LOCAL INVARIANTS OF FOLIATIONS BY REAL HYPERSURFACES

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0. Introduction. In this paper we present complete local invariants under biholomorphic mappings of foliations of complex space by nondegenerate real hypersurfaces. Perhaps the most notable of these invariants is an intrinsic normal direction. Flow along this normal direction provides a map from leaf to leaf which is a contact transformation but, in general, not a CR isomorphism of the leaves of the foliation.

This study of local invariants is intentionally imitative of the work of Chern and Moser [1] — the local invariants of a foliation by real hypersurfaces should be similar to the local invariants of a single real hypersurface. One difference is the existence of an intrinsic normal direction to a foliation. Another involves the fourth-order curvature tensor $S_{\alpha\gamma}{}^{\beta}{}_{\bar{\sigma}}$. For a single real hypersurface, this has trace zero; that is, the Ricci tensor $S_{\gamma\bar{\sigma}} = S_{\alpha\gamma}{}^{\alpha}{}_{\bar{\sigma}} = 0$. This is not true for foliations. For example, while the foliation by hyperquadrics $\{\operatorname{Im} z_{n+1} - \sum |z_j|^2 = \text{constant}\}$ has curvature zero, the foliation by spheres $\{\sum |z_j|^2 = \text{constant}\}$ has positive Ricci curvature. It is natural to ask whether every real hypersurface exists as the leaf of a foliation with vanishing Ricci curvature $S_{\gamma\bar{\sigma}}$. We provide examples showing that this is not true.

It should be noted that Graham and Lee have studied similar objects. In [2] they examine the geometry of a foliation by real hypersurfaces together with a defining function. The local invariants they obtain are quite similar to those obtained here.

1. Local invariants. Let $\mathfrak{F}$ be a foliation by real hypersurfaces of a neighborhood of a point in $\mathbf{C}^{n+1}$, $n \geq 1$. We assume that $\mathfrak{F}$ is given as the level sets of a $\mathbf{C}^{\infty}$ real-valued function $r(z, \bar{z})$ with $dr \neq 0$. The function r may be replaced by $r^*(z, \bar{z}) = f(r(z, \bar{z}))$, where f is any function of one variable with nonzero derivative. In this section, we consider the local equivalence problem for such foliations, to wit: if $\mathfrak{F}$ is a foliation in a neighborhood of a point p and $\mathcal{G}$ is a foliation in a neighborhood of a point q, does there exist a biholomorphic mapping of a neighborhood of p taking p to q and taking the leaves of $\mathfrak{F}$ to the leaves of $\mathcal{G}$? Our solution is obtained via Cartan's Method of Equivalence. This method may be described as follows.

First, we restate the problem as one of equivalence of G-structures. (A G-structure may be viewed as a coframe — a basis for the cotangent space — well-defined up to the action of the group G.) Second, using conditions on the exterior derivatives of the elements of the coframe, we reduce the group to a smaller group. Third, we lift to the principal bundle of all such coframes obtaining a

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new G-structure. This G-structure, we shall find, can be reduced to the identity G-structure. Any equivalence of the original G-structure induces an equivalence of the final $\{e\}$-structure and vice versa. Thus, for example, the coefficients obtained when the exterior derivatives of the elements of the coframe are expressed in terms of elements of the coframe are local invariants and must be preserved by any equivalence.

We start by setting $\theta = i\,\partial r$. Suppose (by changing coordinates if necessary) that $\partial r/\partial z^{n+1} \neq 0$; if we set $\theta^\alpha = dz^\alpha$ then $\{\theta, \theta^\alpha\}$ is a basis for the $(1,0)$ forms at any point. (We shall, in general, take the indices α, β, γ, etc. to range over the integers from 1 to n.) This is our coframe, which depends on our choice of coordinates and our choice of r. If we start with different coordinates and a different r, we get a different coframe $\{\theta^*, \theta^{\alpha *}\}$ satisfying

$$\begin{aligned} \theta &= u\theta^*, \\ \theta^\alpha &= v^\alpha \theta + u_\beta{}^\alpha \theta^{\beta *}, \end{aligned} \tag{1.1}$$

where $u \neq 0$ is real, v^α is complex, and the matrix $u_\beta{}^\alpha$ is complex and invertible. The G-structure (1.1) depends only on the foliation $\mathfrak{F}$; moreover, $\mathfrak{F}$ can be obtained from this G-structure — the leaves of $\mathfrak{F}$ are the submanifolds along which the ideal of differential forms generated by $\theta - \bar{\theta}$ (an invariant of the G-structure) vanishes.

