b,c_1,c_2\in\mathbf{R}$.

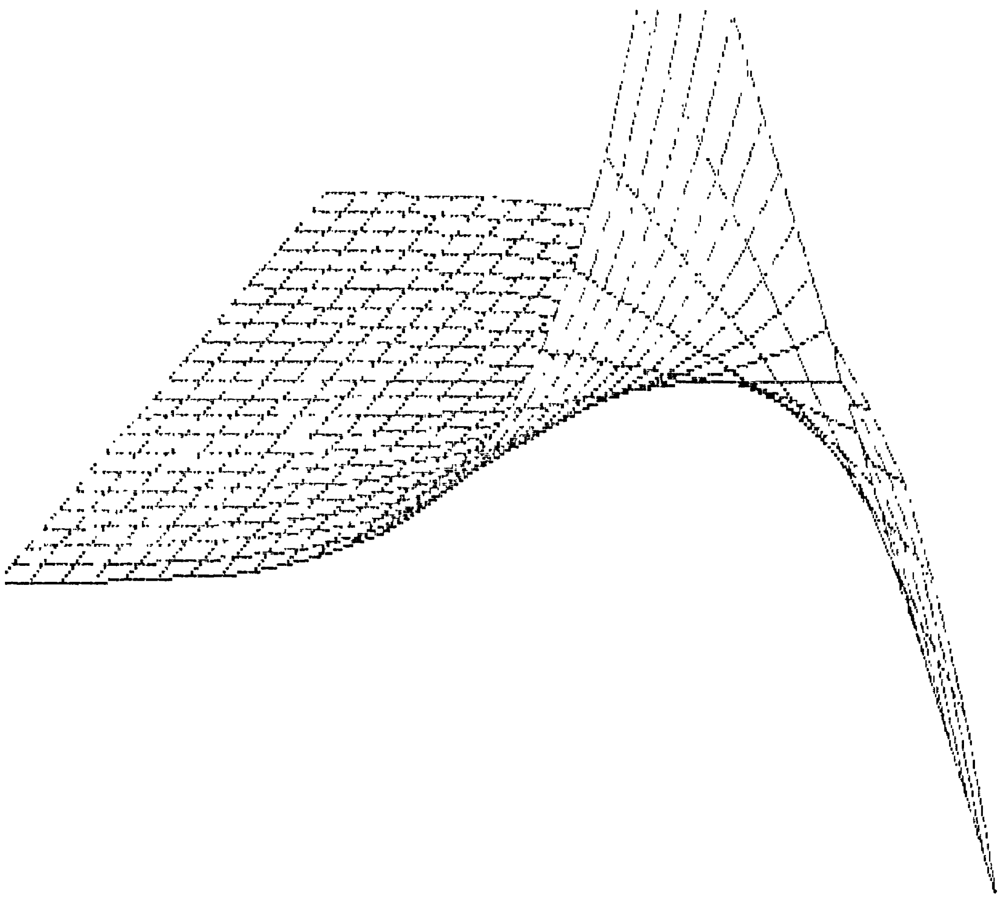

Figure 7 (u, v, ve^u)

REMARK. In the convex case, we have the following theorem of Calabi [2]. If $\phi: M \to \mathbf{A}^3$ is an affine minimal immersion, M convex, ϕM the graph of a function (geometrically complete), and M metrically complete (i.e., complete corresponding to h), then M is a paraboloid. However, the examples of ruled surfaces show that such a theorem cannot be true in the nonconvex case. Not only is there an example of a *flat* manifold which satisfies all the conditions except convexity, namely,

$$x(u, v) = (u, v, uv + g(u)),$$

but there is also an example of an *affine minimal*, *nonflat*, *nonconvex* surface which is *complete* corresponding to both connections and which is globally the graph of a certain function, namely,

$$x(u, v) = (u, v, ve^u).$$

4. Variation Formulas

We start with an affine *nondegenerate* immersion $\phi: M \to \mathbf{A}^{n+1}$; $p \mapsto \phi(p)$, with affine normal vector field ξ. Then $\phi_t: M \to \mathbf{A}^{n+1}$; $p \mapsto \phi_t(p) = \phi(p) + tf(p)\xi(p)$, where $f: M \to \mathbf{R}$ is a C^∞-function on M, with support in a compact domain W of M, and $t \in \mathbf{R}$ is also an affine, nondegenerate immersion for any t sufficiently small. We call ϕ_t a *variation in the affine normal direction* of ϕ. We then define

$$A(t) = \int_W \theta_t,$$

where θ_t is the volume element on M induced by ϕ_t. In the following theorem, we generalize (by deleting the convexity condition) a theorem of Calabi [2].

THEOREM 4.1. *Let $\phi: M \to \mathbf{A}^{n+1}$ be an affine, nondegenerate immersion. Then the following two conditions are equivalent:*

(i) *the immersion ϕ is a minimal immersion;*
(ii) *$A'(0) = 0$ for every variation in the normal direction of ϕ (i.e., the volume of ϕ is extremal).*

In order to prove this theorem we need the following lemma.

LEMMA 4.1. *Let $\phi: M^n \to \mathbf{A}^{n+1}$ be an affine, nondegenerate immersion. Let ϕ_t be a variation in the normal direction of ϕ. Then*

$$A'(0) = \int_M -\frac{n(n+1)}{2+n} f \cdot \operatorname{Tr} S \cdot \theta.$$

Proof. Let $p \in M$. By $\{e_1, e_2, \ldots, e_n\}$ we will always denote a basis of T_pM such that $\theta(e_1, e_2, \ldots, e_n) = 1$. Then $\{\phi_{t*}(e_1), \ldots, \phi_{t*}(e_n), \xi_t\}$ is a basis for $T_p\mathbf{A}^{n+1}$, where ξ_t is the affine normal corresponding to ϕ_t. Denote the dual basis by $\omega_{1t}, \ldots, \omega_{nt}, \omega_t$ and ω_{i0} by ω_i and ω_0 by ω. Then

$$\begin{aligned}\theta_t(e_1, e_2, \ldots, e_n) &= \Omega(\phi_{t*}(e_1), \ldots, \phi_{t*}(e_n), \xi_t)\\ &= \Omega(\phi_*(e_1) + t(\nabla f)(e_1)\xi - tf\phi_*(Se_1), \ldots, \phi_*(e_n)\\ &\quad + t(\nabla f)(e_n)\xi - tf\phi_*(Se_n), \xi_t),\end{aligned}$$

where $(\nabla f)(e_i) = e_i(f)$. From now on, since there is no confusion possible, we will denote $\phi_*(e_i)$ also by e_i. Then we find

$$\begin{aligned}\frac{d}{dt}\theta_t(e_1, e_2, \ldots, e_n)\Big|_{t=0} &= -f\sum_i \Omega(e_1, \ldots, e_{i-1}, Se_i, e_{i+1}, \ldots, e_n)\\ &\quad + \Omega\left(e_1, \ldots, e_n, \frac{d}{dt}\xi_t\Big|_{t=0}\right) \qquad (4.1)\\ &= -nf \operatorname{Tr} S + \omega(\xi_0'),\end{aligned}$$

where $\xi_0' = (d/dt)\xi_t|_{t=0}$ and $\operatorname{Tr} S = (1/n)\sum_i \omega^i(Se_i)$.

In order to compute ξ_0', we will use (2.1). Therefore, we will first compute $(d/dt)h_t(e_i, e_j)|_{t=0}$. To simplify the notations, we will also denote by e_i local extensions of the vectors e_i. Then

$$h_t(e_i, e_j) = \omega_t(D_{\phi_{t*}(e_i)}\phi_{t*}(e_j)) = \omega_t\left(\frac{d}{ds}\phi_{t*}(e_j)\right),$$

where we used a curve α such that $\alpha(t,0)=\phi_t(p)$ and $(d/ds)\alpha(t,0)=\phi_{*t}(e_i)$. Differentiating this and evaluating in $t=0$, we find

$$\frac{d}{dt}h_t(e_i,e_j)\bigg|_{t=0}=\omega'(\xi_t)\,|_{t=0}\,h(e_i,e_j)+(\nabla^2 f)(e_i,e_j)-h(e_i,Se_j)f, \tag{4.2}$$

where $(\nabla^2 f)(e_i,e_j)=e_i((\nabla f)(e_j))-(\nabla f)(\nabla_{e_i}e_j)$. Furthermore, we know that $\omega_t(\xi_t)=1$. By differentiating this and evaluating for $t=0$, we obtain

$$\omega'(\xi_t)\,|_{t=0}+\omega(\xi_0')=0,$$

and thus (4.2) becomes

$$\frac{d}{dt}h_t(e_i,e_j)\bigg|_{t=0}=-h(e_i,e_j)\omega(\xi_0')+(\nabla^2 f)(e_i,e_j)-h(e_i,Se_j)f. \tag{4.3}$$