The next step is to examine the exterior derivatives of θ and θ^α, about which we know two things. First, since the almost complex structure is integrable,

$$d\theta \equiv d\theta^\alpha \equiv 0 \mod(\theta, \theta^\beta). \tag{1.2}$$

Second, since the distribution $\{\theta - \bar{\theta} = 0\}$ is integrable (with integral manifolds the leaves of $\mathfrak{F}$),

$$d(\theta - \bar{\theta}) \equiv 0 \mod(\theta - \bar{\theta}). \tag{1.3}$$

Equation (1.2) implies that

$$d\theta = \theta \wedge \eta + \theta^\alpha \wedge \eta_\alpha \tag{1.4}$$

for some 1-forms η, η_α. Equation (1.3) then becomes

$$\theta \wedge \eta + \theta^\alpha \wedge \eta_\alpha - \theta \wedge \bar{\eta} - \theta^{\bar{\beta}} \wedge \eta_{\bar{\beta}} \equiv 0 \mod(\theta - \bar{\theta}). \tag{1.5}$$

It follows that

$$\begin{aligned} \eta_\alpha &\equiv i g_{\alpha\bar{\beta}} \theta^{\bar{\beta}} + A_{\alpha\beta}\theta^\beta + B_\alpha \theta \mod(\theta - \bar{\theta}), \\ \eta - \bar{\eta} &\equiv B_\alpha \theta^\alpha - B_{\bar{\beta}} \theta^{\bar{\beta}} + iC\theta \mod(\theta - \bar{\theta}), \end{aligned} \tag{1.6}$$

where $A_{\alpha\beta} = A_{\beta\alpha}$, $C \in \mathbf{R}$, and

$$g_{\alpha\bar{\beta}} = g_{\bar{\beta}\alpha}. \tag{1.7}$$

Thus

$$d\theta = i g_{\alpha\bar{\beta}} \theta^\alpha \wedge \theta^{\bar{\beta}} + h_\alpha \theta^\alpha \wedge \bar{\theta} + \theta \wedge \varphi \tag{1.8}$$

for some functions h_α and some 1-form φ.

The matrix $(g_{\alpha\bar\beta})$ is the Levi form of the leaves of $\mathfrak{F}$. We assume the Levi form is nondegenerate. (If $(g_{\alpha\bar\beta})$ is definite, by replacing θ by $-\theta$ if necessary we may assume $(g_{\alpha\bar\beta})$ is positive definite, in which case the change of frame (1.1) should satisfy the additional condition $u>0$.) Let $g^{\alpha\bar\beta}$ be the inverse of $g_{\alpha\bar\beta}$:

$$g^{\alpha\bar\beta}g_{\gamma\bar\beta}=\delta_\gamma{}^\alpha. \tag{1.9}$$

Making the substitution (1.1) we have, modulo (θ),

$$d\theta\equiv ig_{\gamma\bar\sigma}u_\alpha{}^\gamma u_{\bar\beta}{}^{\bar\sigma}\theta^{\alpha*}\wedge\theta^{\bar\beta*}+(uh_\gamma u_\alpha{}^\gamma+iug_{\gamma\bar\beta}u_\alpha{}^\gamma v^{\bar\beta})\theta^{\alpha*}\wedge\bar\theta^*. \tag{1.10}$$

Thus

$$g_{\alpha\bar\beta}{}^*=v^{-1}g_{\gamma\bar\sigma}u_\alpha{}^\gamma u_{\bar\beta}{}^{\bar\sigma}, \tag{1.11}$$

$$h_\alpha{}^*=h_\gamma u_\alpha{}^\gamma+ig_{\gamma\bar\beta}u_\alpha{}^\gamma v^{\bar\beta}. \tag{1.12}$$

If we make a specific choice of $g_{\alpha\bar\beta}$, we may then require that the transformation (1.1) preserve that particular choice:

$$g_{\alpha\bar\beta}=u^{-1}g_{\gamma\bar\sigma}u_\alpha{}^\gamma u_{\bar\beta}{}^{\bar\sigma}; \tag{1.13}$$

that is, $(u_\beta{}^\alpha)$ is conformal unitary with respect to $(g_{\alpha\bar\beta})$. We assume that we have made such a choice of $g_{\alpha\bar\beta}$, either by using (1.11) to normalize $g_{\alpha\bar\beta}$ (e.g., if $g_{\alpha\bar\beta}$ is positive definite, we may take $g_{\alpha\bar\beta}=\delta_{\alpha\beta}$), or by some more arbitrary method (e.g., the $g_{\alpha\bar\beta}$ that appears during a calculation (cf. [1, §6])).

Because $g_{\alpha\bar\beta}$ is invertible, we can choose $v_{\bar\beta}$ in (1.12) so that $h_\alpha{}^*=0$. We then restrict our attention to coframes with $h_\alpha=0$, that is, those satisfying

$$d\theta\equiv ig_{\alpha\bar\beta}\theta^\alpha\wedge\theta^{\bar\beta}\quad \mathrm{mod}(\theta). \tag{1.14}$$

By (1.12), the change of coframe (1.1) taking one coframe to another then satisfies

$$v^\alpha=0. \tag{1.15}$$

The set Y of all such coframes is a principal bundle over $\mathbf{C}^{n+1}$ with group the group of transformations (1.1) satisfying (1.13) and (1.15). Having chosen an arbitrary reference coframe $\{\theta,\theta^\alpha\}$, we can take $u=\bar u$ and $u_\beta{}^\alpha$ as fiber coordinates on Y. The forms

$$\begin{aligned}\omega&=u\theta,\\ \omega^\alpha&=u_\beta{}^\alpha\theta^\beta\end{aligned} \tag{1.16}$$

are intrinsically defined on Y. Abstractly — for example, if $\pi: Y\to\mathbf{C}^{n+1}$ is a projection — then $p=\{\theta,\theta^\alpha\}$ is a point of Y, $X\in TY_p$, and $\omega(p)(X)=\theta(\pi_*X)$.