Then, by combining (4.3) with (4.1) and (2.2), we can compute $\omega(\xi_0')$ as

$$\omega(\xi_0')=\frac{n}{2+n}f\cdot\operatorname{Tr}S+\frac{1}{2+n}\sum_{i=1}^{n}\det(a_{jk}^i), \tag{4.4}$$

where, if $\det(h(e_i,e_j))>0$,

$$a_{jk}^i=h(e_j,e_k)\quad\text{for } k\neq i,$$
$$a_{ji}^i=(\nabla^2 f)(e_j,e_i);$$

and where, if $\det(h(e_i,e_j))<0$,

$$a_{jk}^i=h(e_j,e_k)\quad\text{for } k\neq i,$$
$$a_{ji}^i=-(\nabla^2 f)(e_j,e_i).$$

Substituting this into (4.1), we obtain for the first variation formula that

$$A'(0)=\int_W\left\{-\frac{n(n+1)}{2+n}fH+\frac{1}{2+n}\sum_{i=1}^{n}\det(a_{jk}^i)\right\}\theta.$$

Applying the Codazzi equation, Green's theorem, and the fact that $\operatorname{supp} f\subset W$, we find after a somewhat long (but easy) computation that the second term cancels. So we find that

$$A'(0)=\int_W\left\{-\frac{n(n+1)}{2+n}fH\right\}\theta.$$

Since $\operatorname{supp} f\subset W$, we can also write this as

$$A'(0)=\int_M\left\{-\frac{n(n+1)}{2+n}fH\right\}\theta.$$

This completes the proof of the lemma. □

The proof of Theorem (4.1) now becomes obvious.

From now on, we will always assume that $\phi: M\to \mathbf{A}^{n+1}$ is a nondegenerate, affine *minimal* immersion. In order to compute the formula for the second variation in this case, we need some technical lemmas for which the proofs are straightforward.

LEMMA 4.2.

$$\frac{d^2}{dt^2}\theta_t(e_1, e_2, \dots, e_n)\bigg|_{t=0} = \omega(\xi_0'') - 2\sum_{i=1}^{n} (\nabla f)(e_i)\omega^i(\xi_0') + 2\sum_{i=1}^{n}\sum_{j>i} f^2\{\omega^i(se_i)\omega^j(Se_j) - \omega^j(Se_i)\omega^i(Se_j)\}.$$

LEMMA 4.3.

$$(\nabla^3 f)(X,Y,Z) - (\nabla^3 f)(Y,X,Z) = -(\nabla f)(R(X,Y)Z)) = -h(Y,Z)(\nabla f)(SX) + h(X,Z)(\nabla f)(SY).$$

LEMMA 4.4.

$$\begin{aligned}\frac{d^2}{dt^2}h_t(e_i, e_j)\bigg|_{t=0} &= h(e_i, e_j)(-2\omega'(\xi_0') - \omega(\xi_0''))\\ &\quad -2(\nabla^2 f)(e_i, e_j)\omega(\xi_0') + 2h(e_i, Se_j)\omega(\xi_0')\\ &\quad +2(\nabla f)(e_j)\sum_{k=1}^{n}\omega^k(Se_i)(\nabla f)(e_k)\\ &\quad +2(\nabla f)(e_i)\sum_{k=1}^{n}\omega^k(Se_j)(\nabla f)(e_k)\\ &\quad +2f\sum_{k=1}^{n}\omega^k((\nabla_{e_i}S)e_j)(\nabla f)(e_k).\end{aligned}$$

LEMMA 4.5.