Differentiating (1.16), we find

$$d\omega=ig_{\alpha\bar\beta}\omega^\alpha\wedge\omega^{\bar\beta}+\omega\wedge\varphi \tag{1.17}$$

for some 1-form φ. It follows from (1.3) that $d(\omega-\bar\omega)\equiv 0 \ \mathrm{mod}(\omega-\bar\omega)$. Thus

$$0\equiv\omega\wedge(\varphi-\bar\varphi)\quad \mathrm{mod}(\omega-\bar\omega). \tag{1.18}$$

Therefore

$$\varphi-\bar\varphi=A\omega-\bar A\bar\omega \tag{1.19}$$

for some A. Replacing φ by $\varphi - A\omega$, we still have (1.17) and also have

$$\varphi = \bar{\varphi}. \tag{1.20}$$

This condition defines φ uniquely.

Differentiating the second equation of (1.16), using (1.2) we find that

$$d\omega^\alpha = \omega^\beta \wedge \varphi_\beta{}^\alpha + h^\alpha{}_{\bar\beta}\, \omega \wedge \omega^{\bar\beta} + k^\alpha \omega \wedge \bar\omega \tag{1.21}$$

for some 1-forms $\varphi_\beta{}^\alpha$ and some functions $h^\alpha{}_{\bar\beta}$ and k^α. The functions $h^\alpha{}_{\bar\beta}$ and k^α are well defined and depend on the 3-jet of the defining function (i.e., may be calculated at a point using only the 3-jet of the defining function at the point). The form $\varphi_\beta{}^\alpha$ is well defined up to a substitution:

$$\varphi_\beta{}^\alpha = \varphi_\beta{}^{\alpha *} + A_\beta{}^\alpha{}_\gamma\, \omega^\gamma, \tag{1.22}$$

with $A_\beta{}^\alpha{}_\gamma = A_\gamma{}^\alpha{}_\beta$. Note that ω, ω^α, φ and any choice of $\varphi_\beta{}^\alpha$ together form a coframe for Y. This coframe is well defined up to the transformation (1.22).

Differentiating (1.17),

$$\begin{aligned} 0 = {} & i(dg_{\alpha\bar\beta} - \varphi_{\alpha\bar\beta} - \varphi_{\bar\beta\alpha} + g_{\alpha\bar\beta}\varphi) \wedge \omega^\alpha \wedge \omega^{\bar\beta} + ih_{\alpha\gamma}\, \bar\omega \wedge \omega^\alpha \wedge \omega^\gamma \\ & + \omega \wedge (ih_{\bar\beta\bar\sigma}\, \omega^{\bar\sigma} \wedge \omega^{\bar\beta} + ik_{\bar\beta}\, \bar\omega \wedge \omega^{\bar\beta} - ik_\alpha\, \omega^\alpha \wedge \bar\omega - d\varphi), \end{aligned} \tag{1.23}$$

where we have used $g_{\alpha\bar\beta}, g^{\alpha\bar\beta}$ to raise and lower indices (e.g., $k_\alpha = g_{\alpha\bar\beta} k^{\bar\beta}$). It follows that

$$h_{\alpha\gamma} = h_{\gamma\alpha} \tag{1.24}$$

and

$$dg_{\alpha\bar\beta} - \varphi_{\alpha\bar\beta} - \varphi_{\bar\beta\alpha} + g_{\alpha\bar\beta}\varphi = a_{\alpha\bar\beta\gamma}\, \omega^\gamma + b_{\alpha\bar\beta\bar\sigma}\, \omega^{\bar\sigma} + c_{\alpha\bar\beta}\, \omega, \tag{1.25}$$

with $a_{\alpha\bar\beta\gamma} = a_{\gamma\bar\beta\alpha}$ and $b_{\alpha\bar\beta\bar\sigma} = b_{\alpha\bar\sigma\bar\beta}$. Since the left-hand side of (1.25) is hermitian symmetric, we must have $b_{\alpha\bar\beta\bar\sigma} = a_{\bar\beta\alpha\bar\sigma}$, $c_{\alpha\bar\beta} = 0$. So by setting $A_\beta{}^\alpha{}_\gamma = a_\beta{}^\alpha{}_\gamma$ in (1.22), we obtain (after dropping the *)

$$dg_{\alpha\bar\beta} - \varphi_{\alpha\bar\beta} - \varphi_{\bar\beta\alpha} + g_{\alpha\bar\beta}\varphi = 0. \tag{1.26}$$

This condition uniquely determines $\varphi_\beta{}^\alpha$, thereby providing us with a unique coframe $(\omega, \omega^\alpha, \varphi, \varphi_\beta{}^\alpha\}$ on Y. We summarize this result in the following theorem.

THEOREM. *On the coframe bundle Y there exists a unique coframe $\{\omega, \omega^\alpha, \varphi, \varphi_\beta{}^\alpha\}$ satisfying* (1.17), (1.20), (1.21), *and* (1.26) *and such that for any local section $s: \mathbf{C}^{n+1} \to Y$, the forms $s^*(\omega), s^*(\omega^\alpha)$ are of type* $(1,0)$ *and the leaves of the foliation are given by $s^*(\omega - \bar\omega) = 0$.*

Moreover, any biholomorphic map taking the leaves of one foliation $\mathfrak{F}$ to those of another foliation $\mathfrak{F}'$ induces a bundle map between the corresponding coframe bundles Y and Y', and this bundle map takes the unique coframe on Y to the unique coframe on Y'.

Thus any biholomorphic map taking one foliation to another preserves the forms ω, ω^α, φ, $\varphi_\beta{}^\alpha$ and hence preserves their exterior derivatives. Expressing their exterior derivatives in terms of the forms themselves we obtain coefficients which must also be preserved by the map. These coefficients are then local invariants. For example, $h_{\alpha\beta}$ and k^{α} are local invariants.

To obtain further invariants, we find formulas for $d\varphi$ and $d\varphi_\beta{}^\alpha$. From (1.23) it now follows that

$$d\varphi = ik_{\bar\beta}\bar\omega\wedge\omega^{\bar\beta} + ik_\alpha\bar\omega\wedge\omega^\alpha - ik_\alpha\omega\wedge\omega^\alpha - ik_{\bar\beta}\omega\wedge\omega^{\bar\beta} + il\omega\wedge\bar\omega \tag{1.27}$$

for some real l. This may be written

$$d\varphi = i(k_\alpha\omega^\alpha + k_{\bar\beta}\omega^{\bar\beta})\wedge(\omega-\bar\omega) + il\omega\wedge\bar\omega. \tag{1.28}$$

Differentiating (1.21) we obtain

$$0 = \omega\wedge\omega^{\bar\beta}\wedge Dh^\alpha{}_{\bar\beta} + \omega\wedge\bar\omega\wedge Dk^\alpha - \omega^\beta\wedge\Phi_\beta{}^\alpha, \tag{1.29}$$

where

$$Dh^\alpha{}_{\bar\beta} = dh^\alpha{}_{\bar\beta} + h^\gamma{}_{\bar\beta}\varphi_\gamma{}^\alpha - h^\alpha{}_{\bar\sigma}\varphi_{\bar\beta}{}^{\bar\sigma} - h^\alpha{}_{\bar\beta}\varphi, \tag{1.30}$$

$$Dk^\alpha = dk^\alpha + k^\beta\varphi_\beta{}^\alpha - 2k^\alpha\varphi - h^\alpha{}_{\bar\beta}h^{\bar\beta}{}_\gamma\omega^\gamma, \tag{1.31}$$

$$\Phi_\beta{}^\alpha = d\varphi_\beta{}^\alpha - \varphi_\beta{}^\gamma\wedge\varphi_\gamma{}^\alpha - ih^\alpha{}_{\bar\sigma}g_{\beta\bar\nu}\omega^{\bar\nu}\wedge\omega^{\bar\sigma} + ik^\alpha g_{\beta\bar\sigma}\omega^{\bar\sigma}\wedge(\omega-\bar\omega). \tag{1.32}$$

From (1.29) it follows that

$$-\Phi_\beta{}^\alpha = \mu_\beta{}^\alpha{}_\gamma\wedge\omega^\gamma + V^\alpha{}_{\beta\bar\sigma}\omega\wedge\omega^{\bar\sigma} + T_\beta{}^\alpha\omega\wedge\bar\omega, \tag{1.33}$$

$$Dk^\alpha = -T_\beta{}^\alpha\omega^\beta + B^\alpha{}_{\bar\sigma}\omega^{\bar\sigma} + U^\alpha\omega + W^\alpha\bar\omega, \tag{1.34}$$

$$Dh^\alpha{}_{\bar\beta} = -V^\alpha{}_{\gamma\bar\beta}\omega^\gamma + R^\alpha{}_{\bar\beta\bar\sigma}\omega^{\bar\sigma} + A^\alpha{}_{\bar\beta}\omega + B^\alpha{}_{\bar\beta}\bar\omega \tag{1.35}$$

for some one-forms $\mu_\beta{}^\alpha{}_\gamma$ satisfying

$$\mu_\beta{}^\alpha{}_\gamma = \mu_\gamma{}^\alpha{}_\beta \tag{1.36}$$

and some functions $V^\alpha{}_{\beta\bar\sigma}, T_\beta{}^\alpha, B^\alpha{}_{\bar\sigma}, U^\alpha, W^\alpha, R^\alpha{}_{\bar\beta\bar\sigma}, A^\alpha{}_{\bar\beta}$ satisfying

$$R^\alpha{}_{\bar\beta\bar\sigma} = R^\alpha{}_{\bar\sigma\bar\beta}. \tag{1.37}$$

Note that the symmetry (1.24) of $h_{\bar\alpha\bar\beta}$ implies (using equation (1.30) and equation (1.35))

$$\begin{aligned} V_{\bar\sigma\gamma\bar\beta} &= V_{\bar\beta\gamma\bar\sigma}, \\ R_{\bar\nu\bar\beta\bar\sigma} &= R_{\bar\beta\bar\nu\bar\sigma}, \\ A_{\bar\alpha\bar\beta} &= A_{\bar\beta\bar\alpha}, \\ B_{\bar\alpha\bar\beta} &= B_{\bar\beta\bar\alpha}. \end{aligned} \tag{1.38}$$