$$\omega^k(\xi_0') = \det(c_{ij}^k) + \det(d_{ij}^k),$$

where, if $\det(h(e_i, e_j)) = 1$,

$$\begin{aligned} c_{ij}^k &= d_{ij}^k = h(e_i, e_j) \quad \textit{for } j \neq k,\\ c_{ik}^k &= -e_i(\omega(\xi_0')),\\ d_{ik}^k &= -(\nabla f)(Se_i);\end{aligned}$$

and where, if $\det(h(e_i, e_j)) = -1$, *we have*

$$\begin{aligned} c_{ij}^k &= d_{ij}^k = h(e_i, e_j) \quad \textit{for } j \neq k,\\ c_{ik}^k &= e_i(\omega(\xi_0')),\\ d_{ik}^k &= -(\nabla f)(Se_i).\end{aligned}$$

Proof. First, we take a curve $\alpha(s,t)$ such that $\alpha(0,t) = \phi(p)$ and $\alpha'(0,t) = \phi_{t*}(e_i)$. Then, we know from (2.1) that

$$\omega_t\left(\frac{d}{ds}\xi_t\right) = 0.$$

By differentiating this, we then obtain that

$$\omega(D_{e_i}\xi_0') = \omega_0'(Se_i) = -\sum_{j=1}^{n}\omega^j(Se_i)(\nabla f)(e_j). \tag{4.5}$$

On the other hand, by decomposing N_0' we find that

$$\xi_0' = \sum_{k=1}^{n} \omega^k(\xi_0')e_k + \omega(\xi_0')\xi,$$

and thus

$$\omega(D_{e_i}\xi_0') = \sum_{k=1}^{n} \omega^k(\xi_0')h(e_i, e_k) + e_i(\omega(\xi_0')).$$

Combining this formula with (4.5) then completes the proof. □

By a long but straightforward computation, and by combining the previous results, we have the following lemma.

LEMMA 4.6.

$$\omega(\xi_0'') = -\frac{2}{n+2}\sum_{i=1}^{n}\sum_{j>i} f^2\{\omega^i(Se_i)\omega^j(Se_j) - \omega^j(Se_i)\omega^i(Se_j)\}$$
$$-(n+1)(\omega(\xi_0'))^2 + \frac{2}{n+2}H(f) + \frac{2}{n+2}\alpha(f)$$
$$-\frac{2(n+1)}{n+2}\beta(f) + \frac{2}{n+2}\gamma(f) + \frac{2}{n+2}\delta(f) - 2\epsilon(f),$$

where

$$\alpha(f) = \sum_{j=1}^{n} \det(\alpha_{ik}^j),$$

with

$$\alpha_{ik}^j = h(e_i, e_k) \quad (j \neq k);$$

$$\alpha_{ij}^j = \begin{cases} f(\nabla^2 f)(e_i, Se_j) & \textit{if } \det(h(e_i, e_k)) = 1, \\ -f(\nabla^2 f)(e_i, Se_j) & \textit{if } \det(h(e_i, e_k)) = -1; \end{cases}$$

$$\beta(f) = -\sum_{k=1}^{n} (\nabla f)(e_k)\det(d_{ij}^k);$$

$$\gamma(f) = \sum_{k=1}^{n} \det(\gamma_{ij}^k);$$

with

$$\gamma_{ij}^k = h(e_i, e_j) \quad (j \neq k);$$

$$\gamma_{ik}^k = \begin{cases} (\nabla f)(Se_k)(\nabla f)(e_i) & \textit{if } \det(h(e_i, e_k)) = 1, \\ -(\nabla f)(Se_k)(\nabla f)(e_i) & \textit{if } \det(h(e_i, e_k)) = -1; \end{cases}$$

$$\delta(f) = \sum_{k=1}^{n} \det(\delta_{ij}^k);$$

with

$$\delta_{ij}^k = h(e_i, e_j) \quad (j \neq k),$$

$$\delta_{ik}^k = \begin{cases} (\nabla f)((\nabla_{e_i} S)e_k)\cdot f & \textit{if } \det(h(e_i, e_k)) = 1, \\ -(\nabla f)((\nabla_{e_i} S)e_k)\cdot f & \textit{if } \det(h(e_i, e_k)) = -1; \end{cases}$$

$$\epsilon(f) = -\sum_{k=1}^{n} (\nabla f)(e_k)\det(c_{ij}^k);$$

and

$$H(f)=\left(\sum_{i=1}^{n}\sum_{j>i}\det(H_k^{ij})\right)\cdot\det(h(e_m,e_{m'})),$$

with

$$H_{kl}^{ij}=h(e_k,e_l)\quad(i,j\neq l),$$
$$H_{ki}^{ij}=(\nabla^2 f)(e_k,e_i),$$
$$H_{kj}^{ij}=(\nabla^2 f)(e_k,e_j).$$