Differentiating (1.26) we obtain

$$\begin{aligned} 0 = {}& \omega^\gamma\wedge[\mu_{\alpha\bar\beta\gamma} + V_{\alpha\bar\beta\gamma}\bar\omega + ih_{\alpha\gamma}g_{\mu\bar\beta}\omega^\mu - i(k_\alpha g_{\gamma\bar\beta} + k_\gamma g_{\alpha\bar\beta})(\omega-\bar\omega)] \\ & + \omega^{\bar\sigma}\wedge[\mu_{\bar\beta\alpha\bar\sigma} + V_{\bar\beta\alpha\bar\sigma}\omega - ih_{\bar\beta\bar\sigma}g_{\alpha\bar\nu}\omega^{\bar\nu} - i(k_{\bar\beta}g_{\alpha\bar\sigma} + k_{\bar\sigma}g_{\alpha\bar\beta})(\omega-\bar\omega)] \\ & + (T_{\bar\beta\alpha} - T_{\alpha\bar\beta} - ilg_{\alpha\bar\beta})\omega\wedge\bar\omega. \end{aligned} \tag{1.39}$$

Therefore

$$T_{\bar\beta\alpha}-T_{\alpha\bar\beta}-ilg_{\alpha\bar\beta}=0, \tag{1.40}$$

$$\mu_{\alpha\bar\beta\gamma}=-V_{\alpha\bar\beta\gamma}\bar\omega-ih_{\alpha\gamma}g_{\mu\bar\beta}\omega^{\mu}+i(k_{\alpha}g_{\gamma\bar\beta}+k_{\gamma}g_{\alpha\bar\beta})(\omega-\bar\omega) + S_{\alpha\gamma\bar\beta\mu}\omega^{\mu}+S_{\alpha\gamma\bar\beta\bar\sigma}\omega^{\bar\sigma} \tag{1.41}$$

for some functions $S_{\alpha\gamma\bar\beta\mu}$, $S_{\alpha\mu\bar\beta\bar\sigma}$ satisfying

$$\begin{aligned} S_{\alpha\gamma\bar\beta\mu}&=S_{\alpha\mu\bar\beta\gamma},\\ S_{\alpha\gamma\bar\beta\bar\sigma}&=S_{\bar\beta\bar\sigma\alpha\gamma},\\ S_{\alpha\gamma\bar\beta\bar\sigma}&=S_{\gamma\alpha\bar\beta\bar\sigma}. \end{aligned} \tag{1.42}$$

Thus

$$\begin{aligned} -\Phi_{\beta}{}^{\alpha}=&-S_{\beta\gamma}{}^{\alpha}{}_{\bar\sigma}\omega^{\gamma}\wedge\omega^{\bar\sigma}-V_{\beta}{}^{\alpha}{}_{\gamma}\bar\omega\wedge\omega^{\gamma}+V^{\alpha}{}_{\beta\bar\sigma}\omega\wedge\omega^{\bar\sigma}-ih_{\beta\gamma}\delta_{\mu}{}^{\alpha}\omega^{\mu}\wedge\omega^{\gamma}\\ &+i(k_{\beta}\delta_{\gamma}{}^{\alpha}+k_{\gamma}\delta_{\beta}{}^{\alpha})(\omega-\bar\omega)\wedge\omega^{\gamma}+T_{\beta}{}^{\alpha}\omega\wedge\bar\omega \end{aligned} \tag{1.43}$$

and

$$\begin{aligned} d\varphi_{\beta}{}^{\alpha}=&\varphi_{\beta}{}^{\gamma}\wedge\varphi_{\gamma}{}^{\alpha}+ih^{\alpha}{}_{\bar\sigma}g_{\beta\bar\nu}\omega^{\bar\nu}\wedge\omega^{\bar\sigma}+ih_{\beta\gamma}\delta_{\mu}{}^{\alpha}\omega^{\mu}\wedge\mu^{\gamma}\\ &+S_{\beta\gamma}{}^{\alpha}{}_{\bar\sigma}\omega^{\gamma}\wedge\omega^{\bar\sigma}+V_{\beta}{}^{\alpha}{}_{\gamma}\bar\omega\wedge\omega^{\gamma}-V^{\alpha}{}_{\beta\bar\sigma}\omega\wedge\omega^{\bar\sigma}\\ &-i(k_{\beta}\delta_{\gamma}{}^{\alpha}+k_{\gamma}\delta_{\beta}{}^{\alpha})(\omega-\bar\omega)\wedge\omega^{\gamma}+ik^{\alpha}g_{\beta\bar\sigma}(\omega-\bar\omega)\wedge\omega^{\bar\sigma}-T_{\beta}{}^{\alpha}\omega\wedge\bar\omega. \end{aligned} \tag{1.44}$$

The tensors $S_{\beta\gamma}{}^{\alpha}{}_{\bar\sigma}$, $V_{\beta}{}^{\alpha}{}_{\gamma}, \ldots$ are local invariants. They are not all independent. By differentiating the structure equations we obtain relations among these tensors. (We also obtain further invariants — derivatives of curvature.)

Differentiating (1.44) we obtain further relations. The computations (and even the results) are quite long, so we shall simply state the one result we shall use later.

LEMMA. *If $n>1$ and $S_{\alpha\gamma}{}^{\alpha}{}_{\bar\sigma}$ vanishes identically, then $V_{\alpha}{}^{\alpha}{}_{\gamma}$ vanishes identically.*

Proof. Set $\alpha=\beta$ in (1.44) and sum:

$$\begin{aligned} d\varphi_{\alpha}{}^{\alpha}=&S_{\alpha\gamma}{}^{\alpha}{}_{\bar\sigma}\omega^{\gamma}\wedge\omega^{\bar\sigma}-V_{\alpha}{}^{\alpha}{}_{\gamma}\bar\omega\wedge\omega^{\gamma}+V^{\alpha}{}_{\alpha\bar\sigma}\omega\wedge\omega^{\bar\sigma}\\ &-i(n+1)k_{\gamma}(\omega-\bar\omega)\wedge\omega^{\gamma}+ik_{\bar\sigma}(\omega-\bar\omega)\wedge\omega^{\bar\sigma}-T_{\alpha}{}^{\alpha}\omega\wedge\bar\omega. \end{aligned} \tag{1.45}$$

By assumption, $S_{\alpha\gamma}{}^{\alpha}{}_{\bar\sigma}=0$. If we differentiate (1.45) and calculate modulo $(\omega,\bar\omega)$, we obtain

$$0\equiv ig_{\mu\bar\sigma}V_{\alpha}{}^{\alpha}{}_{\gamma}\omega^{\mu}\wedge\omega^{\bar\sigma}\wedge\omega^{\gamma}-ig_{\gamma\bar\beta}V^{\alpha}{}_{\alpha\bar\sigma}\omega^{\gamma}\wedge\omega^{\bar\beta}\wedge\omega^{\bar\sigma}. \tag{1.46}$$

Thus $0=g_{\mu\bar\sigma}V_{\alpha}{}^{\alpha}{}_{\gamma}-g_{\gamma\bar\sigma}V_{\alpha}{}^{\alpha}{}_{\mu}$. Multiplying by $g^{\mu\bar\sigma}$ and summing over μ and $\bar\sigma$, we have

$$0=(n-1)V_{\alpha}{}^{\alpha}{}_{\bar\sigma}, \tag{1.47}$$

and if $n\neq 1$ the result follows. □

2. Interpretation. In this section we attempt to provide an explanation of the geometric meaning of the third-order invariants $h^{\alpha}{}_{\bar\beta}$ and k^{α}. To do this, first note that the real direction $\omega+\bar{\omega}=\omega^{\alpha}=\omega^{\bar{\beta}}=0$ is well defined in $\mathbf{C}^{n+1}$ (even though the frame $\{\omega, \omega^{\alpha}\}$ is not). By following the integral curves of this direction we obtain a map $F: M \to M'$ between leaves of the foliation.

PROPOSITION. *The map F is a contact transformation; that is, $F^*(\omega|_{M'}) = \lambda\omega|_M$ for some nonzero constant λ.*

Proof. Choose a defining function r. Let X be the vector field defined by

$$(\omega+\bar{\omega})(X)=\omega^{\alpha}(X)=\omega^{\bar{\beta}}(X)=0, \quad dr(X)=1. \tag{2.1}$$

The map F is just the flow of X at a particular time. Thus it suffices to show that the Lie derivative

$$\mathcal{L}_X \omega|_M = \lambda\omega|_M \tag{2.2}$$

for every leaf M. Moreover, it suffices to show this for any particular choice of ω. Choose ω so that $\omega(X)=i/2$. Then $\omega=i\,\partial r$ and $\omega-\bar{\omega}=i\,dr$. So $0=d(\omega-\bar{\omega})=(\omega-\bar{\omega})\wedge\varphi$. Therefore φ is a multiple of $\omega-\bar{\omega}$ and $\varphi|_M=0$. We then have

$$\begin{aligned}\mathcal{L}_X\omega|_M &= i_X d\omega|_M + d(i_X\omega)|_M\\ &= (i/2)\varphi|_M - \varphi(X)\omega|_M\\ &= -\varphi(X)\omega|_M.\end{aligned} \tag{2.3}$$

□

In general F will not be a CR mapping, however. The map F is CR precisely when $\mathcal{L}_X\omega^{\alpha}|_M \equiv 0$ modulo (ω, ω^{β}). But

$$\mathcal{L}_X\omega^{\alpha}|_M \equiv (i/2)h^{\alpha}{}_{\bar{\beta}}\omega^{\bar{\beta}}|_M \text{ modulo } (\omega, \omega^{\gamma}). \tag{2.4}$$

Therefore, *$h^{\alpha}{}_{\bar{\beta}}$ is a measure of how the leaf map F is not a* CR *map.*

The tensor $h^{\alpha}{}_{\bar{\beta}}$ may be thought of as the infinitesimal change in the complex structure on the maximal complex tangent space of the leaf resulting from an infinitesimal movement in the intrinsic normal direction.