Using Green's theorem, the Codazzi equation, Lemma 4.3, and the fact that $\operatorname{supp}(f)\subset W$, we obtain the following formulas.

LEMMA 4.7.

$$\int_W \alpha(f)+\gamma(f)+\delta(f)=0.$$

LEMMA 4.8.

$$\int_W H(f)=\frac{1}{2}(n-1)\int_W \beta(f).$$

Combining all these lemmas, we find the following formula for the second variation:

$$A''(0)=\int_W\left\{-(n+1)(\omega(\xi_0'))^2-\frac{(n+1)}{(n+2)}f^2\right.$$
$$\left.\times\sum_{i,j=1}^{n}\omega^j(Se_i)\omega^i(Se_j)+\frac{(n+1)}{(n+2)}\beta(f)\right\}.$$

Using (4.4), we can write this in the following form:

$$A''(0)=\int_W\left\{-\frac{(n+1)}{(n+2)^2}\left(\sum_{i=1}^{n}\det(a_{jk}^i)\right)^2-\frac{n+1}{n+2}f^2\right.$$
$$\left.\times\sum_{i,j=1}^{n}\omega^j(Se_i)\omega^i(Se_j)+\frac{(n+1)}{(n+2)}\beta(f)\right\}.$$

Next, we introduce the following notation: Since $\sum_{i=1}^{n}\det(a_{jk}^i)$ has all the nice properties of a Laplacian, we denote

$$\sum_{i=1}^{n}\det(a_{jk}^i)=\Delta f.$$

Now, we have proved the following theorem.

THEOREM 4.2. *Let $\phi:M\to\mathbf{A}^{n+1}$ be an affine minimal, nondegenerate immersion. Let ϕ_t be a variation in the normal direction of ϕ. Then the formula of the second variation of the volume is given by*

$$A''(0)=\int_M\left\{-\frac{(n+1)}{(n+2)^2}(\Delta f)^2-\frac{(n+1)}{(n+2)}f^2\right.$$
$$\left.\times\sum_{i,j=1}^{n}\omega^j(Se_i)\omega^i(Se_j)+\frac{(n+1)}{(n+2)}\beta(f)\right\}.$$

In the special case of surfaces in $\mathbf{A}^3$, we obtain the two following corollaries.

COROLLARY 4.1. *Let $\phi: M \to \mathbf{A}^3$ be an affine minimal, nondegenerate, convex (h is positive or negative definite) immersion. Then*

$$A''(0) = \int_M -\frac{3}{16}(\Delta f)^2 + \frac{3}{2} f^2(\det S)$$
$$+\frac{3}{4}\{h(e_2,e_2)(\nabla f)(e_1)(\nabla f)(Se_1) + h(e_1,e_1)(\nabla f)(e_1)(\nabla f)(Se_1)$$
$$-h(e_1,e_2)\{(\nabla f)(e_1)(\nabla f)(Se_2) + (\nabla f)(e_2)(\nabla f)(Se_1)\}\}.$$

COROLLARY 4.2. *Let $\phi: M \to \mathbf{A}^3$ be an affine minimal, nondegenerate, nonconvex immersion. Then*

$$A''(0) = \int_M -\frac{3}{16}(\Delta f)^2 + \frac{3}{2} f^2(\det S)$$
$$+\frac{3}{4}\{-h(e_2,e_2)(\nabla f)(e_1)(\nabla f)(Se_1) + h(e_1,e_2)(\nabla f)(e_1)(\nabla f)(Se_2)$$
$$+h(e_1,e_2)(\nabla f)(Se_1)(\nabla f)(e_2) - h(e_1,e_1)(\nabla f)(e2)(\nabla f)(Se_2)\}.$$