Similarly,

$$\mathcal{L}_X\omega^{\alpha} \equiv ik^{\alpha}\omega \ \operatorname{mod}(\omega^{\alpha}, \omega^{\bar{\beta}}, \omega-\bar{\omega}). \tag{2.5}$$

Thus, *k^{α} is a measure of how the complex normal direction is changing as we flow from leaf to leaf.*

Another interpretation of k^{α} is contained in the following.

PROPOSITION. *$k^{\alpha}=0$ if and only if the foliation $\mathfrak{F}$ has a defining function r which is a solution of the complex Monge-Ampere equation*

$$\det(\partial^2 r/\partial z_i\,\partial\bar{z}_j)=0. \tag{2.6}$$

Proof. Suppose r is a defining function satisfying (2.6). Let $\omega=i\,\partial r$. We saw above that this implies that φ is a multiple of $\omega-\bar{\omega}$. Then $\bar{\partial}r=i\bar{\omega}$ and

$$\partial\bar{\partial}r = i(ig_{\alpha\bar{\beta}}\omega^{\alpha}\wedge\omega^{\bar{\beta}}+\bar{\omega}\wedge\varphi). \tag{2.7}$$

Since φ is a multiple of $\omega-\bar{\omega}$, the defining function r will satisfy (2.6) only when $\varphi=0$. By using (1.28), we obtain $k^{\alpha}=0$.

Conversely, if $k^\alpha = 0$ then

$$d\varphi = il\omega \wedge \bar{\omega}. \tag{2.8}$$

Differentiating,

$$\begin{aligned} 0 &= d^2\varphi \\ &\equiv -lg_{\alpha\bar{\beta}}\omega^\alpha \wedge \omega^{\bar{\beta}} \wedge \omega - lg_{\alpha\bar{\beta}}\omega \wedge \omega^\alpha \wedge \omega^{\bar{\beta}} \end{aligned} \tag{2.9}$$

modulo $(\omega \wedge \bar{\omega})$. Thus we must have $l = 0$, and hence $d\varphi = 0$. By the Frobenius theorem, there is a codimension 1 real submanifold N of Y along which $\varphi = 0$. On this submanifold $d(\omega - \bar{\omega}) = 0$, so $\omega - \bar{\omega} = dR$ for some function R. Choosing a section $s: \mathbf{C}^{n+1} \to Y$ whose image lies in N, let $r = s^*R$. The function r is then a defining function satisfying (2.6) (since we may take a frame with $\omega = i\,\partial r$ and $\varphi = 0$ and apply (2.7)).

3. Ricci flat foliations. The local invariants of foliations described above resemble (intentionally) those of real hypersurfaces described in [1]. One particular parallel is the curvature tensor $S_{\alpha\gamma}{}^{\beta}{}_{\bar{\sigma}}$. As the curvature tensor of a real hypersurface, this must be trace free:

$$S_{\alpha\gamma}{}^{\alpha}{}_{\bar{\sigma}} = 0. \tag{3.1}$$

Calculation of the invariant of the foliation given by $\|z\| = \text{constant}$ shows that this condition need not be satisfied by the curvature tensor $S_{\alpha\gamma}{}^{\beta}{}_{\bar{\sigma}}$ of a foliation. We call a foliation *Ricci flat* if its curvature satisfies (3.1). It is then natural to ask whether every real hypersurface in $\mathbf{C}^{n+1}$ can locally be a leaf of a Ricci flat foliation. In this section, we provide examples of real hypersurfaces in $\mathbf{C}^{n+1}$, $n > 1$, for which there exist no such foliations.

For indeed, if M is a leaf of a Ricci flat foliation, then the local invariants of M satisfy certain relations. It will suffice to construct a real hypersurface whose invariants do not satisfy the necessary relations. These relations are calculated by starting with the local invariants of the foliation and using them to calculate the local invariants of the leaf. This is done very much like the calculations in [1, §6]. (We refer the reader to [1] for the structure equations, curvatures and such we shall use, though we shall also use the Bianchi identities as presented in [3].) The examples are constructed using Moser normal form.

Let M be a leaf of a Ricci flat foliation $\mathfrak{F}$, and let $\{\omega, \omega^\alpha\}$ be an adapted coframe, $g_{\alpha\bar{\beta}}, h^\alpha{}_{\bar{\beta}}, \ldots$ the local invariants in this frame. Then $\{\omega, \omega^\alpha, \varphi\}$ is an adapted frame on M and we wish to calculate the Chern–Moser connection forms $\tilde{\varphi}_\alpha{}^\beta, \tilde{\varphi}^\beta, \tilde{\psi}$ and their curvatures $S_{\alpha\gamma}{}^{\beta}{}_{\bar{\sigma}}, V_\alpha{}^{\beta}{}_{\gamma}, \ldots$. We have, on M,

$$\begin{aligned} d\omega &= ig_{\alpha\bar{\beta}}\omega^\alpha \wedge \omega^{\bar{\beta}} + \omega \wedge \varphi, \\ d\omega^\alpha &= \omega^\beta \wedge \varphi_\beta{}^\alpha + h^\alpha{}_{\bar{\beta}}\omega^{\bar{\beta}} \wedge \omega, \\ d\varphi &= 0. \end{aligned} \tag{3.2}$$

So we take, as a first approximation to $\tilde{\varphi}_\alpha{}^\beta, \tilde{\varphi}^\beta, \tilde{\psi}$:

$$
\begin{aligned}
&\varphi_\beta{}^{\alpha(1)} = \varphi_\beta{}^\alpha,\\
&\varphi^{\alpha(1)} = h^\alpha{}_{\bar\beta}\,\omega^{\bar\beta},\\
&\psi^{(1)} = 0.
\end{aligned}\tag{3.3}
$$

Then a little calculation shows

$$
\begin{aligned}
&d\varphi_\beta{}^{\alpha(1)} - \varphi_\beta{}^{\gamma(1)}\wedge\varphi_\gamma{}^{\alpha(1)} - i\omega_\beta\wedge\varphi^{\alpha(1)} + i\varphi_\beta{}^{(1)}\wedge\omega^\alpha\\
&\quad + i\delta_\beta{}^\alpha(\varphi_\sigma{}^{(1)}\wedge\omega^\sigma) - (1/2)\,\delta_\beta{}^\alpha\psi^{(1)}\wedge\omega\\
&= S_{\alpha\gamma}{}^\beta{}_{\bar\sigma}\,\omega^\gamma\wedge\omega^{\bar\sigma} + V_\alpha{}^\beta{}_\gamma\,\omega^\gamma\wedge\omega - V^\beta{}_{\alpha\bar\sigma}\,\omega^{\bar\sigma}\wedge\omega.
\end{aligned}\tag{3.4}
$$

Using the lemma at the end of Section 1, we see that we can take

$$
\begin{aligned}
&\tilde\varphi_\beta{}^\alpha = \varphi_\beta{}^\alpha,\\
&\tilde\varphi^\alpha = h^\alpha{}_{\bar\beta}\,\omega^{\bar\beta}.
\end{aligned}\tag{3.5}
$$

Then

$$
\begin{aligned}
&d\tilde\varphi^\alpha - \varphi\wedge\tilde\varphi^\alpha - \tilde\varphi^\beta\wedge\tilde\varphi_\beta{}^\alpha - (1/2)\psi^{(1)}\wedge\omega^\alpha\\
&= V^\alpha{}_{\gamma\bar\beta}\,\omega^\gamma\wedge\omega^{\bar\beta} + h^\alpha{}_{\bar\beta}h^{\bar\beta}{}_\gamma\,\omega\wedge\omega^\gamma + (A^\alpha{}_{\bar\beta} + B^\alpha{}_{\bar\beta})\,\omega\wedge\omega^{\bar\beta},
\end{aligned}\tag{3.6}
$$

from which we can derive that we must take

$$\tilde\psi = -(2/n)\,h^{\alpha\beta}h_{\alpha\beta}\,\omega \tag{3.7}$$

and that

$$\tilde P_\gamma{}^\alpha = -h^\alpha{}_{\bar\beta}h^{\bar\beta}{}_\gamma + (1/n)\,h^\mu{}_{\bar\beta}h^{\bar\beta}{}_\mu\,\delta_\gamma{}^\alpha, \tag{3.8}$$

$$\tilde Q_\sigma{}^\alpha = A^\alpha{}_{\bar\sigma} + B^\alpha{}_{\bar\sigma}. \tag{3.9}$$

Differentiating (3.8) and equating coefficients of ω, we obtain

$$0 = \tilde P_\gamma{}^\alpha{}_{,*} + h^{\bar\beta}{}_\gamma\tilde Q_{\bar\beta}{}^\alpha + h^\alpha{}_{\bar\beta}\tilde Q_\gamma{}^{\bar\beta} - (1/n)\,h^{\bar\beta}{}_\mu\tilde Q_{\bar\beta}{}^\mu\delta_\gamma{}^\alpha - (1/n)\,h^\mu{}_{\bar\beta}\tilde Q_\mu{}^{\bar\beta}\delta_\gamma{}^\alpha. \tag{3.10}$$

This is clearly not satisfied if

$$\tilde Q_\alpha{}^{\bar\beta} = 0 \tag{3.11}$$

and

$$\tilde P_\gamma{}^\alpha{}_{,*} \neq 0 \tag{3.12}$$

Thus it suffices for us to construct a real hypersurface satisfying (3.11) and (3.12) at a given point.

THEOREM. *The real hypersurface* $M = \{r(z,\bar z) = 0\} \subset \mathbf{C}^{n+1} = \{(z_1,\dots,z_n,w)\}$ $(n>1)$ *is not the leaf of a Ricci flat foliation in any neighborhood of the origin if*

$$r = (w-\bar w)/2i + \sum|z_\alpha|^2 + (z_1{}^3\bar z_1{}^3 - z_2{}^3\bar z_2{}^3)(w+\bar w) \tag{3.13}$$

or

$$r = (w-\bar w)/2i + \sum|z_\alpha|^2 + (z_1{}^3\bar z_1{}^2\bar z_2 + z_1{}^2z_2\bar z_1{}^3)(w+\bar w). \tag{3.14}$$

Proof. Calculate that (3.11) and (3.12) are satisfied at the origin (cf. [1, §6]). □

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