Starting from a formula such as the one in Corollary 4.1, Calabi proved [2] that in the convex case the second variation is always negative definite; Calabi therefore proposes to call these surfaces maximal surfaces instead of minimal surfaces. Now, we want to show that a theorem such as the one of Calabi is not possible in the nonconvex case. Therefore, we will use the example of the helicoid which can be parameterized in the following form:

$$x(u,v) = (u\cos v, u\sin v, v),$$

where $u, v \in \mathbf{R}$. Then we have

$$\begin{aligned}
&x_u = (\cos v, \sin v, 0), && h(x_u,x_u) = 0;\\
&x_v = (-u\sin v, u\cos v, 1), && h(x_u,x_v) = -1;\\
&\xi = (\sin v, -\cos v, 0), && h(x_v,x_v) = 0;\\
&Sx_u = 0,\ Sx_v = -x_u, && \theta(x_u,x_v) = \Omega(x_u,x_v,\xi) = 1.
\end{aligned}$$

Therefore

$$A''(0) = \int_u \int_v \left\{-\frac{3}{4}\left(\frac{\partial^2 f}{\partial u \partial v}\right)^2 + \frac{3}{4}\left(\frac{\partial f}{\partial u}\right)^2\right\} du\, dv.$$

From [3], we know that we can construct a C^∞ function g on $\mathbf{R}$ such that

$$g(x) = \begin{cases} 1 & \text{for } x \in [-1,1],\\ 0 & \text{for } x \in]-\infty,-2] \text{ and } x \in [2,+\infty[, \end{cases}$$

with $1 \geq g \geq 0$ and $|g'(x)| \leq 3$ for all $x \in \mathbf{R}$. For an arbitrary positive number R, we then define h by $h(x) = g(1/R)x)$. We then have:

(4.5) $$h(x) = 1 \quad \text{for } x \in [-R,R],$$

(4.6) $$h(x)=0 \quad \text{for } x\in\,]-\infty, -2R] \text{ and } x\in[2R, +\infty[,$$

(4.7) $$1\geq h\geq 0,$$

(4.8) $$|h'(x)|\leq 3/R \quad \text{for all } x\in\mathbf{R}.$$

Now we are in a position to show that, for the helicoid, the second variation of the volume has no fixed sign. First, we define $f_1(u, v)$ by

$$f_1(u,v)=u\cdot h(u)h(v).$$

This function is C^∞ and has support in a compact domain. Then we have

$$\begin{aligned}A''(0)=&\int_{-R}^{R}\int_{-R}^{R}\frac{3}{4}\,du\,dv+\int_{-R}^{R}\int_{R}^{2R}\left\{-\frac{3}{4}(h'(v))^2+\frac{3}{4}(h(v))^2\right\}du\,dv\\&+\int_{-R}^{R}\int_{-2R}^{-R}\left\{-\frac{3}{4}(h'(v))^2+\frac{3}{4}(h(v))^2\right\}du\,dv\\&+\int_{-2R}^{-R}\int_{-2R}^{+2R}\left\{-\frac{3}{4}(h(u))^2(h'(v))^2-\frac{3}{4}u^2(h'(u))^2(h'(v))^2\right.\\&\qquad\left.+\frac{3}{4}(h(u))^2(h(v))^2+\frac{3}{4}u^2(h'(u))^2(h(v))^2\right\}du\,dv\\&+\int_{R}^{2R}\int_{-2R}^{+2R}\left\{-\frac{3}{4}(h(u))^2(h'(v))^2-\frac{3}{4}u^2(h'(u))^2(h'(v))^2\right.\\&\qquad\left.+\frac{3}{4}(h(u))^2(h(v))^2+\frac{3}{4}u^2(h'(u))^2(h(v))^2\right\}du\,dv.\end{aligned}$$

By using (4.5), (4.6), (4.7), and (4.8), we obtain the following inequalities:

$$A''(0)\geq 3R^2-\frac{3}{4}\cdot\frac{9}{R^2}4R^2-\frac{6}{4}\frac{9}{R^2}\cdot 4R^2-\frac{6}{4}4R^2\cdot\frac{9}{R^2}\cdot\frac{9}{R^2}(4R^2);$$

$$A''(0)\geq 3R^2-2025.$$

So we see that $A''(0)>0$ for R sufficiently great. On the other hand, if we define a function f_2 by

$$f_2(u,v)=u\cdot e^{\alpha v}\cdot h(u)\cdot h(v),$$

where $\alpha\in\mathbf{R}$, then we find, for the second variation of the helicoid, that

$$\begin{aligned}A''(0)=&\int_{-R}^{R}\int_{-R}^{R}\left\{-\frac{3}{4}\alpha^2e^{2\alpha v}+\frac{3}{4}e^{2\alpha v}\right\}du\,dv\\&+\int_{-R}^{R}\int_{R}^{2R}\left\{-\frac{3}{4}\alpha^2e^{2\alpha v}(h(v))^2-\frac{3}{4}e^{2\alpha v}(h'(v))^2+\frac{3}{4}e^{2\alpha v}(h(v))^2\right\}du\,dv\\&+\int_{-R}^{R}\int_{-2R}^{-R}\left\{-\frac{3}{4}\alpha^2e^{2\alpha v}(h(v))^2-\frac{3}{4}e^{2\alpha v}(h'(v))^2+\frac{3}{4}e^{2\alpha v}(h(v))^2\right\}du\,dv\\&+\int_{-2R}^{-R}\int_{-2R}^{+2R}\left\{-\frac{3}{4}\alpha^2e^{2\alpha v}(h(u))^2(h(v))^2-\frac{3}{4}\alpha^2u^2e^{2\alpha v}(h'(u))^2(h(v))^2\right.\end{aligned}$$

$$-\frac{3}{4}e^{2\alpha v}(h(u))^2(h'(v))^2-\frac{3}{4}u^2e^{2\alpha v}(h'(u))^2(h'(v))^2$$
$$+\frac{3}{4}e^{2\alpha v}(h(u))^2(h(v))^2+\frac{3}{4}u^2e^{2\alpha v}(h'(u))^2(h(v))^2\Big\}\,du\,dv$$
$$+\int_R^{2R}\int_{-2R}^{-2R}\Big\{-\frac{3}{4}\alpha^2e^{2\alpha v}(h(u))^2(h(v))^2-\frac{3}{4}\alpha^2u^2e^{2\alpha v}(h'(u))^2(h(v))^2$$
$$-\frac{3}{4}e^{2\alpha v}(h(u))^2(h'(v))^2-\frac{3}{4}u^2e^{2\alpha v}(h'(u))^2(h'(v))^2$$
$$+\frac{3}{4}e^{2\alpha v}(h(u))^2(h(v))^2+\frac{3}{4}u^2e^{2\alpha v}(h'(u))^2(h(v))^2\Big\}\,du\,dv.$$

By using (4.5), (4.6), (4.7), and (4.8), we obtain the following inequality:

$$A''(0)\leq\frac{1}{2\alpha}\Big\{-\frac{3}{4}\alpha^2+\frac{3}{4}\Big\}\cdot 2R\cdot\{e^{2\alpha R}-e^{-2\alpha R}\}$$
$$+\frac{1}{2\alpha}\Big\{\frac{3}{4}\Big\}\cdot 2R\cdot\{e^{4\alpha R}-e^{2\alpha R}\}+\frac{1}{2\alpha}\Big\{\frac{3}{4}\Big\}\cdot 2R\cdot\{e^{-2\alpha R}-e^{-4\alpha R}\}$$
$$+\frac{1}{2\alpha}\Big\{\frac{3}{2}\Big\}\cdot R\{e^{4\alpha R}-e^{-4\alpha R}\}+\frac{1}{2\alpha}\Big\{\frac{3}{2}\Big\}\cdot 36R\cdot\{e^{4\alpha R}-e^{-4\alpha R}\}.$$

Now, if we choose $\alpha=100$ and $R=1/200$, we obtain

$$A''(0)\leq\frac{1}{80000}\{-29997\}\cdot\{e-e^{-1}\}$$
$$+\frac{3}{80000}\{e^2\}+\frac{3}{80000}\{e^{-2}\}+\frac{111}{80000}\{e^2\}.$$

From this we have

$$A''(0)<0.$$

So for the helicoid we see that the second variation of the volume has no sign.

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